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Monogenic functions and harmonic vectors

Sergiy Plaksa

Dedicated to the memory of Professor Oleksandr Bakhtin

Abstract. We consider special topological vector spaces with a commutative multiplication for some of elements of the spaces and monogenic functions taking values in these spaces. Monogenic functions are understood as continuous and differentiable in the sense of Gâteaux functions. We describe relations between the mentioned monogenic functions and harmonic vectors in the three-dimensional real space and establish sufficient conditions for infinite monogeneity of functions. Unlike the classical complex analysis, it is done in the case where the validity of the Cauchy integral formula for monogenic functions remains an open problem.

Анотація. Розглядаються спеціальні топологічні векторні простори з комутативним множенням для деяких елементів просторів і моногенні функції, що приймають значення в цих просторах. Моногенні функції розуміються як неперервні і диференційовні за Гато функції. Описуються зв'язки між вказаними моногенними функціями і гармонічними векторами в тривимірному дійсному просторі та встановлюються достатні умови для нескінченної моногенності функцій. На відміну від класичного комплексного аналізу, це зроблено у випадку, коли питання про справедливість інтегральної формули Коші для моногенних функцій залишається відкритою проблемою.

Keywords: monogenic function, harmonic function, harmonic vector, Gâteaux derivative, topological vector space, commutative multiplication

Ключові слова: моногенна функція, гармонійна функція, гармонійний вектор, похідна Гато, топологічний векторний простір, комутативне множення

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1. Preliminary information

It is well-known that every doubly continuously differentiable function $u(x, y, z)$ satisfying the three-dimensional Laplace equation

$$\Delta_3 u(x, y, z) := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u(x, y, z) = 0 \tag{1.1}$$

generates the vector-function $\mathbf{V} = \operatorname{grad} u$, which is a solution of the system of equations

$$\operatorname{div}\mathbf{V} = 0, \qquad \operatorname{rot}\mathbf{V} = 0 \tag{1.2}$$

that we rewrite also in the following expanded form:

$$\begin{gathered} \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0, \\ \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} = 0, \qquad \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} = 0, \qquad \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0, \end{gathered} \tag{1.3}$$

where $\mathbf{V} := (v_1, v_2, v_3)$ and $v_k := v_k(x, y, z)$ for $k = 1, 2, 3$ are real scalar functions of Cartesian coordinates x, y, z.

Doubly continuously differentiable functions satisfying equation (1.1) are called spatial *harmonic* functions, and solutions of system (1.2) are called *harmonic* vectors.

It is evident that in the case of plane field, where $v_3 \equiv 0$ and the functions v_1, v_2 do not depend on z, the system (1.3) turns into the classical Cauchy-Riemann conditions for components $u = v_1$, $v = -v_2$ of complex potential

$$F(x + iy) = u(x, y) + iv(x, y)$$

being an analytic function of the complex variable $x + iy$. Moreover, every analytic function $F(x + iy)$ satisfies the two-dimensional Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) F(x + iy) \equiv F''(x + iy)(1^2 + i^2) = 0$$

due to the equality $1^2 + i^2 = 0$ for the unit 1 and the imaginary unit i of the algebra of complex numbers.

Analytic function methods in the complex plane for plane potential fields inspire searching analogous effective methods for solving spatial and multidimensional problems of mathematical physics. Many such methods are based on mappings of hypercomplex algebras.

Gr. Moisil and N. Théodoresco [5], R. Fueter [1] considered generalizations of system (1.3) for functions taking values in the algebra of quaternions.

I.P. Mel'nichenko [4] considered the problem on finding a commutative algebra such that differentiable in some sense functions with values in this algebra have components generating solutions of system (1.3). This problem has relations to the problem considered by P. W. Ketchum [2] on finding commutative harmonic algebras. P. W. Ketchum called a commutative algebra *harmonic* if in this algebra there exists a *harmonic* triad $\{e_1, e_2, e_3\}$ of linearly independent vectors satisfying the equality

$$e_1^2 + e_2^2 + e_3^2 = 0. \tag{1.4}$$

Let

$$E_3 := \{\zeta = xe_1 + ye_2 + ze_3 \mid x, y, z \in \mathbb{R}\}$$

be the linear span of the vectors e_1, e_2, e_3 over the real field $\mathbb{R}$. We will use the same denotation Ω for a domain $\Omega \subset \mathbb{R}^3$ and for the domain in E_3 that is congruent to the domain Ω.

I. P. Mel'nichenko [4] used differentiable in the sense of Gâteaux functions when solving the mentioned problems. Using the Gâteaux differential, I.P. Mel'nichenko suggested to consider the Gâteaux derivative as a function defined in the same domain as a given function.

In what follows, we shall consider concrete Hausdorff topological vector spaces. Let $\mathbb{A}$ be such a space in which a commutative multiplication $\xi\zeta$ is defined for all $\xi \in \mathbb{A}$ and all $\zeta \in E_3 \subset \mathbb{A}$.

Definition 1.1 (cf. I. P. Mel'nichenko [4]). *A function $\Phi\colon \Omega \to \mathbb{A}$ given in a domain $\Omega \subset E_3$ is called* **differentiable in the sense of Gâteaux at a point** *$\zeta \in \Omega$ if there exists an element $\Phi'_G(\zeta) \in \mathbb{A}$ such that*

$$\lim_{\delta\to 0^+} \left(\Phi(\zeta + \delta h) - \Phi(\zeta)\right) \delta^{-1} = h\Phi'_G(\zeta) \qquad \forall h \in E_3.$$

We will call $\Phi'_G(\zeta)$ the **Gâteaux derivative** *of the function Φ at the point ζ.*

To formulate results obtained by I.P. Mel'nichenko [4], consider a concept of monogenic function.

Definition 1.2 (cf. [6,7]). *Say that a function $\Phi\colon \Omega \to \mathbb{A}$ is* **monogenic** *in a domain $\Omega \subset E_3$ if Φ is continuous and differentiable in the sense of Gâteaux at every point $\zeta \in \Omega$.*

Let us formulate in the following statement the results obtained by I.P. Mel'nichenko [4]:

Theorem 1.3 (I.P. Mel'nichenko, [4]). *The commutative associative algebra $\mathbb{A}$ over the complex field $\mathbb{C}$ is harmonic, if the multiplication table for the basis $\{e_1, e_2, e_3\}$ is of the following form:*

$$e_ke_1 = e_k, \qquad k = 1, 2, 3;$$

$$e_2e_2 = -\frac{1}{2}e_1 - \frac{i}{2}(\sin\omega)e_2 + \frac{i}{2}(\cos\omega)e_3,$$

$$e_2e_3 = \frac{i}{2}(\cos\omega)e_2 + \frac{i}{2}(\sin\omega)e_3,$$

$$e_3e_3 = -\frac{1}{2}e_1 + \frac{i}{2}(\sin\omega)e_2 - \frac{i}{2}(\cos\omega)e_3,$$

where i is the imaginary complex unit and $\omega \in \mathbb{C}$. Moreover, if a function $\Phi\colon \Omega \to \mathbb{A}$ is monogenic in a domain $\Omega \subset E_3$, then the components $U_k\colon \Omega \to \mathbb{C}$, $k = 1, 2, 3$, of expansion

$$\Phi(xe_1 + ye_2 + ze_3) = \sum_{k=1}^{3} U_k(x, y, z)e_k, \qquad (x, y, z) \in \Omega,$$

generate the harmonic vectors

$$\mathbf{V}_1 = \Big(\mathrm{Re}U_1, -\frac{1}{2}\mathrm{Re}U_2, -\frac{1}{2}\mathrm{Re}U_3\Big), \quad \mathbf{V}_2 = \Big(\mathrm{Im}V_1, -\frac{1}{2}\mathrm{Im}V_2, -\frac{1}{2}\mathrm{Im}V_3\Big).$$

However, it is impossible to obtain all spatial harmonic functions as components of monogenic functions taking values in finite-dimensional commutative algebras (see, e.g. [3, 12]). Moreover, for each such algebra there are spherical functions that are not components of monogenic functions taking values in this algebra.

In the papers [6, 12], we considered an infinite-dimensional commutative Banach algebra $\mathbb{F}$ over the real field and established that any spherical function is a component of some monogenic function taking values in this algebra.

Note that the algebra $\mathbb{F}$ is contained in a vector space considered by P. W. Ketchum [3]. This vector space is not an algebra, though.

P. W. Ketchum proved that a set of components of functions taking values in the mentioned space includes all analytic solutions of equation (1.1). M. N. Roşculeţ [10] considered an other infinite-dimensional vector space and functions generating solutions of equation (1.1).

In the paper [8], we considered a topological vector space containing the algebra $\mathbb{F}$ and proved that all spatial harmonic functions are components of monogenic functions taking values in this space. We described some relations between the mentioned monogenic functions and harmonic vectors in the three-dimensional real space. We also proved similar results for monogenic functions taking values in a topological vector space whose support coincides with the vector space considered by M. N. Roşculeţ [10]. The consideration of such topological vector spaces is motivated by the need to describe all spatial harmonic functions as components of hypercomplex monogenic functions.

In this paper, we continue studies of relations between monogenic functions and harmonic vectors that were started in the paper [8]. We prove that for every harmonic vector there exist monogenic functions generating this vector in a way similar to that described in Theorem 1.3. Using this fact and some properties of harmonic vectors, we establish sufficient conditions for the Gâteaux derivatives of all orders of a monogenic function to be monogenic. Unlike the classical complex analysis, it is done in the case where the validity of the Cauchy integral formula for monogenic functions remains an open problem.

2. Infinite-dimensional harmonic algebra $\mathbb{F}$ and a topological vector space $\widetilde{\mathbb{F}}$ containing the algebra $\mathbb{F}$

Consider an infinite-dimensional commutative Banach algebra

$$\mathbb{F} := \left\{ g = \sum_{k=1}^{\infty} c_k e_k \mid c_k \in \mathbb{R}, \sum_{k=1}^{\infty} |c_k| < \infty \right\}$$

over the real field $\mathbb{R}$ with the norm

$$\|g\|_{\mathbb{F}} := \sum_{k=1}^{\infty} |c_k|$$

and the basis $\{e_k\}_{k=1}^{\infty}$, where the multiplication table for elements of basis is of the following form:

$$e_n e_1 = e_n, \qquad e_{2n+1} e_{2n} = \frac{1}{2} e_{4n} \quad \forall n \geqslant 1,$$

$$e_{2n+1} e_{2m} = \frac{1}{2}\Big(e_{2n+2m} - (-1)^m e_{2n-2m}\Big) \quad \forall n > m \geqslant 1,$$

$$e_{2n+1} e_{2m} = \frac{1}{2}\Big(e_{2n+2m} + (-1)^n e_{2m-2n}\Big) \quad \forall m > n \geqslant 1,$$

$$e_{2n+1} e_{2m+1} = \frac{1}{2}\Big(e_{2n+2m+1} + (-1)^m e_{2n-2m+1}\Big) \quad \forall n \geqslant m \geqslant 1,$$

$$e_{2n} e_{2m} = \frac{1}{2}\Big(-e_{2n+2m+1} + (-1)^m e_{2n-2m+1}\Big) \quad \forall n \geqslant m \geqslant 1.$$

It is evident that the elements e_1, e_2, e_3 form a harmonic triad of vectors, i.e., they satisfy the equality (1.4).

Note that the algebra $\mathbb{F}$ is isomorphic to the algebra $\mathbf{F}$ of absolutely convergent trigonometric Fourier series

$$g(\tau) = a_0 + \sum_{k=1}^{\infty} \big(a_k i^k \cos k\tau + b_k i^k \sin k\tau\big)$$

with real coefficients a_0, a_k, b_k and the norm $\|g\|_{\mathbf{F}} := |a_0| + \sum\limits_{k=1}^{\infty} \big(|a_k| + |b_k|\big)$, and we have the isomorphism

$$e_{2k-1} \;\leftrightarrow\; i^{k-1}\cos(k-1)\tau, \qquad\qquad e_{2k} \;\leftrightarrow\; i^k \sin k\tau$$

between the elements of bases.

Let us embed the algebra $\mathbb{F}$ in the topological vector space

$$\widetilde{\mathbb{F}} := \left\{ g = \sum_{k=1}^{\infty} c_k e_k : c_k \in \mathbb{R} \right\}$$

with the topology of coordinate-wise convergence.

Note that $\widetilde{\mathbb{F}}$ is not an algebra since the product of elements $g_1, g_2 \in \widetilde{\mathbb{F}}$ is not always defined. However, for all

$$g = \sum_{k=1}^{\infty} c_k e_k \in \widetilde{\mathbb{F}} \qquad \text{and} \qquad \zeta = xe_1 + ye_2 + ze_3,$$

one can define the product

$$\begin{aligned} g\zeta \equiv \zeta g := x \sum_{k=1}^{\infty} c_k e_k + y\Bigg(&-\frac{c_2}{2}e_1 + \left(c_1 - \frac{c_5}{2}\right) e_2 - \frac{c_4}{2}e_3 \\ &+ \frac{1}{2}\sum_{k=2}^{\infty}(c_{2k-1} - c_{2k+3})e_{2k} - \frac{1}{2}\sum_{k=2}^{\infty}(c_{2k-2} + c_{2k+2})e_{2k+1}\Bigg) \\ &+ z\Bigg(-\frac{c_3}{2}e_1 - \frac{c_4}{2}e_2 + \left(c_1 - \frac{c_5}{2}\right) e_3 + \frac{1}{2}\sum_{k=4}^{\infty}(c_{k-2} - c_{k+2})e_k\Bigg). \end{aligned}$$

Thus, the notion of differentiability in the sense of Gâteaux defined in Definition 1.1 is applied to functions $\Phi\colon \Omega \to \widetilde{\mathbb{F}}$ given in a domain $\Omega \subset E_3$ and taking values in the space $\widetilde{\mathbb{F}}$. In the considered case, for a monogenic function $\Phi\colon \Omega \to \widetilde{\mathbb{F}}$, the Gâteaux derivative $\Phi'_G(\zeta)$ takes values in the space $\widetilde{\mathbb{F}}$ for all $\zeta \in \Omega$.

Essentially, P. W. Ketchum [3] considered the space $\widetilde{\mathbb{F}}$ though he did not use the notion of topological vector space as well as the differentiability in the sense of Gâteaux.

3. Some conditions for the existence of harmonic vectors

Before describing the relationship between monogenic functions taking values in the space $\widetilde{\mathbb{F}}$ and harmonic vectors, consider the following property of solutions of system (1.3):

Proposition 3.1. *For every spatial harmonic function $v_1 : \Omega \to \mathbb{R}$ defined on a simply connected domain $\Omega \subset \mathbb{R}^3$, there exists a harmonic vector $\mathbf{V}_0 := (v_1, v_2^0, v_3^0)$ in the domain Ω. Moreover, for any ball $\mho \subset \Omega$ centered at any point $(x_0, y_0, z_0) \in \Omega$ and for any vector $\mathbf{V} := (v_1, v_2, v_3)$ harmonic in $\mho$, the components v_2, v_3 are determined accurate within the real part and the imaginary part of any function $h(t)$ holomorphic in the domain*

$$\{t = z + iy : (x, y, z) \in \mho\}$$

of the complex plane, i.e., for all $(x, y, z) \in \mho$, the following equalities hold:

$$v_2(x, y, z) = v_2^0(x, y, z) + \operatorname{Re} h(z + iy),$$
$$v_3(x, y, z) = v_3^0(x, y, z) + \operatorname{Im} h(z + iy).$$

Proof. Let us show that there exists a spatial harmonic function $u \colon \Omega \to \mathbb{R}$ such that $v_1(x, y, z) = \partial u(x, y, z)/\partial x$ for all $(x, y, z) \in \Omega$.

First of all, for (x, y, z) belonging to a ball $\mho \subset \Omega$ centered at any point $(x_0, y_0, z_0) \in \Omega$, we shall find this function in the form

$$u(x, y, z) = \int\limits_{x_0}^{x} v_1(\tau, y, z) d\tau + \widetilde{u}(z, y). \tag{3.1}$$

Substituting expression (3.1) into equation (1.1) and taking into account that v_1 is a spatial harmonic function in the domain $\mho$, we obtain the two-dimensional Poisson equation

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}\right) \widetilde{u}(z, y) = -\frac{\partial v_1(x, y, z)}{\partial x}\bigg|_{x=x_0}$$

for finding the function $\widetilde{u}$. It is clear that the function $\widetilde{u}$ is determined accurate within a summand that is a harmonic function in the domain $\{(z, y) : (x, y, z) \in \mho\} \subset \mathbb{R}^2$.

Further, we use the standard continuation of harmonic function (3.1) along a curve. We start with a fixed function (3.1) defined in the mentioned spheric neighborhood $\mho$ of a fixed point $(x_0, y_0, z_0) \in \Omega$ and extend this function along any curve via harmonic functions of form (3.1) that are defined in small overlapping balls covering that curve. Since the domain Ω is simply connected, taking into account the uniqueness theorem for spatial harmonic functions (see, e.g., A.F. Timan and V.N. Trofimov [13, p. 88]), as a result we get the function u defined in the domain Ω. By construction, u is a spatial harmonic function in Ω and $v_1(x,y,z) = \partial u(x,y,z)/\partial x$ for all $(x,y,z) \in \Omega$.

Then $\mathbf{V}_0 := \operatorname{grad} u$, i.e.,

$$v_2^0 := \partial u(x,y,z)/\partial y, \qquad v_3^0 := \partial u(x,y,z)/\partial z,$$

for all $(x,y,z) \in \Omega$.

In addition, for any vector $\mathbf{V} := (v_1, v_2, v_3)$ harmonic in the ball $\mho$, consider the harmonic vector $\mathbf{V} - \mathbf{V}_0$ in $\mho$. It is evident that its coordinates $(0, v_2 - v_2^0, v_3 - v_3^0)$ satisfy equations (1.3), which take the form

$$\frac{\partial(v_2 - v_2^0)}{\partial z} = \frac{\partial(v_3 - v_3^0)}{\partial y}, \qquad \frac{\partial(v_2 - v_2^0)}{\partial y} = -\frac{\partial(v_3 - v_3^0)}{\partial z},$$

$$\frac{\partial(v_2 - v_2^0)}{\partial x} = 0, \qquad \frac{\partial(v_3 - v_3^0)}{\partial x} = 0.$$

Therefore, the functions $v_2 - v_2^0$ and $v_3 - v_3^0$ are the real part and the imaginary part of a function $h(t)$ holomorphic in the domain

$$\{t = z + iy : (x,y,z) \in \mho\}$$

of the complex plane, respectively. $\square$

Let us formulate two analogous properties of solutions of system (1.3) that are similarly proved to Proposition 3.1.

Proposition 3.2. *For every spatial harmonic function $v_2 : \Omega \to \mathbb{R}$ in a simply connected domain $\Omega \subset \mathbb{R}^3$, there exists a harmonic vector*

$$\mathbf{V}_0 := (v_1^0, v_2, v_3^0)$$

in the domain Ω. Moreover, for any ball $\mho \subset \Omega$ centered at any point $(x_0, y_0, z_0) \in \Omega$ and for any vector $\mathbf{V} := (v_1, v_2, v_3)$ harmonic in $\mho$, the components v_3, v_1 are determined accurate within the real part and the imaginary part of any function $h(t)$ holomorphic in the domain

$$\{t = x + iz : (x,y,z) \in \mho\}$$

of the complex plane, i.e., for all $(x,y,z) \in \mho$, the following equalities hold:

$$v_3(x,y,z) = v_3^0(x,y,z) + \operatorname{Re} h(x+iz),$$

$$v_1(x,y,z) = v_1^0(x,y,z) + \operatorname{Im} h(x+iz).$$

Proposition 3.3. *For every spatial harmonic function $v_3 : \Omega \to \mathbb{R}$ in a simply connected domain $\Omega \subset \mathbb{R}^3$, there exists a harmonic vector*

$$\mathbf{V}_0 := (v_1^0, v_2^0, v_3)$$

in the domain Ω. *Moreover, for any ball* $\mho \subset \Omega$ *centered at any point* $(x_0, y_0, z_0) \in \Omega$ *and for any vector* $\mathbf{V} := (v_1, v_2, v_3)$ *harmonic in* $\mho$, *the components* v_2, v_1 *are determined accurate within the real part and the imaginary part of any function* $h(t)$ *holomorphic in the domain*

$$\{t = x + iy : (x, y, z) \in \mho\}$$

of the complex plane, i.e., for all $(x, y, z) \in \mho$, *the following equalities hold:*

$$v_2(x, y, z) = v_2^0(x, y, z) + \operatorname{Re} h(x + iy),$$
$$v_1(x, y, z) = v_1^0(x, y, z) + \operatorname{Im} h(x + iy).$$

4. Relation between monogenic functions taking values in the space $\widetilde{\mathbb{F}}$ and harmonic vectors

Now, consider the expansion

$$\Phi(\zeta) = \sum_{k=1}^{\infty} U_k(x, y, z)\ e_k, \tag{4.1}$$

of a function $\Phi\colon \Omega \to \widetilde{\mathbb{F}}$ with respect to the basis $\{e_k\}_{k=1}^{\infty}$, where it is assummed that the functions $U_k : \Omega \to \mathbb{R}$ are differentiable in Ω. This assumption implies the continuity of function (4.1).

It is proved in the paper [8] that in this case, in order that the function Φ be monogenic in a domain $\Omega \subset E_3$, it is necessary and sufficient that the following conditions be satisfied in Ω:

$$\begin{gathered}
\frac{\partial U_1}{\partial x} - \frac{1}{2}\frac{\partial U_2}{\partial y} - \frac{1}{2}\frac{\partial U_3}{\partial z} = 0, \\
\frac{\partial U_3}{\partial y} - \frac{\partial U_2}{\partial z} = 0, \qquad \frac{\partial U_1}{\partial y} + \frac{1}{2}\frac{\partial U_2}{\partial x} = 0, \qquad \frac{\partial U_1}{\partial z} + \frac{1}{2}\frac{\partial U_3}{\partial x} = 0, \\
\frac{\partial U_{2k}}{\partial x} = -\frac{\partial U_{2k-2}}{\partial z} - \frac{\partial U_{2k-1}}{\partial y}, \qquad \frac{\partial U_{2k+1}}{\partial x} = \frac{\partial U_{2k-2}}{\partial y} - \frac{\partial U_{2k-1}}{\partial z}, \\
\frac{\partial U_{2k}}{\partial z} - \frac{\partial U_{2k+1}}{\partial y} = \frac{\partial U_{2k-2}}{\partial x}, \qquad \frac{\partial U_{2k}}{\partial y} + \frac{\partial U_{2k+1}}{\partial z} = \frac{\partial U_{2k-1}}{\partial x},
\end{gathered} \tag{4.2}$$

$k = 2, 3, \ldots$. Note that conditions (4.2) are similar by nature to the Cauchy-Riemann conditions for holomorphic functions of a complex variable.

One can observe that equations (1.3) for the coordinate functions of the vector $\mathbf{V} := (U_1, -\frac{1}{2}U_2, -\frac{1}{2}U_3)$ coincide with the first four equations of system (4.2).

Therefore, the following statement formulated in the paper [8] is true:

Theorem 4.1. *Suppose that in expansion* (4.1) *of a function* $\Phi\colon \Omega \to \widetilde{\mathbb{F}}$ *monogenic in a domain* $\Omega \subset E_3$*, the functions* $U_k : \Omega \to \mathbb{R}$ *are differentiable in* Ω*. Then the function* Φ *generates a harmonic vector*

$$\mathbf{V} := (U_1, -\tfrac{1}{2}U_2, -\tfrac{1}{2}U_3)$$

in the domain Ω*.*

Thus, between harmonic vectors and monogenic functions taking values in the space $\widetilde{\mathbb{F}}$, there is the same relation as in Theorem 1.3 proved by I.P. Mel'nichenko [4].

Some relations between solutions of system (4.2) and spatial potential fields are studied in the papers [6,12], but monogenic functions taking values in the space $\widetilde{\mathbb{F}}$ are not considered there. Based on these relations, we established in the paper [8] that for every spatial harmonic function $U_1 : \Omega \to \mathbb{R}$ in a simply connected domain $\Omega \subset \mathbb{R}^3$, there exists a monogenic function $\Phi\colon \Omega \to \widetilde{\mathbb{F}}$ such that U_1 is the first component of expansion (4.1). Moreover, all components U_k in expansion (4.1) are spatial harmonic functions in Ω. This result is analogous to the corresponding result obtained by P. W. Ketchum [3].

In the next theorem we prove that all harmonic vectors $\mathbf{V} := (v_1, v_2, v_3)$ with continuously differentiable coordinate functions v_1, v_2, v_3 have a relation to monogenic functions taking values in the topological vector space $\widetilde{\mathbb{F}}$.

Theorem 4.2. *Suppose that for a harmonic vector* $\mathbf{V} := (v_1, v_2, v_3)$ *in a simply connected domain* $\Omega \subset \mathbb{R}^3$*, the components* $v_k : \Omega \to \mathbb{R}$ *are continuously differentiable functions in* Ω *for* $k = 1, 2, 3$*. Then there exists a monogenic function* $\Phi\colon \Omega \to \widetilde{\mathbb{F}}$ *such that the functions*

$$U_1 := v_1, \qquad U_2 := -2v_2, \qquad U_3 := -2v_3$$

are the first three components of expansion (4.1)*. Moreover, all components* U_k *in expansion* (4.1) *are spatial harmonic functions in* Ω*.*

Proof. Since the functions v_1, v_2, v_3 are continuously differentiable in the simply connected domain Ω and the equality $\mathrm{rot}\mathbf{V} = 0$ holds in Ω, there exists a function $u\colon \Omega \to \mathbb{R}$ such that $\mathbf{V} = \mathrm{grad}\, u$. Moreover, u is a spatial harmonic function in the domain Ω due to the equality $\mathrm{div}\mathbf{V} = 0$ in Ω. Therefore, v_1, v_2, v_3 are also spatial harmonic functions in Ω.

Let us show that there exists a solution of system (4.2) for components of expansion (4.1) of the sought-for monogenic function Φ, where the spatial harmonic functions $U_1 := v_1$, $U_2 := -2v_2$, $U_3 := -2v_3$ are the first three components of expansion (4.1). It is clear that the first four equations of system (4.2) are satisfied.

Now, let us show that the last four conditions of system (4.2) allow to determine the functions U_{2k}, U_{2k+1}, if the spatial harmonic functions $U_2, U_3, \ldots, U_{2k-1}$ are already determined. Indeed, integrating the fifth equation and the sixth equation of system (4.2), we obtain the expressions

$$U_{2k}(x,y,z) = -\int\limits_{x_0}^{x}\left(\frac{\partial U_{2k-2}(\tau,y,z)}{\partial z} + \frac{\partial U_{2k-1}(\tau,y,z)}{\partial y}\right)d\tau + \widetilde{u}_{2k}(z,y),$$

$$U_{2k+1}(x,y,z) = \int\limits_{x_0}^{x}\left(\frac{\partial U_{2k-2}(\tau,y,z)}{\partial y} - \frac{\partial U_{2k-1}(\tau,y,z)}{\partial z}\right)d\tau + \widetilde{u}_{2k+1}(z,y)$$

for (x,y,z) belonging to a certain ball $\mho \subset \Omega$ centered at any point $(x_0,y_0,z_0) \in \Omega$.

Substituting these expressions into the two latter equations of the system (4.2) and taking into account that U_{2k-2}, U_{2k-1} are spatial harmonic functions in the domain $\mho$, we obtain the following inhomogeneous Cauchy-Riemann system (see, e.g., the monograph by I. N. Vekua [11, p. 23]) for finding the functions $\widetilde{u}_{2k}$, $\widetilde{u}_{2k+1}$:

$$\begin{aligned} \frac{\partial \widetilde{u}_{2k}(z,y)}{\partial z} - \frac{\partial \widetilde{u}_{2k+1}(z,y)}{\partial y} &= \left.\frac{\partial U_{2k-2}(x,y,z)}{\partial x}\right|_{x=x_0}, \\ \frac{\partial \widetilde{u}_{2k}(z,y)}{\partial y} + \frac{\partial \widetilde{u}_{2k+1}(z,y)}{\partial z} &= \left.\frac{\partial U_{2k-1}(x,y,z)}{\partial x}\right|_{x=x_0}. \end{aligned} \tag{4.3}$$

Solutions of system (4.3) are determined accurate within the real part and the imaginary part of any function holomorphic in the domain

$$\{t = z + iy : (x,y,z) \in \mho\}$$

of the complex plane.

Since U_{2k-2}, U_{2k-1} are spatial harmonic functions in the domain $\mho$, taking into account equalities (4.3), it is easy to establish that the functions U_{2k}, U_{2k+1} satisfy equation (1.1) in $\mho$, i.e., U_{2k}, U_{2k+1} are also spatial harmonic functions in the domain $\mho$. Therefore, inasmuch as the domain Ω is simply connected, taking into account the uniqueness theorem for spatial harmonic functions (see, e.g., A. F. Timan and V. N. Trofimov [13, p. 88]), in such a way as it was made for the function u in the proof of Proposition 3.1, it is easy to continue the functions U_{2k}, U_{2k+1} defined in the ball $\mho$ into the domain Ω. □

Remark 4.3. It follows from the proof of Theorem 4.2 that for any fixed $k \geqslant 2$ and for any spatial harmonic functions U_{2k-2}, U_{2k-1} in a ball $\mho \subset \Omega$ centered at any point $(x_0,y_0,z_0) \in \Omega$, the functions U_{2k}, U_{2k+1} satisfying the last four conditions of system (4.2) are determined accurate within the real part and the imaginary part of any function $h_k(t)$ holomorphic in the domain $\{t = z + iy : (x, y, z) \in \mho\}$ of the complex plane, i.e., for all $(x, y, z) \in \mho$, the following equalities hold:

$$U_{2k}(x, y, z) = U^0_{2k}(x, y, z) + \operatorname{Re} h_k(z + iy),$$

$$U_{2k+1}(x, y, z) = U^0_{2k+1}(x, y, z) + \operatorname{Im} h_k(z + iy),$$

where U^0_{2k}, U^0_{2k+1} are functions forming a partial solution of the last four equations of system (4.2).

5. Monogeneity of Gâteaux derivatives

It is evident that if in expansion (4.1) of a function $\Phi\colon \Omega \to \widetilde{\mathbb{F}}$ monogenic in a domain $\Omega \subset E_3$, the functions $U_k : \Omega \to \mathbb{R}$ have continuous second-order partial derivatives in Ω, then function (4.1) is doubly differentiable in the sense of Gâteaux and, therefore, satisfies equalities

$$\Delta_3\Phi(\zeta) \equiv \Phi''_G(\zeta)(e_1^2 + e_2^2 + e_3^2) = 0 \tag{5.1}$$

for all $\zeta = xe_1 + ye_2 + ze_3 \in \Omega$ because the harmonic triad $\{e_1, e_2, e_3\}$ satisfies equality (1.4).

Let us show that equalities (5.1) hold under weaker conditions on the functions U_k in expansion (4.1).

Theorem 5.1. *Suppose that in expansion* (4.1) *of a function* $\Phi\colon \Omega \to \widetilde{\mathbb{F}}$ *monogenic in a domain* $\Omega \subset E_3$*, the functions* $U_k : \Omega \to \mathbb{R}$ *are differentiable in* Ω *and, in addition, the first component* U_1 *is a spatial harmonic function in* Ω*. Then the Gâteaux* n*-th derivative* $\Phi^{(n)}_G$ *is a monogenic function in the domain* Ω *for any* n*.*

Proof. Consider any point $\zeta_0 = x_0e_1 + y_0e_2 + z_0e_3 \in \Omega$ and a ball $\mho \subset \Omega$ centered at the point ζ_0, where $(x_0, y_0, z_0) \in \mathbb{R}^3$.

Taking into account Theorem 4.1 and Proposition 3.1, we can state that the functions $v_1 = U_1$, $v_2 = -U_2/2$, $v_3 = -U_3/2$ satisfying system (1.3) are spatial harmonic functions in the ball $\mho$ as well as the functions U_1, U_2, U_3.

Now, it follows from the proof of Theorem 4.2 that for all $k = 2, 3, \ldots,$ the functions U_{2k}, U_{2k+1}, which form a solution of system (4.2) together with the functions U_1, U_2, U_3, are also harmonic in the domain $\mho$.

Thus, in the expansion

$$\Phi'_G(\zeta) = \frac{\partial\Phi(\zeta)}{\partial x} \equiv \sum_{k=1}^{\infty} \frac{\partial U_k(x, y, z)}{\partial x}\, e_k,$$

all components $\partial U_k(x, y, z)/\partial x$ are continuously differentiable functions satisfying, in turn, a system of form (4.2) in the domain $\mho$. Therefore, the Gâteaux derivative Φ'_G is a monogenic function in the domain $\mho$.

By virtue of the arbitrariness of the choice of the point ζ_0 and the ball $\mho$, we can state that the Gâteaux derivative Φ'_G is a monogenic function in the domain Ω.

Finally, it is established in the same way that the Gâteaux n-th derivative $\Phi_G^{(n)}$ is also a monogenic function in the domain Ω for any n. $\square$

Theorem 5.2. *Suppose that in expansion* (4.1) *of a function* $\Phi\colon \Omega \to \widetilde{\mathbb{F}}$ *monogenic in a domain* $\Omega \subset E_3$, *the functions* $U_k : \Omega \to \mathbb{R}$ *are differentiable in* Ω *and, in addition, the first three components* U_1, U_2, U_3 *are continuously differentiable in* Ω. *Then the Gâteaux n-th derivative* $\Phi_G^{(n)}$ *is a monogenic function in the domain* Ω *for any* n.

Proof. By Theorem 4.1, the monogenic function Φ generates a harmonic vector $\mathbf{V} := (U_1, -\frac{1}{2}U_2, -\frac{1}{2}U_3)$ that implies the equality $\mathrm{rot}\mathbf{V} = 0$ in the domain Ω.

Consider any point $\zeta_0 = x_0e_1 + y_0e_2 + z_0e_3 \in \Omega$, where $(x_0, y_0, z_0) \in \mathbb{R}^3$, and a ball $\mho \subset \Omega$ centered at the point ζ_0. Since the ball $\mho$ is a simply connected domain and the functions U_1, U_2, U_3 are continuously differentiable in Ω, there exists a spatial harmonic function $u\colon \mho \to \mathbb{R}$ such that $\mathbf{V} = \mathrm{grad}\, u$ in $\mho$. Therefore, the functions U_1, U_2, U_3 are also spatial harmonic functions in the ball $\mho$.

Further, Theorem 5.2 is proved in the same way as Theorem 5.1. $\square$

Remark 5.3. Thus, under the conditions of either Theorem 5.1 or Theorem 5.2, equalities (5.1) hold for all $\zeta = xe_1 + ye_2 + ze_3 \in \Omega$. In this case, all components $U_k(x, y, z)$ in expansion (4.1) are spatial harmonic functions, i.e., they satisfy equation (1.1).

Remark 5.4. In [9], we considered a complexification $\widetilde{\mathbb{F}}_{\mathbb{C}}$ of the topological vector space $\widetilde{\mathbb{F}}$ and proved that each monogenic function $\Phi\colon \Omega \to \widetilde{\mathbb{F}}$ can be extended from the given domain $\Omega \subset E_3$ to a monogenic function in some domain of a certain four-dimensional real space $E_4 \subset \mathbb{F}_{\mathbb{C}}$. We proved also analogues of the Cauchy integral theorem and the Cauchy integral formula as well as analogues of the Morera theorem and Taylor theorem for monogenic functions given in domains of the space E_4 and taking values in the space $\widetilde{\mathbb{F}}_{\mathbb{C}}$.

However, unlike the classical complex analysis, an analogue of the Cauchy integral formula, which is proved in [9], can not be applicable for proving Theorem 5.2 because it requires additional assumptions about the function

Φ. In addition, the mentioned formula deals with an integration along a curve not located in E_3.

At the same time, the validity of the Cauchy integral formula using an integration along a curve located in E_3 for monogenic functions $\Phi\colon \Omega \to \widetilde{\mathbb{F}}$ remains an open problem.

6. Infinite-dimensional topological vector space $\widetilde{\mathbb{G}}$ with a commutative multiplication

Let us consider a topological vector space with a commutative multiplication such that analogues of the Cauchy-Riemann conditions for monogenic functions taking values in this space include only equations of form (1.2).

Consider an infinite-dimensional topological vector space

$$\widetilde{\mathbb{G}} := \left\{ g = \sum_{k=-\infty}^{\infty} c_k e_k : c_k \in \mathbb{R} \right\}$$

with the topology of coordinate-wise convergence and the basis $\{e_k\}_{k=-\infty}^{\infty}$. We set that the elements of basis are multiplied by rules

$$e_n e_1 = e_n, \quad e_{2n+1} e_m = e_{2n+m}, \quad e_{2n} e_{2m} = -e_{2n+2m-3} - e_{2n+2m+1}$$

for all integer n and m.

It is evident that the elements e_1, e_2, e_3 form a harmonic triad of vectors, i.e., they satisfy equality (1.4).

Essentially, M. N. Roşculeţ [10] considered the space $\widetilde{\mathbb{G}}$ though he did not use the notion of topological vector space as well as the differentiability in the sense of Gâteaux. It is proved in [10] that every spherical function is a component of the expansion with respect to the basis of the function $\lambda\zeta^n$, where $\lambda \in \widetilde{\mathbb{G}}$.

Note that $\widetilde{\mathbb{G}}$ is not an algebra because the product of elements $g_1, g_2 \in \widetilde{\mathbb{G}}$ is defined not always. But for all

$$g = \sum_{k=-\infty}^{\infty} c_k e_k \in \widetilde{\mathbb{G}} \qquad \text{and} \qquad \zeta = x e_1 + y e_2 + z e_3,$$

one can define the product

$$g\zeta \equiv \zeta g := x \sum_{k=-\infty}^{\infty} c_k e_k + \\ + y \left(\sum_{k=-\infty}^{\infty} c_{2k-1} e_{2k} - \sum_{k=-\infty}^{\infty} (c_{2k-2} + c_{2k+2}) e_{2k+1} \right) + z \sum_{k=-\infty}^{\infty} c_{k-2} e_k.$$

Thus, the notion of differentiability in the sense of Gâteaux defined in Definition 1.1 is applied to functions $\Phi\colon \Omega \to \widetilde{\mathbb{G}}$ given in a domain $\Omega \subset E_3$ and taking values in the space $\widetilde{\mathbb{G}}$. In the considered case, for a monogenic function $\Phi\colon \Omega \to \widetilde{\mathbb{G}}$, the Gâteaux derivative $\Phi'_G(\zeta)$ takes values in the space $\widetilde{\mathbb{G}}$ for all $\zeta \in \Omega$.

7. Relation between monogenic functions taking values in the space $\widetilde{\mathbb{G}}$ and harmonic vectors

Now, consider the expansion

$$\Phi(\zeta) = \sum_{k=-\infty}^{\infty} U_k(x, y, z)\ e_k, \tag{7.1}$$

of a function $\Phi\colon \Omega \to \widetilde{\mathbb{G}}$ with respect to the basis $\{e_k\}_{k=-\infty}^{\infty}$, where it is assummed that the functions $U_k : \Omega \to \mathbb{R}$ are differentiable in Ω. This assumption implies the continuity of function (7.1).

It is proved in the paper [8] that in this case, in order that the function Φ be monogenic in a domain $\Omega \subset E_3$, it is necessary and sufficient that the following conditions be satisfied in Ω for all integer m:

$$\begin{aligned}
&\frac{\partial U_{2m+2}}{\partial x} + \frac{\partial U_{2m+1}}{\partial y} + \frac{\partial U_{2m}}{\partial z} = 0,\\
&\frac{\partial U_{2m}}{\partial y} - \frac{\partial U_{2m+1}}{\partial z} = 0,\\
&\frac{\partial U_{2m+2}}{\partial z} - \frac{\partial U_{2m}}{\partial x} = 0,\\
&\frac{\partial U_{2m+1}}{\partial x} - \frac{\partial U_{2m+2}}{\partial y} = 0.
\end{aligned} \tag{7.2}$$

Thus, every monogenic function $\Phi\colon \Omega \to \widetilde{\mathbb{G}}$ generates a family of harmonic vectors $\mathbf{V} := (U_{2m+2}, U_{2m+1}, U_{2m})$ in the domain Ω for all integer m.

Conditions (7.2) are similar by nature to the Cauchy-Riemann conditions for holomorphic functions of a complex variable and include only equations (1.2) for the vectors $\mathbf{V} := (U_{2m+2}, U_{2m+1}, U_{2m})$ for all integer m.

We established in the paper [8] that for every spatial harmonic function $U_1 : \Omega \to \mathbb{R}$ in a simply connected domain $\Omega \subset \mathbb{R}^3$, there exists a monogenic function $\Phi\colon \Omega \to \widetilde{\mathbb{G}}$ such that U_1 is a component of expansion (7.1). Moreover, all components U_k in expansion (7.1) are spatial harmonic functions in Ω.

In the next theorem, we prove that all harmonic vectors $\mathbf{V} := (v_1, v_2, v_3)$ with continuously differentiable coordinate functions v_1, v_2, v_3 have a relation to monogenic functions taking values in the topological vector space $\widetilde{\mathbb{G}}$.

Theorem 7.1. *Suppose that for a harmonic vector* $\mathbf{V} := (v_1, v_2, v_3)$ *in a simply connected domain* $\Omega \subset \mathbb{R}^3$*, the components* $v_k \colon \Omega \to \mathbb{R}$ *are continuously differentiable functions in* Ω *for* $k = 1, 2, 3$*. Then there exists a monogenic function* $\Phi \colon \Omega \to \widetilde{\mathbb{G}}$ *such that the functions*

$$U_0 := v_3, \qquad U_1 := v_2, \qquad U_2 := v_1$$

are the components of expansion (7.1)*. Moreover, all components* U_k *in expansion* (7.1) *are spatial harmonic functions in* Ω*.*

Proof. Since the functions v_1, v_2, v_3 are continuously differentiable in the simply connected domain Ω and the equality $\mathrm{rot}\mathbf{V} = 0$ holds in Ω, there exists a function $u \colon \Omega \to \mathbb{R}$ such that $\mathbf{V} = \mathrm{grad}\, u$. Moreover, u is a spatial harmonic function in the domain Ω due to the equality $\mathrm{div}\mathbf{V} = 0$ in Ω. Therefore, v_1, v_2, v_3 are also spatial harmonic functions in Ω.

Let us show that there exists a solution of system (7.2) for components of expansion (7.1) of the sought-for monogenic function Φ, where the spatial harmonic functions $U_0 := v_3$, $U_1 := v_2$, $U_2 := v_1$ are the components of expansion (7.1).

First of all, let us show that conditions (7.2) allow to determine the functions U_{2m+1}, U_{2m+2} if the function U_{2m} is already determined for some positive integer m. Indeed, in this case there exists a harmonic vector $\mathbf{V}_{2m} := (v_{2m+2}, v_{2m+1}, U_{2m})$ in the domain Ω due to Proposition 3.3. Then, we set $U_{2m+1} := v_{2m+1}$, $U_{2m+2} := v_{2m+2}$, and these functions are spatial harmonic functions in the domain Ω.

Finally, let us also show that conditions (7.2) allow to determine the functions U_{2m}, U_{2m+1} if the function U_{2m+2} is already determined for some negative integer m. Indeed, in this case there exists a harmonic vector $\mathbf{V}_{2m+1} := (U_{2m+2}, v_{2m+1}, v_{2m})$ in the domain Ω due to Proposition 3.1. Then, we set $U_{2m} := v_{2m}$, $U_{2m+1} := v_{2m+1}$, and these functions are spatial harmonic functions in the domain Ω.

Thus, the functions U_k obtained in such a way satisfy system (7.2) and form function (7.1) monogenic in the domain Ω. □

Let us establish sufficient conditions for infinite monogeneity of a function $\Phi \colon \Omega \to \widetilde{\mathbb{G}}$.

The next theorem is analogous to Theorem 5.1 for monogenic functions in the space $\widetilde{\mathbb{F}}$.

Theorem 7.2. *Suppose that in the expansion* (7.1) *of a function* $\Phi\colon \Omega \to \widetilde{\mathbb{G}}$ *monogenic in a domain* $\Omega \subset E_3$*, the functions* $U_k : \Omega \to \mathbb{R}$ *are differentiable in* Ω *and, in addition, the component* U_{k_0} *is a spatial harmonic function in* Ω *for some integer* k_0*. Then the Gâteaux* n*-th derivative* $\Phi_G^{(n)}$ *is a monogenic function in the domain* Ω *for any* n*.*

Proof. Consider any point $\zeta_0 = x_0e_1 + y_0e_2 + z_0e_3 \in \Omega$ and a ball $\mho \subset \Omega$ centered at the point ζ_0, where $(x_0, y_0, z_0) \in \mathbb{R}^3$.

It follows from system (7.2) that the vector

$$\mathbf{V} := (U_{2m+2}, U_{2m+1}, U_{2m})$$

is harmonic in the domain $\mho$, where $U_{2m} = U_{k_0}$ in the case $k_0 = 2m$ or $U_{2m+1} = U_{k_0}$ in the case $k_0 = 2m + 1$ for some integer m. Moreover, $U_{2m+2}, U_{2m+1}, U_{2m}$ are spatial harmonic functions in $\mho$ due to Proposition 3.3 in the case $k_0 = 2m$ and Proposition 3.2 in the case $k_0 = 2m + 1$.

Now, using Proposition 3.3, we see that for all integer $k > m$ and for the given spatial harmonic function U_{2k}, the functions U_{2k+2}, U_{2k+1} satisfying system (7.2), in which k is substituted instead of m, are also harmonic in the domain $\mho$. Similarly, using Proposition 3.1, we obtain that for all integer $k < m$ and for the given spatial harmonic function U_{2k}, the functions U_{2k-1}, U_{2k-2} satisfying system (7.2), in which $(k-1)$ is substituted instead of m, are also harmonic in $\mho$.

Thus, all components U_k in expansion (7.1) are spatial harmonic functions satisfying system (7.2) in the domain $\mho$. Therefore, in the expansion

$$\Phi'_G(\zeta) = \frac{\partial \Phi(\zeta)}{\partial x} \equiv \sum_{k=-\infty}^{\infty} \frac{\partial U_k(x, y, z)}{\partial x}\, e_k,$$

all components $\partial U_k(x, y, z)/\partial x$ are continuously differentiable functions satisfying, in turn, a system of form (7.2) in the domain $\mho$. Thus, the Gâteaux derivative Φ'_G is a monogenic function in the domain $\mho$.

By virtue of the arbitrariness of the choice of the point ζ_0 and the ball $\mho$, we can state that the Gâteaux derivative Φ'_G is a monogenic function in the domain Ω.

Finally, it is established in the same way that the Gâteaux n-th derivative $\Phi_G^{(n)}$ is also a monogenic function in the domain Ω for any n. □

The next theorem is analogous to Theorem 5.2 for monogenic functions in the space $\widetilde{\mathbb{F}}$.

Theorem 7.3. *Suppose that in the expansion* (7.1) *of a function* $\Phi\colon \Omega \to \widetilde{\mathbb{G}}$ *monogenic in a domain* $\Omega \subset E_3$*, the functions* $U_k : \Omega \to \mathbb{R}$ *are differentiable in* Ω *and, in addition, the components* $U_{2k_0}, U_{2k_0+1}, U_{2k_0+2}$ *are continuously* *differentiable in Ω for some integer k_0. Then the Gâteaux n-th derivative $\Phi_G^{(n)}$ is a monogenic function in the domain Ω for any n.*

Proof. It follows from system (7.2) that the monogenic function Φ generates a harmonic vector $\mathbf{V} := (U_{2k_0+2}, U_{2k_0+1}, U_{2k_0})$ that implies the equality $\mathrm{rot}\mathbf{V} = 0$ in the domain Ω.

Consider any point $\zeta_0 = x_0e_1 + y_0e_2 + z_0e_3 \in \Omega$, where $(x_0, y_0, z_0) \in \mathbb{R}^3$, and a ball $\mho \subset \Omega$ centered at the point ζ_0. Since the ball $\mho$ is a simply connected domain and the functions $U_{2k_0}, U_{2k_0+1}, U_{2k_0+2}$ are continuously differentiable in Ω, there exists a spatial harmonic function $u\colon \mho \to \mathbb{R}$ such that $\mathbf{V} = \mathrm{grad}\, u$ in $\mho$. Therefore, the functions $U_{2k_0}, U_{2k_0+1}, U_{2k_0+2}$ are also spatial harmonic functions in the ball $\mho$.

Further, Theorem 7.3 is proved in the same way as Theorem 7.2. □

Remark 7.4. Thus, under the conditions of either Theorem 7.2 or Theorem 7.3, equalities (5.1) hold for all $\zeta = xe_1 + ye_2 + ze_3 \in \Omega$. In this case, all components $U_k(x, y, z)$ in expansion (7.1) are spatial harmonic functions, i.e., they satisfy equation (1.1).

References

[1] R. Fueter. Die Funktionentheorie der Dirac'schen Differentialgleichungen. *Comment. Math. Helv.*, 16:19–28, 1944. doi:10.1007/BF02568561.

[2] P. W. Ketchum. Analytic functions of hypercomplex variables. *Trans. Amer. Math. Soc.*, 30(4):641–667, 1928. doi:10.2307/1989440.

[3] P. W. Ketchum. A Complete Solution of LaPlace's Equation by an Infinite Hypervariable. *Amer. J. Math.*, 51(2):179–188, 1929. doi:10.2307/2370704.

[4] I. P. Mel'nichenko. The representation of harmonic mappings by monogenic functions. *Ukr. Math. J.*, 27(5):499–505, 1975. doi:10.1007/BF01089142.

[5] Gr. C. Moisil and N. Theodoresco. Fonctions holomorphes dans l'espace. *Mathematica (Cluj)*, 5:142–159, 1931.

[6] S. A. Plaksa. Commutative algebras associated with classic equations of mathematical physics. In *Advances in Applied Analysis, Trends in Mathematics*, pages 177–223. Springer, Basel, 2012. doi:10.1007/978-3-0348-0417-5.

[7] S. A. Plaksa and V. S. Shpakivskyi. Constructive description of monogenic functions in a harmonic algebra of the third rank. *Ukr. Math. J.*, 62(8):1251–1266, 2011. doi:10.1007/s11253-011-0427-x.

[8] S. A. Plaksa and V. S. Shpakivskyi. A description of spatial potential fields by means of monogenic functions in infinite-dimensional spaces with a commutative multiplication. *Bulletin Soc. Sci. et Lettr. Łódź*, 62(2):55–65, 2012.

[9] S. A. Plaksa and V. S. Shpakivskyi. An extension of monogenic functions and spatial potentials. *Lobachevskii Journal of Mathematics*, 38(2):330–337, March 2017. doi:10.1134/s1995080217020160.

[10] M. N. Roşculeţ. Algèbres infinies associées à des équations aux dérivées partielles, homogènes, aux ceofficients constants d'ordre quelconque. *Acad. R. P. Romîne. Stud. Cerc. Mat.*, 6:567–643, 1955.

[11] I. N. Vekua. *Generalized analytic functions*. Pergamon Press, London-Paris-Frankfurt; Addison-Wesley Publishing Company, Inc., Reading, Mass., 1962.

[12] И. П. Мельниченко and С. А. Плакса. *Коммутативные алгебры и пространственные потенциальные поля*. Ин-т математики НАН Украины, Киев, 2008.

[13] А. Ф. Тиман and В. Н. Трофимов. *Введение в теорию гармонических функций*. Наука, Москва, 1968.

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Sergiy Plaksa
Institute of Mathematics of the National Academy of Sciences of Ukraine;
University of Padova, Italy
Email: plaksa62@gmail.com
ORCID: 0000-0002-2828-0329