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Topological structure of optimal flows on the Girl's surface

Maria Loseva, Alexandr Prishlyak

Abstract. We investigate the topological structure of flows on the Girl's surface which is one of two possible immersions of the projective plane in three-dimensional space with one triple point of self-intersection. First, we describe the cellular structure of the Boy's and Girl's surfaces and prove that there are unique images of the project plane in the form of a 2-disk, in which the opposite points of the boundary are identified and this boundary belongs to the preimage of the 1-skeleton of the surface. Second, we describe three structures of flows with one fixed point and no separatrices on the Girl's surface and prove that there are no other such flows. Third, we prove that Morse-Smale flows and they alone are structurally stable on the Boy's and Girl's surfaces. Fourth, we find all possible structures of optimal Morse-Smale flows on the Girl's surface. Fifth, we obtain a classification of Morse-Smale flows on the projective plane immersed on the Girl's surface. And finally, we describe the isotopic classes of these flows.

Анотація. В роботі досліджується топологічна структура потоків на поверхні Гьоли, яка є одним з двох занурень проєктивної площини в тривимірний простір з однією потрійною точкою самоперетину. Спочатку описано структуру кліткових комплексів поверхонь Боя и Гьоли та доведено існування та єдиність зображень проєктивної площини у вигляді 2-диска, у якого ототожнені протилежні точки межі, а сама межа належить прообразу 1-кістяка поверхні. Далі, описано три структури потоків з однією нерухомою точкою та без сепаратрис на поверхні Гьоли та доведено, що інших таких потоків немає. В роботі також доведено, що потоки Морса-Смейла і лише вони є структурно стійкими на поверхнях Боя и Гьоли та знайдено всі можливі структури оптимальних потоків Морса-Смейла на поверхні Гьоли. Крім того, отримано класифікацію

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потоків Морса-Смейла на проєктивній площині, що занурюються на поверхню Гьоли, та описано ізотопічні класи таких потоків.

Introduction

The Boy's [2] and Girl's [5,6] surfaces are certain immersions of the real projective plane in a three-dimensional space. They are unique possible immersions with a single triple point and connected set of self-intersections.

We consider the Boy's and Girl's surfaces as stratified sets and CW-complexes. A Morse-Smale flow is one of the main instruments in the Morse theory on manifolds and stratified sets.

A smooth flow on a closed manifold is always generated by a smooth vector field, but on a manifold with boundary (a stratified set) vector field has to be tangent to the boundary (every strata). Then all the concepts that are introduced for dynamical systems (flow) can be applied to vector fields. Let f^t be the flow generated by a vector field X.

Two vector fields (flows) X, Y on M are called *topologically equivalent* if there exists a homeomorphism $h\colon M \to M$ sending each trajectory of X onto a trajectory of Y preserving their orientations.

A flow f^t on a closed manifold is called *structurally stable* if, for any sufficiently close flow g^t, there exists a homeomorphism h sending orbits of the system g^t to orbits of the system f^t, [21].

Fixed (singular) points and closed trajectories of a dynamical system are called *critical elements*.

A singular point p of a vector field $X = \{X_1, \ldots, X_n\}$ is called *non-degenerate* or *hyperbolic*, if in some local coordinates $(x_1, \ldots, x_n)$ at p the Jacobi matrix $\left(\frac{\partial X_i}{\partial x_j}\right)_{i,j=1}^{n}$ does not contain eigenvalues with zero real part. We not give the definition for a periodic orbit, e.g. [41], because it is not used in the paper.

A stable manifold $S(p)$ of a critical element p is the following set

$$S(p) = \{x \in M : \lim_{t\to\infty} d(f^t(x), p) = 0\}.$$

Similarly, the set $U(p) = \{x \in M : \lim_{t\to-\infty} d(f^t(x), p) = 0\}$ is called the *unstable manifold* of p.

A point $p \in M$ is *wandering* for f^t if there exists a neighborhood V of p and a number $n > 0$ such that $f^t(V) \cap V = \varnothing$ for any $t > n$.

A fixed point $y \in M$ is called an α-limit (resp. an ω-limit) point for $x \in M$ if there exists a sequence $\{t_i\}$ such that $t_i \to +\infty$ (resp. $t_i \to -\infty$) and $f^{t_i}(x) \to y$.

A flow is called *Morse-Smale*, e.g. [41], whenever

(1) its non-wandering set consists of finitely many singular points and periodic orbits,
(2) each of which is hyperbolic,
(3) their stable and unstable manifolds intersect transversally.

If M is a closed two-manifold, then a flow is structurally stable if and only if it is Morse-Smale, [20].

The set of all α-limit points (resp. ω-limit points) of x is called the α-limit (ω-limit) set. Condition (1) of the definition of Morse-Smale dynamical system can be replaced by the following condition

(1′) for each point $x \in M$ its α-limit and ω-limit sets are contained in the union of critical elements.

For vector fields on two-dimensional manifolds there are three types of non-degenerated (hyperbolic) singular points: sinks, sources and saddles. Condition (3) in this case is equivalent to the statement that there are no trajectories whose α-limit and ω-limit sets are saddle points.

There is a lot of papers on structural classification of Morse-Smale vector fields on surfaces. Most known are [3, 14, 15, 18, 22].

A Morse-Smale flow without closed orbit is called a *Morse* flow. We say that flow is *optimal* if it has the lowest number of fixed points among all flows of that type on the surface. The Morse flow on the closed surface is optimal if and only if it has only one sink and one source, [12]. Such a flow is also called a *polar* Morse flow. The topological structure of polar (optimal) Morse flows on closed 2- and 3-manifolds was described in [4, 11, 12, 17, 24, 27, 28, 32, 36].

For flows on a stratified set, their restriction to each two-dimensional stratum has the same structure as flows on a surface with boundary. This situation has been investigated in [30, 31, 34, 35, 37, 38]. Structural stability of Morse-Smale flows was proved for closed manifolds in [20, 42] and for manifolds with boundary in [23, 40].

Morse fields are topologically equivalent to gradient fields of Morse functions [42]. Moreover, the structure of the Morse field coincides with the structure of the Morse function, in which all points of the same index have equal function values [1]. The structure of an arbitrary Morse function can be described by the structure of the Morse flow with the given value of the function in fixed points. The classification of functions on two-manifolds was obtained in [7–10, 13, 16, 26, 29, 39]

The formula for the sum of indices of fixed point is useful for calculating their number for flow on a stratified set, [30].

In [33], it was described the structure of next optimal flows on the Boy's surface:

(1) flows with one fixed point,

(2) Morse-Smale flows,
(3) Morse-Smale projective flows.

The purpose of our article is to describe the structure of such flows on the Girl's surface, as well as to study these flows with respect to homotopy, that is, to describe the linearly connected components of the set of such flows. The constructed flow invariants are graphs embedded in the surface. To encode them, one can use a rotation system as in topological graph theory or a graph with a list of words as in [25].

Our paper has the following structure. In Section 1, we describe the cellular structure of the Boy's and Girl's surfaces and prove that there are unique representations of the projective plane in the form of a 2-disk, which has opposite boundary points identified and this boundary belongs to the inverse image of the 1-skeleton of the surface. In Section 2, we find three flow structures with one fixed point and no separatrices on the Girl's surface and prove that there are no other such flows. Section 3 is devoted to the proof that Morse-Smale flows and only they are structurally stable on Boy's and Girl's surfaces. In Section 4, all possible structures of optimal Morse-Smale flows on the Girl's surface are described. In Section 5, we construct a classification of optimal Morse-Smale projective flows on the Girl's surface. Finally, in Section 6, for each of the Boy's and Girl's surfaces, we prove formulas relating the numbers of topologically non-equivalent flows, symmetric flows, and non-isotopic flows.

1. Natural structures of CW-complexes on the Boy's and Girl's surfaces

Here we describe the natural CW-structures of the Boy's and Girl's surfaces, as well as the resulting CW-structures of the projective plane. In this case, we use a model of the projective plane in the form of a 2-disk, in which opposite points on the boundary are identified (glued). The corresponding CW-structures are planar models of these surfaces.

On each surface there is one triple dot, which we call *null-point*, and three loops formed by double points. Denote them by A, B, C. These loops lie at the coordinate angles of different planes and touch the coordinate axes at their common point. We set the direction of movement along them so that the corresponding speed vectors at the null-point coincide when moving from C to B, from B to A, and from A to C. The structure of a CW-complex for each surface consists of one 0-cell – null-point, three 1-cells – A, B, C and four 2-cells. The neighborhoods of the 1-skeleton are shown in Figure 1.1. They determine the gluing of 2-cells.

We enumerate the angles at the null-point of this neighborhood with integers from 1 to 12 as in Figure 1.2.

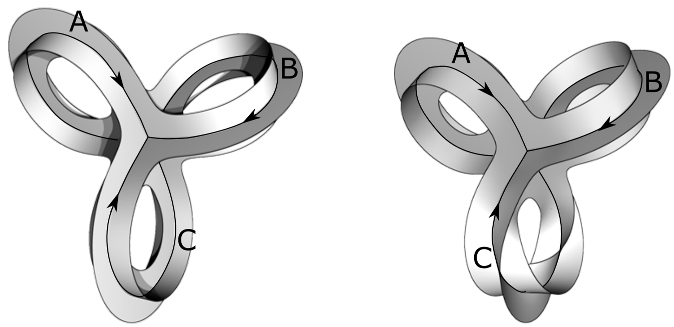

FIGURE 1.1. Neighborhoods of self-intersections on the Boy's (left) and Girl's (right) surfaces

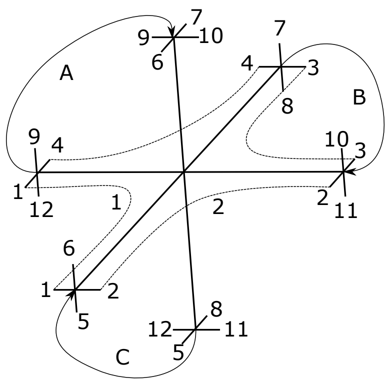

FIGURE 1.2. Angle numbering

Two one-dimensional cells on the projective plane, which are mapped to the 1-cell A on the Boy's and Girl's surfaces, will be denoted by $\mathbf{A}$ and $\mathbf{A}'$, so that $\mathbf{A}$ contains the intersection of angles 1 and 4, and $\mathbf{A}'$ contains the intersection of 9 and 12. By analogy, $\mathbf{B}$ contains the intersection of 10 and 11, and $\mathbf{B}'$ of 2 and 3, $\mathbf{C}$ of 1 and 2, and $\mathbf{C}'$ of 5 and 6.

The gluing of each 2-cell is determined by the sequence of angles and 1-cells that occur when bypassing its boundary. So in the Boy's surface there are three small 2-cells that lie in the coordinate angles and are bounded by 1-cells:

$$9\,\mathbf{A}'\,9, \quad 3\,\mathbf{B}'\,3, \quad 5\,\mathbf{C}'\,5, \quad 1\,\mathbf{A}\,6\,\mathbf{C}'\,8\,\mathbf{B}\,11\,\mathbf{C}\,2\,\mathbf{B}'\,4\,\mathbf{A}\,7\,\mathbf{B}\,10\,\mathbf{A}'\,12\,\mathbf{C}\,1.$$

After gluing three small cells to this nonagon to the hatched sides, we get a hexagon as in Figure 1.3 on the left.

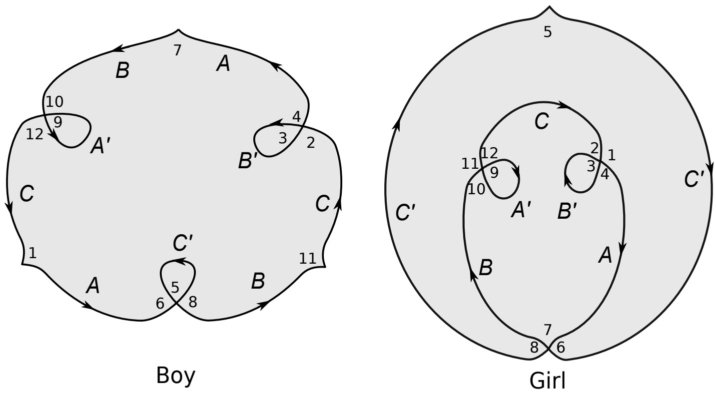

Figure 1.3. Planar model of the Boy's and Girl's surfaces

On the Girl's surface, the 1-cells are connected by the angles:

$\mathbf{A}$: $1-6,\ 4-7,$	$\mathbf{A}'$: $9-9,\ 12-10,$
$\mathbf{B}$: $7-10,\ 8-11,$	$\mathbf{B}'$: $3-3,\ 4-2,$
$\mathbf{C}$: $12-2,\ 11-1,$	$\mathbf{C}'$: $5-6,\ 8-5.$

We get the following boundaries of 2-cells:

$$9\,\mathbf{A}'\,9,\quad 3\,\mathbf{B}'\,3,\quad 1\,\mathbf{A}\,6\,\mathbf{C}'\,5\,\mathbf{C}'\,8\,\mathbf{B}\,11\,\mathbf{C}\,1,\quad 2\,\mathbf{C}\,12\,\mathbf{A}'\,10\,\mathbf{B}\,7\,\mathbf{A}\,4\,\mathbf{B}'\,2.$$

After gluing all sides except $\mathbf{C}'$, we get the planar model of the Girl's surface in Figure 1.3 on the right.

Lemma 1.1. *Figure 1.3 gives a unique (up to homeomorphism) representation of the projective plane in the form of a 2-disk with glued opposite boundary points and such that the boundary is mapped to the 1-skeleton of the Boy's and Girl's surfaces, respectively.*

Proof. If we glue the other sides of the first planar model then we obtain the Möbius band that cannot be placed on the plane. In the second model, any other gluing leads to gluing of $\mathbf{C}'$ and then its neighborhood is homeomorphic to the Möbius band. □

We introduce further notations for regions on the Girl's surface:

- LD (Left Disk) is a region, which contains 9 ($9\mathbf{A}'9$-cell),
- RD (Right Disk) contains 3 ($3\mathbf{B}'3$-cell),
- BR (Boundary Region) contains 5 ($1\mathbf{A}6\mathbf{C}'5\ldots$-cell),
- CR (Central Region) contains 7 ($7\mathbf{A}4\mathbf{B}'2\ldots$-cell).

CW-structures on Boy's and Girl's surfaces form their stratifications, where the stratas correspond to the cells.

2. Flows with a single fixed point

By analogy with the Boy‘s surface, the null-point is a fixed point for a flow on the Girl's surface [33]. The 1-stratas (1-cells) consist of fixed points and flow trajectories. A flow cannot have closed trajectories or oriented cycles, because if such exist, there will be a second fixed point inside them. So, each trajectory begins and ends at the null-point. Cutting the surface by 1-cells and separatrices, we get regions that can be of two types:

(1) elliptical: all its trajectories begin and end in one corner (loops)
(2) polar: trajectories begin and end in different corners.

Here *a separatrix* is a trajectory belonging to the boundary of a hyperbolic corner and the interior of a 2-strata. So, we do not regard trajectories of 1-stratas as separatrices.

Our aim is to find all the flows (up to topological equivalence) without separatrices. Since axial symmetry with respect to the vertical axis is a topological equivalence, each flow with a direction of C from 2 to 12 is equivalent to a flow with a direction of C from 12 to 2. Therefore, we fix one direction of C: from 12 to 2.

By $-A$ and $-B$ we denote the orientations inverse to the given orientations A and B as in Figure 1.3 right. There are four possible flows with different orientation of 1-strata A and B:

(1) with given orientations A and B,
(2) with orientations $-A$ and B,
(3) with orientations A and $-B$,
(4) with orientations $-A$ and $-B$.

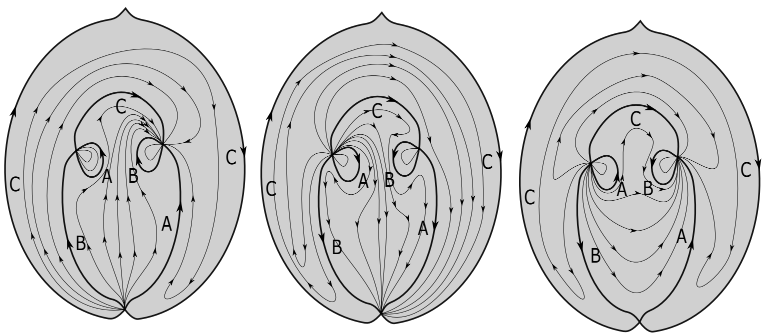

Figure 2.1. Flows with one fixed point

In all the cases, LD and RD are elliptical regions.

In the case (1), BR has angle 8 as the source and angle 6 as a sink and it is therefore polar. However, the CR region has two sources – angles 4 and

12, as well as two sinks – angles 2 and 10, so there is at least one separatrix in that region connecting 4 and 12 or 2 and 10.

In the case (2), in the BR there is a source angle 8 and a sink in angle 1, in CR – 7 is a source, and 2 is a sink. So both regions are polar and there is only one structure of such a flow.

In the case (3), the angle 11 is a source and 6 is a sink for BR, while 12 is a source and 7 is a sink for the region CR. So, there is one flow structure.

In the last case (4), 11 is a source and 1 is a sink for BR, while 10 is a source, 4 is a sink for CR. So, there is only one flow structure.

To sum up, we have

Theorem 2.1. *There are three different flow structures with one fixed point and no separatrices on the Girl's surface (see Figure 2.1).*

3. Structural stability of Morse-Smale flows

A function (a map, a flow) is smooth on the Boy's or Girl's surface if it smooth in any 2-stratum and for any point p belonging to a 1-stratum or 0-stratum, for each 2-stratum S containing p in its boundary, there is a neighborhood U of p such that for any component C of $U \cap S$ there is a 2-manifold M $p \in \overline{C} \subset M$, $\partial M = \varnothing$ and the function (the map, the flow) can be extended to a smooth function (a map, a flow) on M.

A vector field (flow) F is structural stable if in the space of all vector field on the Boy's (Girl's) surface with C^1 topology there exist a neighborhood U that any $G \in U$ is topologically equivalent to F.

A point p belonging to a 1-stratum or 0-stratum is *hyperbolic* if for each 2-stratum S containing p in its boundary, there is a neighborhood U of p such that the restrictions of flow to any component C of $U \cap S$ can be extended to a flow on a manifold M with inner hyperbolic point $p \in \overline{C} \subset M$, $\partial M = \varnothing$.

A smooth flow on the Boy's or Girl's surface is called a *Morse-Smale flow* whenever

(1) all its critical elements (singular points and closed trajectories) are hyperbolic,
(2) the limit sets of each trajectory are critical elements,
(3) there are no separatrices connecting the saddles.

By *a separatrix* we mean a trajectory that belongs to the interior of a 2-stratum and a stable or unstable manifold of a saddle point.

To prove structural stability of Morse-Smale flows on the Boy's and Girl's surfaces we need the following lemma:

Lemma 3.1. *Suppose that the angle α in a neighborhood of null-point is given by the inequalities $x \geqslant 0, y \geqslant 0$ with respect to some coordinate system (x,y). Then in the angle α, a smooth vector field F of a flow is given by $\{xf(x,y),\, yg(x,y)\}$, where f and g are smooth functions. In the case of a Morse-Smale vector field, we have that $f(0,0) \neq 0$ and $g(0,0) \neq 0$.*

Proof. Let $F = \{X(x,y), Y(x,y)\}$ be the coordinate functions of F. The assumption that F is tangent to the axis x means that $Y(x,0) = 0$. As in Hadamard's Lemma,

$$Y(x,y) = \int_0^1 \frac{dY(x,ty)}{dt}dt = \int_0^1 \frac{\partial Y(x,ty)}{\partial y} y dt = yg(x,y),$$

where $g(x,y) = \int_0^1 \frac{\partial Y(x,ty)}{\partial y}dt$. Similarly, $X(x,y) = x\ f(x,y)$. Then

$$\frac{\partial X}{\partial x}(0,0) = f(0,0), \quad \frac{\partial X}{\partial y}(0,0) = 0, \quad \frac{\partial Y}{\partial x}(0,0) = 0, \quad \frac{\partial Y}{\partial y}(0,0) = g(0,0),$$

and the condition of non-degeneration of the singular point implies that $f(0,0) \neq 0$ and $g(0,0) \neq 0$. □

Theorem 3.2. *A smooth flow on the Boy's or Girl's surface is structurally stable if and only if it is a Morse-Smale flow.*

Proof. It is necessary to show that there does not exist a small bifurcation of Morse-Smale flow and that it exists for other flows. If a bifurcation does not occur at 0-stratum, then it is the same as on a surface with a boundary. Therefore, we can take advantage of the structural stability of Morse-Smale flows on a surface with a boundary, [19]. Consider a flow in an angle with a vertex in the 0-strata. For any flow in the angle we have a representation of the form $\{xf(x,y),\, yg(x,y)\}$ as in the Lemma 3.1. With a small change of the flow, the condition $f(0,0) \neq 0$ and $g(0,0) \neq 0$ does not change. Then the Morse-Smale flows are structurally stable in 0-stratum. Using the family of flows $\{x\big(c + f(x,y)\big),\, y\big(c + g(x,y)\big)\}$, with small changes to the parameter c, any flow can be brought in the neighborhood of $(0,0)$ to the Morse-Smale flow. It remains to prove that there are no other flows, with the same structure as Morse-Smale flows and without bifurcations. Let the flow $\{xf(x,y),\, yg(x,y)\}$ has $f(0,0) = 0$ and $(0,0)$ is an isolated fixed point like for the Morse-Smale flows. Consider the family

$$\big\{x(c^2 - cx + f(x,y) - f(c,y)),\, yg(x,y)\big\}, \quad c \in [0,1].$$

Then with any $c > 0$ the resulting flow has two fixed points $(0,0)$ and $(c,0)$. Therefore, it is not topologically equivalent to the original flow (at $c = 0$). If $g(0,0) = 0$, then the bifurcation formula is similar:

$$\big\{xf(x,y),\, y(c^2 - cy + g(x,y) - f(x,c))\big\}, \quad c \in [0,1].$$ □

4. Optimal Morse-Smale flows

As with the Boy's surface, [33], each Morse-Smale flow has at least one fixed point in every 1-cell. If a 1-stratum consisted of a single trajectory, then it would be a homoclinic trajectory. It is well known that Morse-Smale flows do not contain such trajectories, since such trajectories must lie in the α or ω-limit sets of trajectories intersecting some neighborhood of the corresponding strata, which contradicts the definition of a Morse-Smale flow.

Say that a fixed point is a *potential source (sink)* if it is simultaneously an α-limit (resp. ω-limit) point for two trajectories belonging to the same 1-cell. We indicate potential sources in red as well as by the letter R. Potential sinks well be colored in green (G). Separatrices will be colored in the same color as their limit points.

Theorem 4.1. *On the Girl's surface each optimal Morse-Smale flow has exactly* 4 *fixed points: null-point and one point in each of three* 1*-cells. In total, there are* 534 *non-homeomorphic and* 1058 *non-homotopic optimal Morse-Smale flows.*

Proof. Let us designate the singular points of the flow on the planar model by a, b, c, d, e, f, g as in Figure 4.1. Since c, d and g are glued together at the same point, they have the same color. Assume for definiteness that they are green. Similarly, points b and f are colored with one color, as well as one color of points a and e. Therefore, three options are possible:

(i) a is red and b is green,
(ii) both a and b are red,
(iii) both a and b are green.

In the first case (i) (a is red and b is green, see Figure 4.1), to calculate the total number of flow structures n, we will find the number of flow structures n_b in BR and the number of flow structures n_c in CR. Then $n = n_b\, n_c$. If there are no red separatrices in BR, then each of the points b, c, d, g can be a sink. So we have four different flow structures in BR in this case.

If there is a red separatrix, then it should be unique and end up at a. If it starts at angle 8, then it splits BR into two regions in each of which have two green points: b, g for the inner region and c, d for the outer region. Thus, in each of these areas there can be two sinks and the total number of flows with a separatrix $8 \to a$ is $2 \cdot 2 = 4$. Separatrices $11 \to a$ and $5 \to a$ split BR into two regions, one with a green point and the other with three green points. Then, the total number of options for each of them is $1 \cdot 3 = 3$. Adding everything together, we get that $n_b = 4 + 4 + 3 + 3 = 14$.

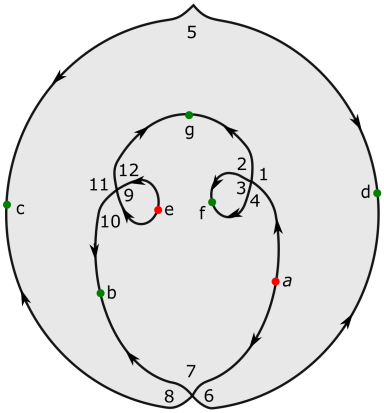

FIGURE 4.1. Morse-Smale flows, case (i)

If there are no red separators in CR, then the three green points on the boundary give 3 options. If there is one separatrix $2 \to e$, $2 \to a$, $a \to e$ or $e \to a$, then it splits CR into two regions with one and two green points. Thus, we have $4 \cdot 2 = 8$ variants of flows with one separatrix. If there are two red separatrices in CR ($2 \to a$, $2 \to e$), then only one such flow is possible. In total, we have $n_c = 3 + 8 + 1 = 12$. The total number of non-homeomorphic flows for the first diagram is $n = 14 \cdot 12 = 168$.

In the cases (ii) and (iii), it is necessary to take into account cases that are symmetric with respect to the vertical axis of symmetry. We denote by b_s the number of symmetric flows in BR and by b_n the number of non-symmetrical ones. Similarly, we denote by c_s and c_n the corresponding numbers for the region CR. Then the total number of flow structures can be calculated by the formula

$$n = b_s\, c_s + b_s\, c_n + b_n\, c_s + 2b_n\, c_n.$$

Consider the case (ii) (a and b are red, see Figure 4.2, left). If there is no red separatrix in BR, then the version with a sink in g is symmetrical, and versions with a sink in c or d are not symmetrical. The last two versions are homeomorphic to each other. If there is only one red separator, it may end in a or b. These cases are symmetrical (homeomorphic), so we consider only the first of them. Then there are two cases of separatrix $5 \to a$ or $b \to a$, which divide BR into two regions with one and two possible options for a sink. So, we have 4 non-symmetrical flow structures. If there are two red separators in BR ($5 \to a$, $5 \to b$), then the flow is unique and symmetrical. Summing up, we get $b_s = 2$, $b_n = 5$.

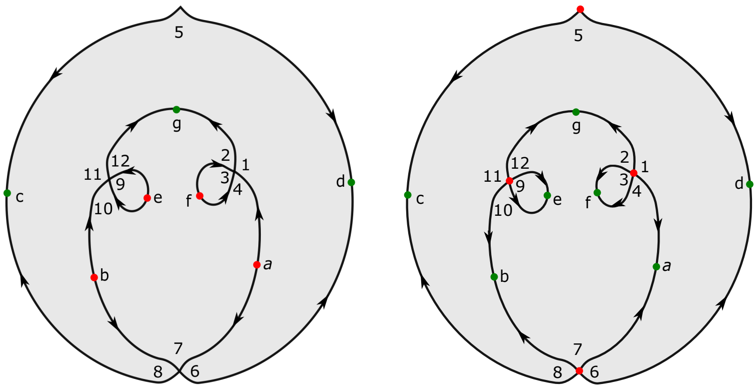

FIGURE 4.2. Morse-Smale flows, cases (ii) and (iii)

In CR, the symmetric flow must contain the green separatrix $g \to 7$. Then the sources can be in 1) a and b or 2) e and f. Thus, $c_s = 2$. Another option for arranging sources (e.g., a, e) for this separatrix is non-symmetrical. For the separatrix $g \to$ we have three versions of the sources (a, b, e), and if there are no separators, then two non-homeomorphic versions of the source (b, e). Summing up, we get $c_n = 6$.

Then the total number of non-homeomorphic flow structures for the second option is $n = 2 \cdot 2 + 2 \cdot 6 + 5 \cdot 2 + 2 \cdot 5 \cdot 6 = 86$.

For the third case (iii) (a and b are green, see Figure 4.2, right), symmetric flows for each of the two regions should have a sink in g, and for non-symmetric ones there are two non-homeomorphic possibilities for the location of the sink. Therefore, $b_s = 1$, $b_n = 2$, $c_s = 1$, $c_n = 2$, and

$$n = 1 \cdot 1 + 2 \cdot 1 + 1 \cdot 2 + 2 \cdot 2 \cdot 2 = 13.$$

Summing up all three options, we have $168 + 86 + 13 = 267$ flows with green c, d and the same number with red, that is, only 534 non-homeomorphic ms-flows. □

5. PROJECTIVE MORSE-SMALE FLOWS

A Morse-Smale flow on the real projective plane is *projective* if it projects to a flow on the Girl's (or Boy's) surface.

Theorem 5.1. *An optimal projective Morse-Smale (OPMS) flow of the Girl's surface has three sources, three sinks and five saddles on the projective plane. There are* 118 *non-homeomorphic OPMS flows on the Girl's surface.*

Proof. Consider the first case where 0-cell is the source for all three corresponding vertices on the projective plane. Three cases are possible for e and f:

(i) they are both sinks,
(ii) e is a sink, f is a saddle,
(iii) they are both saddles.

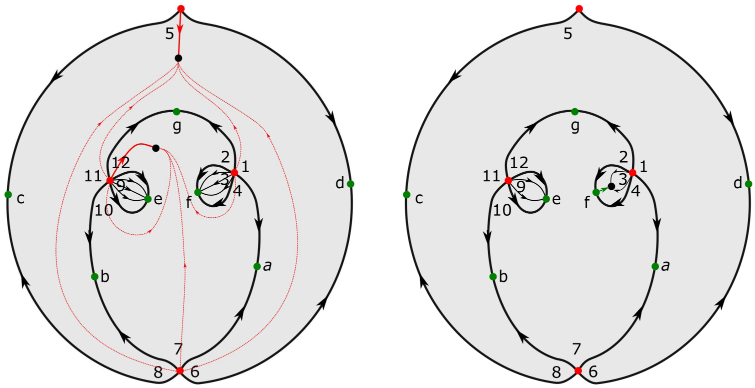

FIGURE 5.1. Projective Morse-Smale flows, cases (i) and (ii)

Consider case (i), (see Figure 5.1, left). Since two sinks are known (e and f), the third sink lies within or at the boundary of the BR. This situation splits into the four subcases.

Subcase (a): the third sink is inside BR. Then BR has symmetrical structure, and therefore the other two sinks must be separated by a stable manifold in the CR. Assume that one of its separatrices come out of 12. Then for the other there are three options: 4, 7, 10. The options when the first of the separators comes out of 2 are homeomorphic to those considered. So there are three non-homeomorphic flow structures in this case.

Subcase (b): $c = d$ are sinks. Since they must be separated by a stable manifold of saddle in BR, one of the separatrices included in BR starts at 5, and for the other there are four possibilities. There are three options for CR, as in the previous case. So, for the subcase (b) we have $3 \cdot 4 = 12$ non-homeomorphic flows.

Subcase (c): g is a sink. In this case, one of the separatrices starts at 12 and the other starts at 2. Such non-homomorphic pairs are possible, specifying two saddles inside the CR:

(1) $2 - 4$, $10 - 12$ (symmetrical flow),
(2) $2 - 7$, $7 - 12$ (symmetrical flow),

(3) $2 - 4$, $7 - 12$,
(4) $2 - 4$, $4 - 12$,
(5) $2 - 12$, $2 - 4$,
(6) $2 - 12$, $2 - 7$,
(7) $2 - 12$, $2 - 10$.

So we have 7 flow structures.

Subcase (d): a is a sink. The following options are possible:

1) $4 - 7$, $2 - 4$,
2) $4 - 7$, $12 - 4$,
3) $4 - 7$, $2 - 7$,
4) $4 - 7$, $12 - 7$,
5) $4 - 7$, $2 - 10$,
6) $4 - 7$, $12 - 10$,
7) $4 - 10$, $4 - 2$,
8) $4 - 10$, $4 - 12$
9) $4 - 10$, $10 - 2$,
10) $4 - 10$, $10 - 12$,
11) $4 - 12$, $7 - 12$,
12) $4 - 12$, $10 - 12$,
13) $2 - 4$, $2 - 7$,
14) $2 - 4$, $12 - 7$,
15) $2 - 4$, $2 - 10$,
16) $2 - 4$, $12 - 10$.

We have 16 possibilities. Thus in total, there are $3 + 12 + 7 + 16 = 38$ flow structures (two of them are symmetrical).

Consider the second case (ii) when e is a sink, f is a saddle (Figure 5.1, right). The situation is similar to the case (i). We consider the variations of the location of the sink in BR. There are five cases:

(a) if the sink is internal, then we have one flow structures,
(b) if $c = d$ are sinks, the we have 4 flow structures (inner saddle in BR, one separtrix enters it from 5, and for the second point there are 4 options),
(c) if g is a sink, then we have 4 flow structures (inner saddle in CR, one separtrix enters it from 12, and for the second point there are 4 options),
(d) if b is a sink, then we have 4 cases ($10 - 2$, 4, 7, or 12),
(e) if a is a sink, then we have $2 \cdot 3 = 6$ cases (7 or $10 - 2$, 4 or 12).

Thus, in total, there are $1 + 4 + 4 + 4 + 6 = 19$ flow structures.

In the third case (iii) (e, f are saddles, Figure 5.2) the sink lies on the boundary of BR and CR. There are two possible subcases of the location of the sink: points a and g. In each case, the flow is unique in each region. Therefore, there are two flow structures (the first of which is symmetrical).

Summing up, there are $2\,(38{+}19{+}2) = 118$ projective non-homeomorphic flows. □

6. Flow isotopy

Two flows are called *isotopic* if there exists a topological equivalence between them isotopic to the identity map.

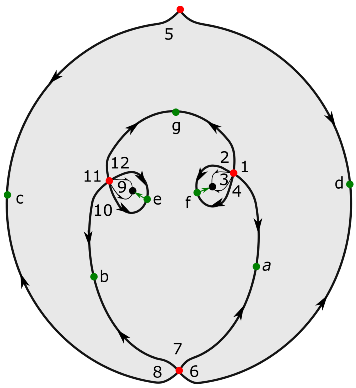

FIGURE 5.2. Projective Morse-Smale flows, case (iii)

Lemma 6.1. *A topological equivalence ϕ of flows on the Boy's or Girl's surface M is isotopic to the identity map iff it sends each 1-stratum X into itself and the restriction of $\phi|_X$ preserves orientation of X.*

Proof. *Necessity.* Note that each homeomorphism of M is cellular, i.e. it sends i-cells to i-cells, $i = 0, 1, 2$. In particular, if $\phi_t : M \to M$, $t \in [0; 1]$, is an isotopy of M, then all ϕ_t yield the same permutation of cells, and map them with the same orientation. If in addition ϕ_0 is the identity map, then each ϕ_t leaves invariant each cell and preserves its orientation.

Sufficiency. Since each 2-strata is determined by the list of 1-stratas included in its boundary, it follows from the preservation of 1-strats that each 2-strata is mapped onto itself. Any homeomorphism of a segment that preserves the orientation is isotopic to the identity. Thus, at the boundary of each 2-strata we have an isotopy between the given and the identical mapping. It can be extended to the isotopy of the 2-strata in the same way as from the boundary of the 2-disk to its interior. Together, we obtain the desired isotopy of the surface. □

Each homeomorphism of a Boy's or Girl's surface induces a permutation on a set of 1-stratas, and after such permutations the lists of words defining 2-strata must be preserved. Then on the Boy's surface, except for the identical one, the following substitutions are possible.

(1) Two letters change between places (3 ways of choosing two letters). In this case, we have a mapping that interchanges two 1-stratas and reverses the orientation on the third 1-strata.
(2) Letters are rearranged cyclically (2 ways).

On the Girl's surface, one substitution is possible: the letters **A** and **B** are swapped. This observation together with Lemma 6.1 imply that on the

Boy's surface each class of topological equivalence contains no more than 6 flows that are not isotopic each other, while on the Girl's surface, there are no more than two such flows.

A flow is called *symmetric* if its topological equivalence to itself exists, which is not isotopic to the identity mapping.

Theorem 6.2. *Let V be one of the sets of vector fields (flows) which we considered above, n is the total number of topological structures from V and n_s is the number of symmetric structures from V, then the number of non-isotopic structures is given by the formula*

$$m = 2n - n_s$$

on the Girl's surface and by formula

$$m = 3 \cdot (2n - n_s)$$

for optimal flows on the Boy's surface.

Proof. If the flow diagram is not symmetrical, then symmetry about the vertical axis result in a non-isotopic flow. Therefore, the total number of flow structures must be multiplied by two. In this case, we counted the symmetric flows twice. Hence, the number of symmetric threads must be subtracted from the total number. Taking into account that there are no $\mathbb{Z}_3$ invariant flows among the optimal flows on the Boy's surface, we see that the total number of flows must be multiplied by three. □

By analogy, how this was done in Theorem 4.1, we check each of the described structures on symmetry. We do the same for structures on the Boy's surface in [33]. Then, using the formulas in the last theorem, we calculate the number of non-isotopic structures. The resulting number of structures are given in the Table 6.1.

surface	1 fixed point	Morse-Smale	projective
Boy's	18/108	342/2004	80/438
Girl's	3/6	534/1058	118/230

Table 6.1. Numbers of optimal flows structures up to homeomorphism/isotopy

References

[1] A.V. Bolsinov and A.T. Fomenko. *Integrable Hamiltonian systems. Geometry, Topology, Classification.* A CRC Press Company, Boca Raton London New York Washington, D.C., 2004. 724 p.

[2] W. Boy. Über die Curvatura integra und die Topologie geschlossener Flächen. *Math. Ann.*, 57(2):151–184, 1903. doi:10.1007/BF01444342.

[3] G. Fleitas. Classification of gradient-like flows on dimensions two and three. *Bol. Soc. Brasil. Mat.*, 6(2):155–183, 1975. doi:10.1007/BF02584782.

[4] O. A. Giryk. Classification of polar Morse-Smale vector fields on two-dimensional manifolds. *Methods Funct. Anal. Topology*, 2(1):23–37, 1996.

[5] S. Goodman and M. Kossowski. Immersions of the projective plane with one triple point. *Differential Geom. Appl.*, 27(4):527–542, 2009. doi:10.1016/j.difgeo.2009.01.011.

[6] S. Goodman, A. Mellnik, and C. H. Séquin. Girl's surface. In George W. Hart and Reza Sarhangi, editors, *Proceedings of Bridges 2013: Math.*, pages 383–388, Phoenix, Arizona, 2013. Tessellations Publishing. URL: http://archive.bridgesmathart.org/2013/bridges2013-383.html.

[7] B. I. Hladysh and A. O. Prishlyak. Functions with nondegenerate critical points on the boundary of the surface. *Ukr. Math. J.*, 68(1):29–40, 2016. doi:10.1007/s11253-016-1206-5.

[8] B. I. Hladysh and A. O. Prishlyak. Topology of functions with isolated critical points on the boundary of a 2-dimensional manifold. *SIGMA*, 13:1–17, 2017. doi:10.3842/SIGMA.2017.050.

[9] B. I. Hladysh and A. O. Prishlyak. Deformations in the general position of optimal functions on oriented surfaces with boundary. *Ukr. Math. Journ.*, 71(8):1173–1185, dec 2020. doi:10.1007/s11253-019-01706-8.

[10] Bohdana Hladysh and Alexandr Prishlyak. Simple Morse functions on an oriented surface with boundary. *Zh. Mat. Fiz. Anal. Geom.*, 15(3):354–368, 2019.

[11] O. A. Kadubovskyj. Classification of Morse-Smale vector fields on 2-manifolds. *Visn., Mat. Mekh., Kyïv. Univ. Im. Tarasa Shevchenka*, 14:85–88, 2005.

[12] Z. Kibalko, A. O. Prishlyak, and R. Shchurko. Trajectory equivalence of optimal Morse flows on closed surfaces. *Proc. Int. Geom. Cent.*, 11(1):12–26, 2018. doi:10.15673/tmgc.v11i1.916.

[13] A. S. Kronrod. On functions of two variables. *Uspehi Matem. Nauk (N.S.)*, 5(1(35)):24–134, 1950.

[14] V. E. Kruglov, D. S. Malyshev, and O. V. Pochinka. Multicolored graph as a complete topological invariant for Ω-stable flows without periodic trajectories on surfaces. *Mat. Sb.*, 209(1):100–126, 2018. doi:10.4213/sm8797.

[15] E. Leontovich and A. Mayer. On a scheme determining the topological structure of the separation of trajectories. *Dokl. Akad. Nauk SSSR (N.S.)*, 103:557–560, 1955.

[16] D. P. Lychak and A. O. Prishlyak. Morse functions and flows on nonorientable surfaces. *Methods Funct. Anal. Topol.*, 15(3):251–258, 2009.

[17] S. Maksymenko. Local inverses of shift maps along orbits of flows. *Osaka J. Math.*, 48(2):415–455, 2011. URL: http://projecteuclid.org/euclid.ojm/1315318347.

[18] A. A. Oshemkov and V. V. Sharko. On the classification of Morse-Smale flows on two-dimensional manifolds. *Mat. Sb.*, 189(8):93–140, 1998. doi:10.1070/SM1998v189n08ABEH000341.

[19] J. Palis. On Morse-Smale dynamical systems. *Topology*, 8:385–404, 1968. doi:10.1016/0040-9383(69)90024-X.

[20] J. Palis and S. Smale. Structural stability theorems. In *Global Analysis (Proc. Sympos. Pure Math., Vols. XIV, XV, XVI, Berkeley, Calif., 1968)*, pages 223–231. Amer. Math. Soc., Providence, R.I., 1970.

[21] M. M. Peixoto. On structural stability. *Annals of Mathematics*, 69(1):199 – 222, 1959. doi:10.2307/1970100.

[22] M. M. Peixoto. On the classification of flows on 2-manifolds. In *Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, pages 389–419. Academic Press, New York, 1973.

[23] P. B. Percell. Structural stability on manifolds with boundary. *Topology*, 12:123–144, 1973. doi:10.1016/0040-9383(73)90002-5.

[24] D. Poltavec. Equivalent polar Morse-Smale systems on two dimensional manifolds of genus 3. In *Abstract of International Conference on Topology and its Applications, Kiev*, page 29, 1995.

[25] A. O. Prishlyak. On embedded graphs in a surface. *Uspekhi Mat. Nauk*, 52(4(316)):211–212, 1997. doi:10.1070/RM1997v052n04ABEH002074.

[26] A. O. Prishlyak. Conjugacy of Morse functions on surfaces with values on a straight line and circle. *Ukr. Mat. Zh.*, 52(10):1421–1425, 2000. doi:10.1023/A:1010461319703.

[27] A. O. Prishlyak. Morse-Smale vector fields without closed trajectories on three-dimensional manifolds. *Mat. Zametki*, 71(2):254–260, 2002. doi:10.1023/A:1013963315626.

[28] A. O. Prishlyak. Topological equivalence of Morse-Smale vector fields with beh 2 on three-dimensional manifolds. *Ukr. Mat. Zh.*, 54(4):492–500, 2002. doi:10.1023/A:1021035327909.

[29] A. O. Prishlyak. Topological equivalence of smooth functions with isolated critical points on a closed surface. *Topology Appl.*, 119(3):257–267, 2002. doi:10.1016/S0166-8641(01)00077-3.

[30] A. O. Prishlyak. On the sum of indices of flow with isolated fixed points on a stratified set. *Mat. Fiz. Anal. Geom.*, 10(1):106–115, 2003.

[31] A. O. Prishlyak. Topological classification of m-fields of two- and three-dimensional manifolds with boundary. *Ukraïn. Mat. Zh.*, 55(6):799–805, 2003. doi:10.1023/B:UKMA.0000010596.67853.89.

[32] A. O. Prishlyak. Complete topological invariants of morse-smale flows and handle decompositions of 3-manifolds. *J. Math. Sci., New York*, 144(5):4492–4499, 2007. doi:10.1007/s10958-007-0287-y.

[33] A. O. Prishlyak and L. Di Beo. Flows with minimal number of singularities in the boy's surface. *Proc. Int. Geom. Cent.*, 15(1):32–49, 2022. doi:10.15673/tmgc.v15i1.2225.

[34] A. O. Prishlyak and M. V. Loseva. The topology of Morse-Smale flows with singularities on the boundary of 2-dimensinal disk. *Proc. Intern. Geom. Center*, 9(2):32–41, 2016. doi:10.15673/tmgc.v9i2.279.

[35] A. O. Prishlyak and M. V. Loseva. Optimal Morse-Smale flows with singularities on the boundary of a surface. *J. Math. Sci., New York*, 243(2):279–286, 2019. doi:10.1007/s10958-019-04539-9.

[36] A. O. Prishlyak and M. V. Loseva. Topology of optimal flows with collective dynamics on closed orientable surfaces. *Proc. Int. Geom. Cent.*, 13(2):50–67, 2020. doi:10.15673/tmgc.v13i2.1731.

[37] A. O. Prishlyak, A. Prus, and S. Guraka. Flows with collective dynamics on a sphere. *Proc. Int. Geom. Cent.*, 14(1):61–80, 2021. doi:10.15673/tmgc.v14i1.1902.

[38] A. O. Prishlyak and A. A. Prus. Three-color graph of the morse flow on a compact surface with boundary. *Journal of Mathematical Sciences*, 249(4):661–672, aug 2020. doi:10.1007/s10958-020-04964-1.

[39] G. Reeb. Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique. *C. R. Acad. Sci. Paris*, 222:847–849, 1946.

[40] C. Robinson. Structural stability on manifolds with boundary. *Journal of Differential Equations*, 37(1):1–11, jul 1980. doi:10.1016/0022-0396(80)90083-2.
[41] S. Smale. Morse inequalities for a dynamical system. *Bull. Amer. Math. Soc.*, 66:43–49, 1960. doi:10.1090/S0002-9904-1960-10386-2.
[42] S. Smale. On gradient dynamical systems. *Ann. of Math. (2)*, 74:199–206, 1961. doi:10.2307/1970311.

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Maria Loseva
Taras Shevchenko National University of Kyiv
Email: mv.loseva@gmail.com
ORCID: 0000-0002-2282-206X

Alexandr Prishlyak
Taras Shevchenko National University of Kyiv
Email: prishlyak@knu.ua
ORCID: 0000-0002-7164-807X