

Singularity classes of special multi-flags, I

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Abstract. The construction of the geometric *singularity classes* of special multi-flags was exhaustively done in [P. Mormul, SIGMA, 5:Paper 102, 22 pages, 2009] for special 2-flags, i.e. when flag's width, typically denoted by m , was 2. Now analogous singularity classes are being constructed for special flags of all widths $m \geq 2$, compatible with and extending by far the construction in the mentioned paper.

Анотація. Конструкція геометричних *класів сингулярностей* спеціальних мульти-прапорів була вичерпно описана в роботі автора [P. Mormul, SIGMA, 5:Paper 102, 22 pages, 2009] для спеціальних 2-прапорів, у яких ширина прапора, зазвичай позначена m , дорівнювала 2. В даній статті будуються аналогічні класи сингулярностей для спеціальних прапорів будь-якої ширини $m \geq 2$, сумісних і значно розширюючих конструкцію в згаданій вище роботі.

1. HISTORICAL PRELIMINARIES

In the 21st century' differential geometry and singularity theory increasingly popular are various *monster towers* – towers of manifolds hosting, regular in certain sense, yet very far from being homogeneous *distributions* – subbundles in the tangent bundles. The benchmark meta-example in this line of research is the Goursat Monster Tower (GMT for short). Its stages host locally universal Goursat distributions of all possible coranks. (Goursat distributions are particular rank-2 subbundles in the tangent bundle to a manifold having the flag of consecutive Lie squares – the *derived* flag – regular and growing in ranks – slowly! – always only by one). Historically of key importance has been the local classification problem for Goursat distributions, with roots in the 19th century, in the years 1889–1898 (F. Engel, E. von Weber). 80+ years later in [6] produced were local (pseudo-)normal

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forms of Goursat distributions – so-called Kumpera-Ruiz (pseudo-)normal forms («KR forms» in what follows). In [7] constructed was a fundamental stratification of germs of Goursat distributions (or – the same thing – of points in the particular stages of the GMT) into invariant *sandwich* (or Kumpera-Ruiz) *classes* («KR classes» in what follows). There holds a perfect matching between the KR (pseudo-)normal forms and the KR strata – the sandwich classes. (See the entire page 466 and in particular [7, Proposition 2.1]). It is natural to label the KR strata by words over the alphabet $\{1, 2\}$, «1» standing for every flag’s member in regular position, «2» for every member in singular position. (First two letters are invariably ‘1.1’ because of the Darboux and Engel theorems in dimension 3 and 4, respectively).

However, Goursat is but a harbinger of more general objects and constructions. Namely, another series of interesting meta-objects are so-called Special Multi-Flag Monster Towers (*SmFMT*) initialized (under different name!) around the turn of the century by Kumpera and Rubín in [5]. The stages of the *SmFMT* host distributions locally universal as concerns generating *special m -flags* of arbitrary lengths. Special 2-flags were already addressed at length in [10]. The present text is the part I of a (long since announced) generalization of [10] to special flags of arbitrary width $m \geq 2$.

So, for the reader’s convenience, what are those flags? After simplifications by several authors, the most compact version of the definition was given in the years 2009–2010 by Adachi [1] and (independently) by Shibuya and Yamaguchi – in Drapeau Theorem in [13]. Namely, these flags are nested sequences of subbundles in the tangent bundle to a manifold, say, M that originate from a rank- $(m + 1)$ distribution D and that satisfy

$$TM = D^0 \supset D^1 \supset D^2 \supset \dots \supset D^{r-1} \supset D^r = D, \quad (1.1)$$

where

- $\text{rk} D^j = \dim M - jm = m + 1 + (r - j)m$,
- $D^j + [D^j, D^j] = D^{j-1}$ for $j = r, r - 1, \dots, 1$,
- and such that D^1 **possesses a corank one involutive subdistribution** F ($[F, F] \subset F \subset D^1$).

Note parenthetically that, when all members of the derived flag (1.1) are regular distributions of constant ranks (like in the present case), then simply

$$D^{j-1} = [D^j, D^j].$$

(More information about the historically earlier versions of the definition can be found in an exhaustive reference paper [9]). The importance of such a single involutive subdistribution F seems strange at first sight. But it is self-explanatory, because, whenever $m \geq 2$, such an F is unique. This is in

sharp contradistinction to the Goursat case $m = 1$ when there exist plenty of analogous involutive subdistributions of the respective flag's member D^1 .

2. THE AIM AND MAIN THEOREM

The aim of the present work is to stratify the stages of the $SmFMT$ in function of the local geometry of a flag in question. This task was already realized in [10] for the special 2-flags. Each stage of the S2FMT has been stratified into *singularity classes*, encoded by words over the alphabet $\{1, 2, 3\}$ of length equal to the stage number (= flag's length) starting with «1» and such that the first «3» (if any) goes only after the first «2» (if any).

The merit of [10] was that, along with the strata themselves, there were also constructed wide fans of local polynomial normal forms for the underlying rank-3 distributions, faithfully corresponding to the strata. That is, polynomial visualizations (or: night glasses) of the strata. Much like (much earlier) the KR normal forms having been visualizing the KR strata in the GMT.

In the years 2010s there had appeared several proposals, [3, 4, 12] among them, aiming at stratifying the singularities of special multi-flags. The proposals [3] and [4] were based on a generalization of a deep curve-singularities-related approach to Goursat flags in [8]. In [4] the singularities of special 2-flags of lengths not exceeding 4 were analyzed. The authors arrived at the very conclusions of [12] save one observation in the latter that the classification under consideration ceases to be stable with respect to flag's width $m \geq 2$ from the length 4 onwards (cf. [12, section 7.7]). Also, in [4]

- (a) there was still used the original rather unwieldy definition of *special* multi-flags, while the posterior compact definition given in Drapeau Theorem (ii) in [13] was at hand, and
- (b) [13, Remark 4.5] went completely unnoticed.

The approaches of [4] and [3] were summarized and unified by Castro *et al* in [2]. The stratifications emerging from that latter contribution, of the stages of the Simple Towers (the algebraic geometry' name for the $SmFMT$'s) into so-called *RV strata* appear to be – in more than one sense – orthogonal to the singularity classes' stratifications in [10] and in the present text. A first attempt at drawing comparison between the two approaches is in the recent work [11]. In it (on p.345), the two exemplary (and still fairly elementary) singularity classes of special 2-flags, discussed also *here* below in Section 7, are *dissected* by RV strata (dissected, not built up of!). Even the dissection of, say, the class 1.2.1.1.3 by RV strata is not, as of

now, precisely described. It is clear by now that the singularity classes are *not* directly expressible in terms of the RV strata, and vice versa.

Also, the mere cardinalities in both approaches quickly diverge. For instance, for $m = 2$ and $r = 7$ there are 365 singularity classes against 1935 RV strata, while the number of local geometries in that length is already *infinity*, as proven on pages 39–41 in [12]. Further comparison results are badly needed.

In the present work we go far beyond [10] and construct, for every $m \geq 2$, stratifications of stages of the $SmFMT$ into – again – *singularity classes* that coincide with those built for $m = 2$ in [10]. This time they are labelled by words of length equal to flag’s length over the alphabet

$$\{1, 2, \dots, m, m + 1\}$$

subject to certain limitations. The words start with a «1» and any letter ℓ bigger than 1 shows up (if any) only when the letter $\ell - 1$ has shown up somewhere earlier. For instance, the word 1.2.1.4, ($m \geq 3$) is not allowed and is to be replaced by 1.2.1.3. This limitation (observed already in the above-recalled visualizations of the 1- and 2-flags) is called in this work the least upward jumps rule or, in short, the «lujr». The singularity class, to-be-constructed below in sections 4.1-4.2, of the germ at p of a distribution D will be denoted in what follows by $\mathfrak{w}(D, p)$.

The strata are being precisely constructed only now, while the polynomial normal forms visualizing them *post factum*, are fairly old, defined still in [9]. They are traditionally called *Extended Kumpera-Ruiz* (“EKR” for short) and labelled likewise by strings of letters from the pool

$$\{\mathbf{1}, \mathbf{2}, \dots, \mathbf{m}, \mathbf{m} + \mathbf{1}\}$$

(pay attention to the **bold** characters in this place). The EKR forms are precisely reproduced in Section 6 (which culminates in Theorem 6.1.1 excerpted verbatim from [9]).

The very message of this work is that the EKR (pseudo-)normal forms correspond faithfully to the singularity classes. In fact, passing over (for a time) some details and anticipating, we formulate in the present paper

Theorem 2.1. *Let D be the germ at $p \in M$ of a rank- $(m + 1)$ distribution generating a special m -flag of length $r \geq 1$, belonging to a fixed singularity class $j_1.j_2\dots.j_r$ ($= \mathfrak{w}(D, p)$). Then the EKR pseudo-normal forms of D subject to the least upward jumps rule are uniquely of the type $\mathbf{j}_1.\mathbf{j}_2\dots\mathbf{j}_r$. (Here the respective bold and italic characters stand for numerically identical integer values).*

In other words, the singularity class of a germ of a special m -flag which is already given in an EKR form $\mathbf{j}_1.\mathbf{j}_2\dots\mathbf{j}_r$ subject to the least upward jumps

rule, is $j_1.j_2\dots j_r$. That is, abusing notation a bit,

$$\mathfrak{w}(\mathbf{j}_1.\mathbf{j}_2\dots\mathbf{j}_r) = j_1.j_2\dots j_r.$$

Germ's singularity class that is to emerge from a long and involved definition of section 4.2, will acquire in this theorem an utmostly concrete visualization. Additionally, Theorem 2.1 shows that all singularity classes are non-empty (and says *much* more about them). Speaking uniquely about the EKR's satisfying the «lujr», whenever one finds an EKR for a germ of special m -flag, one simultaneously stumbles upon its singularity class. (A historically first nontrivial example of retrieving the singularity classes from bare EKR's is reproduced towards paper's end in Section 7).

The initial part of Theorem 2.1 for $m = 2$ was proved in [10, Theorem 1]. A proof for arbitrary width $m \geq 2$ builds upon that one, but needs also finer new tools. It is being postponed to the forthcoming part II of the present work.

3. SANDWICH DIAGRAM AND SANDWICH CLASSES

The structure of special m -flags of length r is geometrically visualized in a *Sandwich Diagram*, so called after a similar diagram originally assembled in [7] for Goursat distributions or 1-flags.

$$\begin{array}{ccccccc} D^0 & \supset & D^1 & \supset & D^2 & \supset & \dots \supset D^{r-1} & \supset & D^r \\ & & \cup & & \cup & & \cup & & \cup \\ & & F & \supset & L(D^1) & \supset & \dots \supset L(D^{r-2}) & \supset & L(D^{r-1}) & \supset & L(D^r). \end{array}$$

The objects in diagram's lower line are, apart from F (and F is absent for Goursat, because not unique then!), the *Cauchy characteristic* modules of vector fields of the respective flag's members, here automatically happening to be regular (and involutive by Jacobi) distributions. The inclusions in the lower line are due to the Jacobi identity as well. All vertical inclusions are of codimension 1 while all *drawn* horizontal inclusions are of codimension m . The squares formed by these inclusions can indeed be perceived as certain «sandwiches» indexed by squares' upper right vertices: D^2, D^3, \dots

For instance, in the utmost left sandwich the distributions F and D^2 are as if fillings, while D^1 and $L(D^1)$ constitute the covers (of dimensions differing by $m + 1$, though). At that, the sum ($= m + 1$) of codimensions, in D^1 , of F and D^2 equals the dimension of the quotient space $D^1/L(D^1)$, so that it is natural to ask how the m -dimensional *space* $F/L(D^1)$ and the *line* $D^2/L(D^1)$ are mutually positioned in $D^1/L(D^1)$. Similar alternatives also arise (independently!) in further sandwiches 'indexed' by the upper right 'vertices' D^3, D^4, \dots, D^r .

One thus divides all existing germs of special m -flags of length r on M into 2^{r-1} pairwise disjoint «sandwich classes» depending on the geometry

of the distinguished spaces in the sandwiches (at the reference point, say p , for each germ), and label those aggregates of points of the underlying manifold M by words $\mathfrak{w}(= \mathfrak{w}_2(D, p))$ of length r over the alphabet $\{1, 2\}$

- starting on the left with 1,
- having the second letter 2 iff $D^2(p) \subset F(p)$,
- and for $3 \leq j \leq r$ having the j -th letter 2 iff $D^j(p) \subset L(D^{j-2})(p)$.

Note that one particular sandwich class has the label

$$\mathfrak{w}_2 = 1.1\dots 1.1.$$

It is the only singularityless class, called «jet-like» in a local jargon.

The Sandwich Diagram (SD) is the key object. Virtually all that follows will ultimately (but in a complex enough way!) boil down to possible inclusions, of objects accordingly and frequently modified by Lie algebra' techniques, in the members of the SD.

4. CONSECUTIVE REFINEMENTS OF THE SANDWICH CLASSES

The partition into sandwich classes \mathfrak{w}_2 recalled in Section 3 is the first step of the construction to be presented in this paper. Agree that the class' label \mathfrak{w}_2 is the outcome of a local refinement of the trivial word $\mathfrak{w}_1 = 1.1\dots 1.1 = \mathfrak{w}_1(D, p)$ for all $p \in M$. (Needless to say, this first local refinement depends already on points). The next step consists in refining the sandwich classes with labels \mathfrak{w}_2 to classes labelled by words \mathfrak{w}_3 over a wider alphabet $\{1, 2, 3\}$. Words starting (obviously) from 1 and such that a first 3 (if any) goes only after a first 2 (if any – a prototype of the «lujr» restriction pervading the entire present text, cf. Section 2 above). This task was carried through in [10] precisely for special 2-flags ($m = 2$). Yet the procedure of arriving at the word \mathfrak{w}_3 goes virtually unchanged for all widths $m \geq 2$, as is noted in [10, Remarks 4(b) and 5, page 13].

Attention. In what follows the reader is invited to pay high attention to various stades of the construction of words of given length r (typically $r \geq m+1$), ultimately over the alphabet $\{1, 2, \dots, m, m+1\}$, with sometimes not all these letters being used. Because at *all* stades of the construction those words will be written $j_1.j_2\dots j_r$. Initially they will be over $\{1, 2\}$ only, as mentioned above, and under general name \mathfrak{w}_2 . Then – over $\{1, 2, 3\}$, under general name \mathfrak{w}_3 . And so on until the eventual words $j_1.j_2\dots j_r$ over the alphabet $\{1, 2, \dots, m, m+1\}$.

Definition 4.0.1. In sections 4.1 and 4.2 directly below there is constructed a finite sequence of words $\{\mathfrak{w}_j(D, p)\}_{j \geq 1}$. Either

- (i) this sequence is of full length $m + 1$, i.e. ends on a word $\mathfrak{w}_{m+1}(D, p)$ and then the *singularity word* of a distribution D at point p , denoted by $\mathfrak{w}(D, p)$, is $\mathfrak{w}_{m+1}(D, p)$, or else
- (ii) there appears in this sequence **for the first time** a pair of words $\mathfrak{w}_{k-1}(D, p)$ and $\mathfrak{w}_k(D, p)$ such that $\mathfrak{w}_k(D, p)$ differs from $\mathfrak{w}_{k-1}(D, p)$ by at most one new letter k .

Then the singularity word $\mathfrak{w}(D, p)$ is $\mathfrak{w}_k(D, p)$.

(The alternatives (i) and (ii) may not be mutually excluding: the eventual word $\mathfrak{w}_{m+1}(D, p)$ may feature just one letter $m + 1$).

Reiterating now, the words written $j_1.j_2\dots.j_r$ will depend on the construction stage in question! At a given stage k we will write

$$\mathfrak{w}_k(D, p) = j_1.j_2\dots.j_r$$

and at the next stage $k + 1$ will again write

$$j_1.j_2\dots.j_r = \mathfrak{w}_{k+1}(D, p),$$

although the latter word will sometimes differ from the former.

In section 4.3 there is given one extended (if virtual) example of a sequence of our words' refinements implied by (and also illustrating) the recursion procedure presented in sections 4.1 and 4.2.

4.1. First step beyond the sandwich classes. One keeps the germ of a rank- $(m + 1)$ distribution D generating a special m -flag of length r on a manifold M fixed. Suppose that in the sandwich class

$$\mathfrak{w}_2(D, p) = j_1.j_2\dots.j_r$$

of D at p there appears somewhere, for the first time when going from the left, the letter $j_f = 2$ (j_f is assuredly not the first letter on the left) **and** that there are other letters $2 = j_\nu$, $f < \nu$, as well. (If not, then the singularity word $\mathfrak{w}(D, p)$ is already defined as $\mathfrak{w}_2(D, p)$, cf. Definition 4.0.1 above, and there is nothing more to do). One is going to specify each such j_ν to either 2 or 3, while the initial 2 in the j_f -th position will (by virtue of our definition) stay put. Let the **nearest 2 standing to the left** to j_ν be $2 = j_\alpha$, $f \leq \alpha < \nu$. These two «neighbouring» letters 2 are separated by $l = \nu - \alpha - 1 \geq 0$ letters 1. We know after Section 3 that in the α -th sandwich there holds the inclusion

$$F(p) \supset D^2(p), \text{ when } \alpha = 2, \text{ or } L(D^{\alpha-2})(p) \supset D^\alpha(p), \text{ when } \alpha > 2. \quad (4.1)$$

The core of the refinement consists in taking the *small flag* (that is, the same thing, the Lie flag) of the original flag's member D^ν ,

$$D^\nu = V_1 \subset V_2 \subset V_3 \subset V_4 \subset V_5 \subset \dots,$$

$V_{i+1} = V_i + [D^\nu, V_i]$, and then focusing on this new flag's member V_{2l+3} . Now substituting $V_{2l+3}(p)$ for $D^\alpha(p)$ in inclusion (4.1) means specifying j_ν to 3. In other words, $j_\nu = 2$ in $\mathfrak{w}_2(D, p)$ is being specified to $j_\nu = 3$ in a new word if and only if $F(p) \supset V_{2l+3}(p)$ (when $\alpha = 2$) or when $L(D^{\alpha-2})(p) \supset V_{2l+3}(p)$ (when $\alpha > 2$) holds. In the end of the day all non-first letters 2 (if any) in $\mathfrak{w}_2(D, p)$ are, one independently of another, specified to 2 or 3. (The first appearing from the left letter 2 remains, as the reader remembers, unchanged). Altogether one obtains a word over $\{1, 2, 3\}$, denoted by $\mathfrak{w}_3(D, p)$ and satisfying the «lujr» in the by-now-classical terminology of [9]. Pay attention that it may happen that either

$$\mathfrak{w}_3(D, p) = \mathfrak{w}_2(D, p),$$

i.e. no letter «3» is being produced in the process of refining the sandwich class at p , or else there is only one new letter «3» in $\mathfrak{w}_3(D, p)$. In such a situation the procedure of constructing the singularity class at point p is finished, and that singularity class is but $\mathfrak{w}_3(D, p)$.

What is important for the following is the sequence of Lie algebra operations transforming each distribution D^ν as above into «its» *module* V_{2l+3} . This transformation is being named $\sigma_2[\nu]$ after we **widely extrapolate** its meaning and also start to denote by $\alpha_2(\nu)$ the position α of the letter «2» closest on the left in $\mathfrak{w}_2(D, p)$ to $j_\nu = 2$. From now on $\sigma_2[\nu]$ is an operator

- a) acting on arbitrary modules of vector fields tangent to M and
- b) depending on the word $\mathfrak{w}_2(D, p)$ only, and attached to the ν -th place in $\mathfrak{w}_2(D, p)$.

Reiterating the situation in the word $\mathfrak{w}_2(D, p)$ from this wider point of view, $\mathfrak{w}_2(D, p)$ features, apart from the first «2» on the left, letters «2» like $j_\nu = 2$ and the nearest to j_ν on the left $2 = j_{\alpha_2(\nu)}$. These two positions ν and $\alpha_2(\nu)$ define the quantity

$$2l + 3 = 2(\nu - \alpha_2(\nu) - 1) + 3$$

and then, for an arbitrary module U of vector fields on M ,

$$\sigma_2[\nu](U) := \text{the } (2l + 3)\text{-rd term of } U\text{'s small flag.}$$

(In particular $\sigma_2[\nu](D^\nu) = V_{2l+3}$).

Note also that *after* the refinement (i.e. in terms of the newly created word $\mathfrak{w}_3(D, p)$) the operators $\sigma_2[\nu]$ are defined for all ν such that the letter j_ν in $\mathfrak{w}_3(D, p)$ is 2 (excepting the first 2 on the left in $\mathfrak{w}_3(D, p)$) or else j_ν is 3. In a wider perspective, the refinement of $\mathfrak{w}_2(D, p)$ to $\mathfrak{w}_3(D, p)$ ought to be viewed as the beginning of a *recursive procedure*. The details are as follows.

4.2. Recursive refining procedure leading from the word $\mathfrak{w}_k(D, p)$ to $\mathfrak{w}_{k+1}(D, p)$ for $3 \leq k \leq m$. We assume throughout this section that the word $\mathfrak{w}_k(D, p)$, $3 \leq k \leq m$, has at least *two*, newly appeared letters « k » in comparison with the previous word $\mathfrak{w}_{k-1}(D, p)$. Otherwise the entire refining procedure has ended already (cf. Definition 4.0.1 above), outputting the singularity word $\mathfrak{w}(D, p)$. Consequently, for a given k , $3 \leq k \leq m$, we assume already known:

- (a) the word $\mathfrak{w}_k(D, p)$ over the alphabet $\{1, 2, \dots, k\}$, satisfying the «lujr» and featuring at least two letters k ,
- (b) the operators $\sigma_{k-1}[\nu]$ and the integers $\alpha_{k-1}(\nu)$ ($2 \leq \alpha_{k-1}(\nu) < \nu$) for all indices ν such that $j_\nu = k-1$ in $\mathfrak{w}_k(D, p)$ (excepting the first letter « $k-1$ » on the left in $\mathfrak{w}_k(D, p)$) or else $j_\nu = k$ in $\mathfrak{w}_k(D, p)$.

(As for the beginning of recursion $k = 3$, the needed data has been gathered and/or defined in the course of section 4.1).

We are going to refine the «lujr» word $\mathfrak{w}_k(D, p)$ over $\{1, 2, \dots, k\}$ to a «lujr» word $\mathfrak{w}_{k+1}(D, p)$ over $\{1, 2, \dots, k, k+1\}$ and proceed **separately with each non-first-on-the-left letter** $j_\nu = k$ in $\mathfrak{w}_k(D, p)$. In fact, let then $j_\mu = k$ be the nearest to the left to j_ν such letter. We write

$$j_{s_1}, j_{s_2}, \dots, j_{s_\rho}, \quad \mu < s_1 < s_2 < \dots < s_\rho < \nu, \quad \rho \geq 0,$$

all intermediate letters in $\mathfrak{w}_k(D)$ that are **bigger than 1 and smaller than k** . ($\rho = 0$ means that there is no such letters and $j_\mu = k = j_\nu$ are either genuine neighbours in $\mathfrak{w}_k(D, p)$ or are separated only by a string of 1's). When formally degraded back to $\mathfrak{w}_2(D, p)$, the letters $j_{s_1}, \dots, j_{s_\rho}, j_\nu$ are **all and the only letters «2»** past j_μ and not farther than j_ν in the word $\mathfrak{w}_2(D, p)$. So we know what are the operators

$$\sigma_2[s_1], \dots, \sigma_2[s_\rho], \sigma_2[\nu].$$

Now comes the first punch line in the recursion step. The new operator $\sigma_k[\nu]$ and new/old integer $\alpha_k(\nu)$ are being defined. Namely,

$$\sigma_k[\nu]: = \sigma_{k-1}[\mu] \circ \sigma_2[s_1] \circ \dots \circ \sigma_2[s_\rho] \circ \sigma_2[\nu]$$

(the operator $\sigma_{k-1}[\mu]$ is already defined at this moment, because $j_\mu = k$ in $\mathfrak{w}_k(D, p)$ ¹) and

$$\alpha_k(\nu): = \alpha_{k-1}(\mu)$$

(clearly $\nu > \mu > \alpha_{k-1}(\mu) = \alpha_k(\nu)$).

At this moment the way to define the upgrading is open and the second punch line comes. In fact, the letter $j_\nu = k$ in $\mathfrak{w}_k(D, p)$ is being upgraded

¹Observe that, possibly, not all operators $\sigma_{k-1}[\mu]$ from the recursion premise are being used in the recursion step.

to $j_\nu = k + 1$ in $\mathfrak{w}_{k+1}(D, p)$ if and only if

$$\sigma_k[\nu](D^\nu)(p) \subset L(D^{\alpha_k(\nu)-2})(p) \quad \text{when } \alpha_k(\nu) > 2,$$

or, when $\alpha_k(\nu) = 2$, if and only if

$$\sigma_k[\nu](D^\nu)(p) \subset F(p).$$

That is to say, the upgrading of j_ν from k to $k+1$ occurs when the respective inclusion holds in the $\alpha_k(\nu)$ -th sandwich evaluated at p . If the inclusion does not hold, then $j_\nu = k$ stays put in $\mathfrak{w}_{k+1}(D, p)$.

In this way all non-first on the left letters k in $\mathfrak{w}_k(D, p)$ are served and inspected. The resulting word $\mathfrak{w}_{k+1}(D, p)$ may either stay equal to $\mathfrak{w}_k(D, p)$, or may have only one letter k upgraded to $k+1$ (in both these situations this tantamounts to the recursion procedure's end, when the singularity class $\mathfrak{w}(D, p)$ is already ascertained), or else may have at least two letters k upgraded to $k+1$. If $k < m$, then the procedure does not stop yet and the further recursion step $k+1 \rightarrow k+2$ is to occur (not excluding, for instance, the possibility $\mathfrak{w}_{k+1}(D, p) = \mathfrak{w}_{k+2}(D, p)$).

This recursion goes (or may sometimes go) until the last step $m \rightarrow m+1$, provided the length r is big enough: $r \geq m+1$. When $r = m+1$, the particular word $1.2.3\dots m.(m+1)$, clearly satisfying the «lujr», may very well happen. As long as $r < m+1$, the biggest letter that may show up in our words is r , in the only word $1.2.3\dots(r-1).r$.

Remark 4.2.1. By the very nature of the above construction (and the one recalled in section 4.1) all constructed words $\mathfrak{w}_k(D, p)$, hence also the final singularity words $\mathfrak{w}(D, p)$, are satisfying the «lujr».

4.3. *Ad hoc*, examples illustrating the recursion of section 4.2.

The recursive procedure of section 4.2 is involved enough. An extended example below is intended to show it (or emulate it) in action. Suppose that $D = D^{12}$ is a rank-4 distribution on a 40-dimensional manifold M ($40 = 4 + 12 \cdot 3$) generating a special 3-flag in TM which features at certain point $p \in M$ the following singularities uncovered step by step via the procedure from section 4.2:

$$\begin{aligned} \mathfrak{w}_2(D, p) &= 1.2.2.2.1.2.2.1.1.2.2.2, \\ \mathfrak{w}_3(D, p) &= 1.2.3.2.1.2.3.1.1.3.2.3, \\ \mathfrak{w}_4(D, p) &= \mathfrak{w}(D, p) = 1.2.3.2.1.2.3.1.1.4.2.3. \end{aligned}$$

The patterns governing appearances of the 2's in the word \mathfrak{w}_2 are clear – it is pure sandwich diagram-related linear algebra. The inclusions underlying the appearances of the 3's in \mathfrak{w}_3 are less straightforward, but well-known after the paper [10], and also recapitulated in section 4.1 above. So, in the

notation used in section 4.2, the refinement $\mathfrak{w}_2(D, p) \rightarrow \mathfrak{w}_3(D, p)$ is made on the basis of the following inclusions and non-inclusions at p :

$$\begin{aligned} \sigma_2[3](D^3)(p) &\subset F(p), \\ \sigma_2[4](D^4)(p) &\not\subset L(D^1)(p), \\ \sigma_2[6](D^6)(p) &\not\subset L(D^2)(p), \\ \sigma_2[7](D^7)(p) &\subset L(D^4)(p), \\ \sigma_2[10](D^{10})(p) &\subset L(D^5)(p), \\ \sigma_2[11](D^{11})(p) &\not\subset L(D^8)(p), \\ \sigma_2[12](D^{12})(p) &\subset L(D^9)(p). \end{aligned}$$

Concerning the refinement $\mathfrak{w}_3(D, p) \rightarrow \mathfrak{w}_4(D, p)$, one is handling – section 4.2 in hand – all 3's in $\mathfrak{w}_3(D, p)$ save the leftmost one that is to stay put in this eventual refinement. The above-proposed $\mathfrak{w}_4(D, p)$ is then got on the basis of

$$\begin{aligned} \sigma_2[3] \circ \sigma_2[4] \circ \sigma_2[6] \circ \sigma_2[7](D^7)(p) &\not\subset F(p), \\ \sigma_2[7] \circ \sigma_2[10](D^{10})(p) &\subset L(D^4)(p), \\ \sigma_2[10] \circ \sigma_2[11] \circ \sigma_2[12](D^{12})(p) &\not\subset L(D^5)(p). \end{aligned}$$

4.4. How far back does one go in the Sandwich Diagram when ascertaining in the recursion a given word $\mathfrak{w}_k(D, p)$? In the recursion described in section 4.2, the decision whether upgrade or not a letter $j_\nu = k$ in $\mathfrak{w}_k(D, p)$ is taken in the sandwich number $\alpha_k(\nu)$. How much smaller is $\alpha_k(\nu)$ compared to ν ? For the sake of better visualization, let us talk about «transports» of modules of vector fields, although nothing is genuinely transported in section 4.2. Handling $j_\nu = k$, one first transports D^ν , transforming it in the process into the module

$$\sigma_2[s_1] \circ \cdots \circ \sigma_2[s_\rho] \circ \sigma_2[\nu](D^\nu) =: V$$

to the sandwich number μ , $\mu < \nu$, $j_\mu = k$ in $\mathfrak{w}_k(D, p)$. After which the operator $\sigma_{k-1}[\mu]$ takes care of V and sends it, possibly far, back to the sandwich number $\alpha_{k-1}(\mu) = \alpha_k(\nu)$. Concerning transports alone, that much is told in the recursion' description. Let us shadow now step by step the backward travel of the module V .

For the sake of precision, note that, when a sequence of words' refinements

$$\mathfrak{w}_2 \rightarrow \mathfrak{w}_3 \rightarrow \cdots \rightarrow \mathfrak{w}_{k-1} \rightarrow \mathfrak{w}_k$$

(all words are meant for D at p) is assumed known, the backward «forgetting» mappings

$$\mathfrak{w}_k \rightarrow \mathfrak{w}_{k-1} \rightarrow \cdots \rightarrow \mathfrak{w}_3 \rightarrow \mathfrak{w}_2$$

are clearly defined. (Attention: in what follows the same symbols j stand for different values, depending on the actual word they are in). After transforming D^ν into V one stands at letter $j_\mu = k$ in the word \mathfrak{w}_k . One goes down to \mathfrak{w}_{k-1} not changing the position. So now $j_\mu = k - 1$ in \mathfrak{w}_{k-1} and j_μ is not the first letter $k - 1$ in \mathfrak{w}_{k-1} . Let

$$j_{\mu_1} = k - 1, \quad \mu_1 < \mu,$$

be the closest to j_μ such letter in \mathfrak{w}_{k-1} . Then, descending now to \mathfrak{w}_{k-2} , $j_{\mu_1} = k - 2$ in \mathfrak{w}_{k-2} and j_{μ_1} is not the first letter $k - 2$ in \mathfrak{w}_{k-2} . So one picks $j_{\mu_2} = k - 2$ ($\mu_2 < \mu_1$) the closest to j_{μ_1} in \mathfrak{w}_{k-2} , then descends to \mathfrak{w}_{k-3} , and so on and on.

Eventually one has $j_{\mu_{k-3}} = 3$ in \mathfrak{w}_3 , descends to $j_{\mu_{k-3}} = 2$ in \mathfrak{w}_2 and finds its closest to the left in \mathfrak{w}_2 neighbour $j_{\mu_{k-2}} = 2$ with, clearly,

$$\mu_{k-2} < \mu_{k-3} < \cdots < \mu_2 < \mu_1 < \mu.$$

Thus found number μ_{k-2} is nothing but the number $\alpha_{k-1}(\mu) = \alpha_k(\nu)$ featuring in the recursion step $k \rightarrow k + 1$ in section 4.2.

Repeating one last time after section 4.2, it is precisely in the sandwich of $D^{\alpha_k(\nu)}$ where the disputed value $j_\nu = k$ in the word $\mathfrak{w}_k(D, p)$ is being upgraded or not. That is to say, j_ν becomes $k + 1$ in the subsequent word $\mathfrak{w}_{k+1}(D, p)$ if and only if

$$\sigma_k[\nu](D^\nu)(p) \subset L(D^\tau)(p) \tag{4.2}$$

where $\tau = \alpha_k(\nu) - 2$ when $\alpha_k(\nu) > 2$, or else iff

$$\sigma_k[\nu](D^\nu)(p) \subset F(p) \tag{4.3}$$

when $\alpha_k(\nu) = 2$.

5. SINGULARITY CLASSES EMANATING FROM THE SINGULARITY WORDS \mathfrak{w}

In more general framework, what can be said about the underlying manifold M of dimension $m + 1 + rm$? With no loss of generality it may be the model manifold $P^r(\mathbb{R}^{m+1})$ from [10, section 2.1] (compare in this respect [9, Theorems 2 and 3]). That is, the total space of the r times Cartan prolonged full tangent bundle $T\mathbb{R}^{m+1}$ to \mathbb{R}^{m+1} . The outcome of that sequence of r Cartan prolongations is a rank- $(m + 1)$ distribution D generating a *locally universal* special m -flag of length r . Meaning that such

an object locally models every special m -flag existing on $(m + 1 + rm)$ -dimensional manifolds; compare especially [9, Remark 3] where the symbol $SmFMT$ has been used for the first time ever.

This important fact does not in itself explain how rich are local geometries of special multi-flags. The (singularity) words $\mathfrak{w}(D, p)$ defined in Section 4 are clearly invariants of the local classification, but maybe some or many of them do not materialize at all? At any rate one gets, however, a partition of the underlying manifold into invariant sets of constant singularity \mathfrak{w} -words. Abusing notation a bit

$$j_1.j_2\dots j_r := \{p \in P^r(\mathbb{R}^{m+1}) : \mathfrak{w}(D, p) = j_1.j_2\dots j_r\}. \quad (5.1)$$

From now on, by $j_1.j_2\dots j_r$ (a word over $\{1, 2, \dots, m, m + 1\}$ satisfying the «lujr») we understand precisely the locus (5.1) of points featuring this one singularity word. We call this locus the «singularity class $j_1.j_2\dots j_r$ ».

Remark 5.1. Reiterating such a central fact, for each $r \geq 1$, by the very nature of definitions in sections 4.1 and 4.2, the partition of the r -th stage $P^r(\mathbb{R}^{m+1})$ of $SmFMT$ into singularity classes is **invariant** with respect to the automorphisms of the locally universal distribution generating in $P^r(\mathbb{R}^{m+1})$ the special m -flag of length r .

In the end of the day all singularity classes will turn out to be smooth manifolds in each monster's stage $P^r(\mathbb{R}^{m+1})$, and of explicitly computable codimensions². Possessing neat polynomial (pseudo-)normal forms yielding also handy sets of equations describing these classes. Forming altogether an utmostly elegant regular stratification of each stage $P^r(\mathbb{R}^{m+1})$. How can one see all this?

First and foremost, for special multi-flags there exist direct generalizations of classical Kumpera-Ruiz coordinates in the stages of the Goursat Monster Tower (for their history see, for instance [9, Corollary 1] and the first paragraph in [10, section 3.3]). Those coordinates serve as genuine «night glasses» allowing to watch the flags. The next Section is entirely devoted to them.

6. EXTENDED KUMPERA-RUIZ (EKR) PSEUDO-NORMAL FORMS

In this section we are going to produce a huge variety of polynomially written germs at $0 \in \mathbb{R}^N$, N possibly very large and always $1 \pmod{m}$, of rank- $(m + 1)$ distributions. Often certain variables x will appear in them in a «shifted» form $X = c + x$. And always a capital letter X , typically with indices, will mean such a shifted variable, not excluding, incidentally, the value $c = 0$. For each $k \in \{1, 2, \dots, m + 1\}$ we are going to define an

² Compare a forerunning [10, Proposition 4].

operation \mathbf{k} (the numerical value of \mathbf{k} is identical with that of k) producing new rank- $(m+1)$ distributions from older ones. The technical writing of its outcome, not operation's formal definition, will depend on how many operations were done *before* \mathbf{k} .

More specifically, the outcome of the operation \mathbf{k} being performed as operation number l on a distribution (Z_1, \dots, Z_{m+1}) defined in the vicinity of $0 \in \mathbb{R}^s(y_1, \dots, y_s)$ is a new rank- $(m+1)$ distribution defined in the vicinity of $0 \in \mathbb{R}^{s+m}(y_1, \dots, y_s, x_1^l, \dots, x_m^l)$, generated by the vector field

$$Z'_1 = x_1^l Z_1 + \dots + x_{k-1}^l Z_{k-1} + Z_k + X_k^l Z_{k+1} + \dots + X_m^l Z_{m+1}, \quad (6.1)$$

$$Z'_2 = \frac{\partial}{\partial x_1^l}, \quad \dots, \quad Z'_{m+1} = \frac{\partial}{\partial x_m^l}.$$

It is important that these local generators are written precisely in this order, yielding together a new longer object $(Z'_1, Z'_2, \dots, Z'_{m+1})$.

For instance, when $k = 1$, the sum Z'_1 starts with Z_1 and all new incoming variables in Z'_1 are (in general) shifted. When $k = m+1$, the sum Z'_1 ends with Z_{m+1} and all new variables are bare (with no shifting constants).

6.1. Definition of the EKR pseudo-normal forms. Extended KR pseudo-normal forms (EKR for short), of length $r \geq 1$, denoted by $\mathbf{j}_1 \cdot \mathbf{j}_2 \dots \mathbf{j}_r$, where $\mathbf{j}_1, \dots, \mathbf{j}_r \in \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{m} + \mathbf{1}\}$, are defined inductively, starting from the empty label distribution

$$(\partial_0, \partial_1, \dots, \partial_m) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1^0}, \dots, \frac{\partial}{\partial x_m^0} \right)$$

in the vicinity of $0 \in \mathbb{R}^{m+1}(t, x_1^0, \dots, x_m^0)$. Then, assuming $\mathbf{j}_1 \dots \mathbf{j}_{r-1}$ already known, $\mathbf{j}_1 \dots \mathbf{j}_{r-1} \cdot \mathbf{j}_r$ is the outcome of the operation \mathbf{j}_r performed as the operation number r over the distribution $\mathbf{j}_1 \dots \mathbf{j}_{r-1}$.

It is a matter of straightforward check by backward induction

$$r \rightarrow r-1 \rightarrow r-2 \rightarrow \dots \rightarrow 1 \rightarrow 0$$

that each EKR generates on \mathbb{R}^{m+1+rm} a special m -flag of length r . (The Cauchy characteristic subdistributions in general situation behave like in [10, Proposition 2] for $m = 2$). Yet the number $(m+1)^r$ of such multi-parameter local models is much too high. (For Goursat the relevant number of KR models of length r is $\frac{1}{4}2^r$, because then $\mathbf{j}_1 = \mathbf{j}_2 = \mathbf{1}$ – there is no second sandwich!) A remedy to that redundancy was proposed in [9]³. In fact, the following theorem excerpted from [9] is a cornerstone in our process of visualizing special multi-flags.

³ It will be shown in the forthcoming paper's part II that the redundancy is thus completely eliminated – see the central here Theorem 2.1 invoiced already in Section 2.

Theorem 6.1.1. *Let a rank- $(m+1)$ distribution D on a manifold M^{m+1+rm} ($m \geq 2$) generate a special m -flag of length $r \geq 1$. For every point $p \in M$, D in a neighborhood of p is equivalent, by a local diffeomorphism that sends p to 0 , to a certain EKR $\mathbf{j}_1.\mathbf{j}_2\dots\mathbf{j}_r$ in a neighborhood of $0 \in \mathbb{R}^{m+1+rm}$. Moreover, that EKR can be taken such that $j_1 = 1$ and, for $l = 1, \dots, r-1$, if $j_{l+1} > \max(j_1, \dots, j_l)$, then $j_{l+1} = 1 + \max(j_1, \dots, j_l)$ (the least upward jumps rule in the words $\mathbf{j}_1.\mathbf{j}_2\dots\mathbf{j}_r$).*

It is of utmost importance that each EKR is to be understood as the germ at 0 . A reference point $p \in M$ is thus becoming $0 \in \mathbb{R}^{m+1+rm}$. One of consequences of it is that the leading vector field Z'_1 in (6.1) projects *singularly* onto $\text{span}(Z_1, Z_2, \dots, Z_{k-1})$. Note also that constants in every such an EKR polynomially visualizing a given germ (D, p) are *not*, in general, defined uniquely, very much like in the Goursat situation when $m = 1$. (Cf. for instance [9, Proposition 1]).

Remark 6.1.2. The advantages of such «night glasses» in each $SmFMT$ are numerous. The entire bottom row in the Sandwich Diagram gets a crystal clear description, like in [10, Proposition 2] when $m = 2$. The distributions generating special m -flags appear to be effectively locally nilpotentizable – in [9, Theorem 4]. More surprisingly, perhaps, the very basis $(Z_1, \dots, Z_m, Z_{m+1})$ in any EKR turns out to generate a finite-dimensional nilpotent Lie algebra over the reals, of effectively recursively computable *nilpotency order*.

The following two questions impose by themselves at this point.

- [†] Is an EKR model for (D, p) emerging from Theorem 6.1.1 unique up to the values of certain constants showing up in it?
- [‡] Is the singularity word $\mathfrak{w}(D, p)$ expressible in terms of the EKR(s) visualizing the germ (D, p) ?

Answer (in suspension) to questions [†] and [‡] is included in Theorem 2.1 formulated already earlier in the introductory Section 2.

7. EXAMPLE OF DISCERNMENT INSIDE $\mathfrak{w}_2 = 1.2.1.2$

Remark 7.1. A prototype of the example presented below was the discernment, in [9, Proposition 1(iv)], of the mere EKR's 1.2.2 and 1.2.3 visualizing parts of the third stage of the S2FMT. That was done (20 years ago, there were no singularity classes at that time) by means of the *small growth vector* (of a distribution at a point).

In the example we assume without loss of generality that flag's width $m = 2$, because our (local) aim is to analyze the refinement of $j_4 = 2$ in the word $\mathfrak{w}_2 = 1.2.1.2$ to 2 or 3 in the word \mathfrak{w}_3 ; compare (again!) [10,

Remark 5]. Note also that what follows is but a tiny specimen of the arguments behind [10, Theorem 1].

On the manifold $\mathbb{R}^{2+1+4\cdot 2}(t, x_0, y_0, x_1, y_1, \dots, x_4, y_4)$ we will propose two non-equivalent families of EKR's, both sitting in the sandwich class 1.2.1.2. One is called D and features (all members of the family) the singularity class $\mathfrak{w}_3(D) = 1.2.1.2$, and the other is called E and features the class $\mathfrak{w}_3(E) = 1.2.1.3$. Moreover, we will see the geometrical distinction between D and E at work. This time, for bigger transparency, we use the vertical writing of the most involved vector field's generators.

The first family of germs at $0 \in \mathbb{R}^{11}$ is generated by the following vector fields:

$$D = \left(\begin{array}{c} x_4 \left[\begin{array}{c} 1 \\ x_2 \left[\begin{array}{c} x_1 \\ y_1 \end{array} \right] \\ 1 \\ y_2 \\ X_3 \\ Y_3 \\ 1 \\ Y_4 \\ 0 \\ 0 \end{array} \right] \end{array} \right), \quad \partial_{x_4}, \quad \partial_{y_4} \end{array} \right).$$

Here, a variable x to the left of a bracket means (also in the sequel) multiplying by x all entries subsumed by that bracket. Thus, this is a 3-parameter family of rank-3 distributions (they are by far *not*, all pairwise nonequivalent – EKR's are only *pseudo-normal forms*). And the second (2-parameter and not all non-equivalent as well) family of germs reads as follows:

$$E = \left(\begin{array}{c} x_4 \left[\begin{array}{c} 1 \\ x_2 \left[\begin{array}{c} x_1 \\ y_1 \end{array} \right] \\ 1 \\ y_2 \\ X_3 \\ Y_3 \\ y_4 \\ 1 \\ 0 \\ 0 \end{array} \right] \end{array} \right), \quad \partial_{x_4}, \quad \partial_{y_4} \end{array} \right). \tag{7.1}$$

Remark 7.2. Only recapitulating here, what is (or are) the singularity class(es) of the EKR families **1.2.1.2** and **1.2.1.3**? Clearly, the answer is already in [10, Corollary 3] (a splinter of the main Theorem **2.1** there). They both, and only they, represent the sandwich class $\mathfrak{w}_2 = 1.2.1.2$. At the same time that class is the union of the singularity classes $\mathfrak{w}_3 = 1.2.1.2$ and $\mathfrak{w}_3 = 1.2.1.3$. (In the end of the day, clearly,

$$\mathfrak{w}_3(\mathbf{1.2.1.2}) = 1.2.1.2 \quad \text{and} \quad \mathfrak{w}_3(\mathbf{1.2.1.3}) = 1.2.1.3.$$

In the present Section, however, we purposely argue elementarily).

So, in the passing from a sandwich to singularity class(es), the first letter 2 on the left stays put while the specification of the second letter 2 is subtler, (section **4.1**).

For that second 2 in 1.2.1.2, the values of the integers μ and l are $\mu = 2$ and $l = 1$. Because μ takes the smallest possible value, the covariant subdistributions F_D and F_E of, respectively, D^1 and E^1 (after the standard indexation of the big flags of D and E) enter into play. And, because we work with EKR's, these covariant subdistributions are both equal to

$$F = (dt, dx_0, dy_0)^\perp$$

(the common covariant subdistribution F is spanned by all the versors in \mathbb{R}^{11} save $\partial_t, \partial_{x_0}, \partial_{y_0}$).

Let

$$D = D_1 \subset D_2 \subset D_3 \subset \dots$$

$$E = E_1 \subset E_2 \subset E_3 \subset \dots$$

be the respective small flags. Since $2l + 3 = 5$, the algorithm of finding the singularity class requires to analyze the positions at 0 of D_5, E_5 , and F . In order to do that it is helpful to watch carefully the early small flags' members D_2 and E_2 :

$$D_2 = \left(\begin{array}{c} x_4 \left[\begin{array}{c} 1 \\ x_2 \left[\begin{array}{c} x_1 \\ y_1 \\ 1 \\ y_2 \\ X_3 \\ Y_3 \\ 1 \\ Y_4 \\ 0 \\ 0 \end{array} \right] \\ \end{array} \right] \\ \end{array} \right. \quad \left. \begin{array}{c} x_2 \left[\begin{array}{c} 1 \\ x_1 \\ y_1 \\ 1 \\ y_2 \\ X_3 \\ Y_3 \\ 0 \\ 0 \\ 0 \end{array} \right] \\ \end{array} \right) \quad \left(\begin{array}{cccc} \partial_{x_3} & \partial_{y_3} & \partial_{x_4} & \partial_{y_4} \end{array} \right),$$

$$E_2 = \left(\begin{array}{c} x_4 \left(\begin{array}{c} 1 \\ x_2 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \\ 1 \\ y_2 \\ X_3 \\ Y_3 \\ y_4 \\ \mathbf{1} \\ 0 \\ 0 \end{array} \right) \\ x_2 \begin{pmatrix} 1 \\ x_1 \\ y_1 \\ 1 \\ y_2 \\ X_3 \\ Y_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array} \partial_{x_3} \partial_{y_3} \partial_{x_4} \partial_{y_4} \right).$$

Note that the first generators on the left in these descriptions are superfluous for D_2 (resp. E_2) as such. Yet we are to compute parts of the *small* flags of the departure objects D and E . Because of that these vector fields *are* important for our computations.

The main observation is that by multiplying the first and second generators in D_2 (resp. E_2) we get $\partial_{x_2} + Y_4 \partial_{y_2} \in D_3$ (resp. $y_4 \partial_{x_2} + \partial_{y_2} \in E_3$). Because of the special role played by the variable x_2 , this leads in two more steps to

$$\partial_t + x_1 \partial_{x_0} + y_1 \partial_{y_0} + Y_4 \partial_{y_1} \in D_5.$$

Thus

$$D_5(0) \not\subset (\partial_{x_1}, \partial_{y_1}, \partial_{x_2}, \partial_{y_2}, \partial_{x_3}, \partial_{y_3}, \partial_{x_4}, \partial_{y_4}) = F(0).$$

While the same operations performed for E lead to a benign inclusion

$$y_3 (\partial_t + x_1 \partial_{x_0} + y_1 \partial_{y_0}) + \partial_{y_1} \in E_5$$

(meaning that this vector field from the module E_5 evaluated at 0 sits in $F(0)$). Does, however, one has the expected full inclusion $E_5(0) \subset F(0)$? Yes, because having (7.1) and the module E_2 explicited above, a quick count gives

$$E_3 = \left(E_2, x_4 \partial_{x_2}, x_4 \partial_{y_2}, y_4 \partial_{x_2} + \partial_{y_2} \right).$$

And then, multiplying this E_3 two times by E in (7.1), one safely gets

$$E_5(0) \subset (\partial_{x_1}, \partial_{y_1}, \partial_{x_2}, \partial_{y_2}, \partial_{x_3}, \partial_{y_3}, \partial_{x_4}, \partial_{y_4}) = F(0)$$

(the covariant subdistribution F is one and the same for both models D and E). Therefore, by the definition of singularity classes for the special 2-flags, the second letter 2 in the sandwich word $\mathfrak{w}_2 = 1.2.1.2$ is being specified: to 2 for D , and to 3 for E . (The first letter 2 univocally stays put). That is to say, the 3-parameter family of germs D has $\mathfrak{w}_3(D) = 1.2.1.2$, whereas the 2-parameter family E has $\mathfrak{w}_3(E) = 1.2.1.3$.

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