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# Canonical quasi-geodesic mappings of special pseudo-Riemannian spaces

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**Abstract.** Studying diffeomorphisms of pseudo-Riemannian spaces, that belong to the class of quasi-geodesic mappings with the reciprocity condition and almost geodesic mappings of the 2nd type, we arrived at the quasi-geodesic mapping $(QGM)$ $f\colon \left(V_n, g_{ij}, F_i^h\right) \to \left(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h\right)$ of the spaces with generalized-recurrent affinor structure [I. N. Kurbatova and M. Pistruil, 2020].

Quasi-geodesic mappings are divided into two types: general and canonical. In this article, canonical quasi-geodesic mappings of recurrent-parabolic spaces are considered [I. N. Kurbatova and D. V. Lozienko, 2017]. Namely, the basic questions of the theory of such mappings are solved. Using methods developed in the theory of geodesic mappings [N. S. Sinyukov, 1979], the basic equations of canonical $QGMs$ of recurrent-parabolic spaces are reduced to a form that allows effective study. The fundamental theorems of the theory of canonical quasi-geodesic mappings of recurrent parabolic spaces are proved.

Also considered the canonical quasi-geodesic mapping of the recurrent-parabolic space $\left(V_n, g_{ij}, F_i^h\right)$ onto the semisymmetric space $\left(\overline{V}_n, \overline{g}_{ij}, F_i^h\right)$.

**Анотація.** Досліджуючи дифеоморфізми псевдоріманових просторів, що належать до класу квазі-геодезичних відображень з умовою взаємності та майже геодезичних відображень 2-го типу, автори прийшли до квазі-геодезичного відображення $f\colon \left(V_n, g_{ij}, F_i^h\right) \to \left(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h\right)$ просторів з особливою афінорною структурою, яку назвали узагальнено-рекурентною [І. М. Курбатова та М. Пiструiл, 2020].

Квазі-геодезичні відображення можуть бути двох типів: загального виду і канонічні. В представленій статті досліджуються канонічні квазі-геодезичні відображення рекурентно-параболічних просторів [І. М. Курбатова та Д. В. Лозієнко, 2017]. А саме, вирішуються базові питання

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теорії таких відображень. За допомогою методів теорії геодезичних відображень [Н. С. Синюков, 1979], основні рівняння канонічних квазі-геодезичних відображень рекурентно-параболічних просторів приводяться до вигляду, який допускає ефективне дослідження. Доведено теореми, які дозволяють для будь-якого рекурентно-параболічного простору $(V_n, g_{ij}, F_i^h)$ або знайти всі простори $(\overline{V}_n, \overline{g}_{ij}, F_i^h)$, на які $V_n$ допускає канонічне квазі-геодезичне відображення, або довести, що таких просторів немає. Також розглянуто канонічне квазі-геодезичне відображення рекурентно-параболічного простору $\left(V_n, g_{ij}, F_i^h\right)$ на напівсиметричний простір $\left(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h\right)$.

## 1. Introduction

We have studied diffeomorphisms of pseudo-Riemannian spaces that belong to the intersection of classes of quasi-geodesic mappings [5–7, 9, 10, 12] with the reciprocity condition and almost-geodesic mappings of the second type [2–4, 11, 13–15]. We have obtained the basic equations of such a mapping $f\colon (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$ in the common coordinate system $(x^i)$ with respect to the mapping $f$ [9]:

$$\overline{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_{(i}(x)\delta_{j)}^h + \phi_{(i}(x)F_{j)}^h(x), \tag{1.1}$$

$$F_i^h(x) = \overline{F}_i^h(x),$$

$$g_{i\alpha}F_j^\alpha = -g_{j\alpha}F_i^\alpha, \qquad \overline{g}_{i\alpha}F_j^\alpha = -\overline{g}_{j\alpha}F_i^\alpha,$$

$$F_{(i,j)}^h = q_{(i}F_{j)}^h, \tag{1.2}$$

$$F_\alpha^h F_i^\alpha = e\delta_i^h, \qquad e = 0, \pm 1, \tag{1.3}$$

$$i, h, j, \ldots = 1, 2, \ldots, n,$$

where $\Gamma_{ij}^h$, $\overline{\Gamma}_{ij}^h$ are the Christoffel symbols of $V_n$, $\overline{V}_n$, respectively, $\psi_i(x)$, $\phi_i(x)$, $q_i(x)$, $p_i(x)$ are certain covectors, $F_i^h(x)$ is affinor, brackets $(i, j)$ denote the symmetrization with respect to the corresponding indices, and comma «,» is a sign of the covariant derivative in respect to the connection of $V_n$.

If in (1.1) $\phi_i = 0$ and $\psi_i = 0$, then the quasi-geodesic mapping degenerates into an affine mapping, and for $\phi_i = 0$ and $\psi_i \neq 0$, into a geodesic one. For $\phi_i \neq 0$ and $\psi_i = 0$, the quasi-geodesic mapping is called *canonical*.

An affinor structure that satisfies condition 1.3 is called:

(1) *elliptic* if $e = -1$,

(2) *hyperbolic* if $e = +1$,

(3) *m-parabolic* when $e = 0$, $\operatorname{rank} F = m$, $2m < n$,

(4) *parabolic* when $e = 0$, $\operatorname{rank} F = m$, $2m = n$.

1.1. If in conditions (1.2) $q_i = 0$, the affinor $F_i^h$ defines a $K$-structure [1].

In [5, 14] were studied quasi-geodesic mappings of Riemannian spaces $(V_n, g_{ij}, F_i^h)$ with a $K$-structure of elliptic and hyperbolic types (i.e., this $K$-structure satisfies (1.3) for $e = \pm 1$).

In [10], a recurrent-parabolic structure was introduced, which is determined by the conditions:

$$F_\alpha^h F_i^\alpha = 0, \qquad g_{i\alpha} F_j^\alpha = -g_{j\alpha} F_i^\alpha, \qquad F_{i,j}^h = q_j F_i^h. \tag{1.4}$$

The article [8] is devoted to some issues that concern canonical quasi-geodesic mappings of recurrent-parabolic spaces.

1.2. We call an affinor structure $F_i^h$ that satisfies conditions (1.2) a *generalized-recurrent structure* (of elliptic, hyperbolic, or parabolic type) [9].

It is obvious that the $K$-structure and the recurrent-parabolic structure are the special cases of generalized-recurrent structure.

In [9] the properties of a generalized-recurrent structure of parabolic type were studied. We call the vector $q_i$ in (1.2) the *generalized recurrence vector* of the structure $F_i^h$, and in the case $F_{i,j}^h = q_j F_i^h$, the *recurrence vector*. Note that under the condition that the vector $q_i$ is gradient, the affinor $\tilde{F}_i^h = e^{-q} F_i^h$, where $q_i = \frac{\partial q(x)}{\partial x^i}$, defines a $K$-structure in the generalized-recurrent space $(V_n, g_{ij}, F_i^h)$, and a Kählerian structure in the recurrent-parabolic space.

1.3. Further, let us define an operation of contraction with an affinor, which is called *conjugation* with respect to the corresponding index and is denoted as follows:

$$T^{\cdots}_{j_1 \cdots j_{k-1} \alpha j_{k+1} \cdots j_r} F_i^\alpha = T^{\cdots}_{j_1 \cdots j_{k-1} \bar{i} j_{k+1} \cdots j_r},$$
$$T^{j_1 \cdots j_{k-1} \alpha j_{k+1} \cdots j_r}_{\cdots} F_\alpha^h = T^{j_1 \cdots j_{k-1} \bar{h} j_{k+1} \cdots j_r}_{\cdots}.$$

1.4. The integrable parabolic structure $F_i^h$ in some neighborhood of the point $V_n$ can be reduced to the form

$$(F_i^h) = \begin{pmatrix} 0 & 0 \\ I_m & 0 \end{pmatrix},$$

where $I_m$ is the identity matrix of order $m = \frac{n}{2}$.

Further, the auxiliary tensor $A_i^h$, will be useful to us, which is determined in the adapted coordinate system by the matrix

$$(A_i^h) = \begin{pmatrix} 0 & I_m \\ 0 & 0 \end{pmatrix}.$$

It is easy to check that

$$F_\alpha^\beta A_\beta^\alpha = m, \qquad A_\alpha^h A_i^\alpha = 0, \qquad F_\alpha^h A_i^\alpha + A_\alpha^h F_i^\alpha = \delta_i^h. \tag{1.5}$$

1.5. The integrability conditions for the equations $F_{i,j}^h = q_j F_i^h$ give

$$F_{i,[jk]}^h = q_{[j,k]} F_i^h,$$

i.e. based on the Ricci identity

$$R_{ijk}^{\overline{h}} - R_{\overline{i}jk}^{h} = F_i^h q_{[j,k]}, \tag{1.6}$$

and in particular

$$R_{\overline{i}jk}^{\overline{h}} = 0.$$

Here $R_{ijk}^h$ are components of the Riemann tensor of the space $V_n$ and square brackets denote the operation of alternation with respect to the corresponding indices.

Let us conjugate (1.6) in the index $k$ and contract it with $A_h^i$ with respect to the indices $h$, $i$. Taking into account (1.5),we obtain

$$R_{\overline{k}j} = \frac{n}{4}\, q_{\overline{k}j},$$

where $q_{jk} = q_{[j,k]}$.

According to the last relation and (1.4), contraction (1.6) with respect to the indices $h$, $k$, and symmetrization with respect to $i$, $j$, gives us

$$R_{j\overline{i}} = -R_{i\overline{j}}$$

for $n \neq 4$. Here $R_{ij}$ are the components of the Ricci tensor of the space $V_n$.

1.6. In [9] we proved that the integrable affinor structure of the generalized-recurrent space $(V_n, g_{ij}, F_i^h)$ is characterized by the following properties:

$$F_{i,\alpha}^\alpha = 0,$$
$$F_{\overline{j},i}^h = F_{j,\overline{i}}^h = F_{j,i}^{\overline{h}} = 0, \qquad q_{\overline{i}} = 0. \tag{1.7}$$

Note that, in contrast to hyperbolic and elliptic types, an integrable generalized-recurrent structure of parabolic type (in particular, a parabolic $K$-structure) need not be Kählerian, i.e., relations (1.7) do not imply that the affinor $F_i^h$ is covariantly constant.

Further in this paper, we consider only the integrable affinor structure.

1.7. In [8] it is proved that the image of a recurrent-parabolic space under a canonical quasi-geodesic mapping is also a recurrent-parabolic space with the same recurrence vector $q_i$, i.e.

$$F^h_{i|j} = q_j F^h_i, \tag{1.8}$$

where «|» is the sign of the covariant derivative with respect to the connection of the space $\overline{V_n}$, and that the vector $\phi_i$ in equations (1.1) satisfies the condition

$$\phi_{\bar{i}} = 0. \tag{1.9}$$

In this paper, we consider the canonical quasi-geodesic mapping ($QGM$) of recurrent-parabolic spaces with an integrable affinor structure.

The investigation is carried out in tensor form, locally, in the class of real sufficiently smooth functions.

## 2. A new form of the basic equations of the canonical $QGM$ of recurrent-parabolic spaces

2.1. A recurrent-parabolic space $(V_n, g_{ij}, F^h_i)$ admits a canonical $QGM$ onto the space $f\colon (V_n g_{ij}, F^h_i) \to (\overline{V}_n, \overline{g}_{ij}, \overline{F}^h_i)$ if and only if in the coordinate system $(x^i)$ the basic equations of this mapping are satisfied:

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \phi_i(x) F^h_j + \phi_j(x) F^h_i, \tag{2.1}$$

$$F^h_i(x) = \overline{F}^h_i(x),$$

$$\phi_{\bar{i}} = 0, \tag{2.2}$$

$$F^h_\alpha F^\alpha_i = 0, \tag{2.3}$$

$$g_{i\alpha} F^\alpha_j = -g_{j\alpha} F^\alpha_i, \qquad \overline{g}_{i\alpha} F^\alpha_j = -\overline{g}_{j\alpha} F^\alpha_i, \tag{2.4}$$

$$F^h_{i,j} = F^h_{i|j} = q_j F^h_i, \tag{2.5}$$

$$i, h, j, \ldots = 1, 2, \ldots, n.$$

In other words, the mapping $f\colon (V_n, g_{ij}, F^h_i) \to (\overline{V}_n, \overline{g}_{ij}, \overline{F}^h_i)$ is a canonical $QGM$ if and only if under conditions (2.2)–(2.5) in the space $(V_n, g_{ij}, F^h_i)$ the system of nonlinear differential equations in partial derivatives of the first order (2.1) with respect to the components of the tensor $\overline{g}_{ij}(x)$ and the vector $\phi_i \neq 0$ has a solution.

The modern theory of differential equations does not provide regular methods for studying the conditions for the existence and uniqueness of solutions to such a system. Using methods that developed in the theory of geodesic mappings of Riemannian spaces [13], we reduce the basic equations (2.1)–(2.5) to a form that allows an effective study.

The following holds:

**Theorem 2.1.1.** *A recurrent-parabolic space* $(V_n, g_{ij}, F_i^h)$ *admits a non-trivial canonical* $QGM$ *if and only if it contains a nonsingular symmetric tensor* $a_{ij}$ *of type* $(0,2)$ *satisfying the equations*

$$a_{ij,k} = \lambda_i F_{jk} + \lambda_j F_{ik},$$
$$\lambda_{\bar{i}} = 0$$

*and*

$$a_{i\alpha} F_j^\alpha = -a_{j\alpha} F_i^\alpha, \qquad \det \|a_{ij}\| \neq 0,$$

*for some covector* $\lambda_i \not\equiv 0$.

**Proof.** Since $\overline{g}_{ij|k} = 0$ in $\overline{V}_n$, equation (2.1) can be written in an equivalent form:

$$\begin{aligned} \overline{g}_{ij,k} &= \phi_i \overline{F}_{jk} + \phi_j \overline{F}_{ik}, \\ \overline{F}_{ik} &= \overline{g}_{i\alpha} F_k^\alpha. \end{aligned} \tag{2.6}$$

Let us introduce the nondegenerate tensor

$$a_{ij} = \overline{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}. \tag{2.7}$$

Since $\overline{g}_{i\alpha}\overline{g}^{\alpha h} = \delta_i^h$, we have $\overline{g}_{i\alpha,k}\overline{g}^{\alpha h} = -\overline{g}_{i\alpha}\overline{g}^{\alpha h}_{,k}$. Therefore, from (2.7) and (2.6) it follows:

$$a_{ij,k} = -\overline{g}_{\gamma\delta,k}\overline{g}^{\gamma\alpha}\overline{g}^{\delta\beta} g_{\alpha i} g_{\beta j} = -\left(\phi_\gamma \overline{F}_{\delta k} + \phi_\delta \overline{F}_{\gamma k}\right)\overline{g}^{\gamma\alpha}\overline{g}^{\delta\beta} g_{\alpha i} g_{\beta j}.$$

In this way

$$a_{ij,k} = \lambda_i F_{jk} + \lambda_j F_{ik}, \tag{2.8}$$

where

$$\lambda_i = -\phi_\gamma \overline{g}^{\gamma\alpha} g_{\alpha i}. \tag{2.9}$$

It is easy to check that in view of (2.2) and (2.4)

$$\lambda_{\bar{i}} = 0 \tag{2.10}$$

and

$$a_{i\alpha} F_j^\alpha = -a_{j\alpha} F_i^\alpha, \qquad \det \|a_{ij}\| \neq 0. \tag{2.11}$$

Thus, if a pseudo-Riemannian space $(V_n, g_{ij}, F_i^h)$ with a recurrent-parabolic structure $F_i^h$ admits a non-trivial canonical $QGM$ on $(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$, then it necessarily contains a nonsingular symmetric tensor $a_{ij}$ that satisfies (2.8), (2.10), (2.11) for some nonzero vector $\lambda_i$.

The converse is also true. Indeed, if $a_{ij}$ and $\lambda_i$ satisfy (2.8), (2.10), (2.11), then (2.2), (2.4), (2.6) hold for $\overline{g}_{ij} = a^{\alpha\beta} g_{\alpha i} g_{\beta j}$ and $\phi_i = -\lambda_\alpha g^{\alpha\beta}\overline{g}_{\beta i}$. ☐

Note that Theorem 2.1.1 was formulated in [8] without proof. We considered it appropriate to present its proof here.

The equation (2.8) is a new form of the basic equations of the theory of canonical $QGM$s of recurrent-parabolic spaces.

## 3. Fundamental theorems of canonical $QGM$s of recurrent-parabolic spaces

3.1. Let a recurrent-parabolic space $(V_n, g_{ij}, F_i^h)$ be given, i.e., its metric tensor $g_{ij}(x)$ and affinor $F_i^h(x)$ satisfying conditions (2.3)–(2.5) are known. The question of the existence of a canonical $QGM$ of the space $(V_n, g_{ij}, F_i^h)$ is reduced to the study of differential equations (2.8) with respect to the tensor $a_{ij}$ and the vector $\lambda_i$, which satisfy conditions (2.10) and (2.11).

To study this question, we consider the integrability conditions for the equations (2.8), which, taking into account the Ricci identity and (2.5), have the form:

$$a_{\alpha(i}R^{\alpha}_{j)kl} = F_{l(i}L_{j)k} - F_{k(i}L_{j)l}, \tag{3.1}$$

where

$$L_{il} = \lambda_{i,l} + \lambda_i q_l. \tag{3.2}$$

Note that from (3.2), in view of (2.5), (2.10), it follows:

$$L_{\bar{i}l} = 0. \tag{3.3}$$

Let's introduce the tensor

$$A^{ij} = A^i_{\alpha} g^{\alpha j}.$$

We contract (3.1) with $A^{kj}$ with respect to the indices $k$, $j$ and conjugate with respect to the index $i$. Taking into account (3.3), we obtain

$$L_{i\bar{l}} = \tfrac{2}{n+2}\, a_{\alpha(i}R^{\alpha}_{\gamma)\beta\bar{l}}A^{\beta\gamma}. \tag{3.4}$$

Contraction (3.1) with $A^{kj}$ over indices $k, j$ gives us

$$a_{\alpha(i}R^{\alpha}_{\gamma)\beta l}A^{\beta\gamma} = \tfrac{n}{2}L_{il} + L_{i\nu}A^{\overline{\nu}}_{l} - \mu F_{il}.$$

Then based on (3.4):

$$\tfrac{n}{2}L_{il} = a_{\alpha(i}\tilde{R}^{\alpha\gamma}_{\gamma)l} + \mu F_{il}, \tag{3.5}$$

where

$$\tilde{R}^{hk}_{il} = R^h_{i\beta\nu}A^{\beta k}\big(\delta^{\nu}_{l} - \tfrac{2}{n+2}A^{\overline{\nu}}_{l}\big),$$
$$\mu = L_{\alpha\beta}A^{\beta\alpha}.$$

From (3.5) we find

$$\tfrac{n}{2}\lambda_{i,l} = -\tfrac{n}{2}\lambda_i q_l + \mu F_{il} + a_{\alpha(i}\tilde{R}^{\alpha\gamma}_{\gamma)l}. \tag{3.6}$$

3.2. Integrability conditions for (3.6), taking into account (2.8), have the form:

$$\begin{aligned}\tfrac{n}{2}\lambda_\alpha R^\alpha_{ilk} &= \tfrac{n}{2}(\lambda_{i,[l}q_{k]} + \lambda_i q_{[k,l]}) + \mu_{,[l}F_{k]i} + \mu q_{[l}F_{k]i}\\ &+ (\lambda_\alpha F_{ik} + \lambda_i F_{\alpha k})\tilde{R}^{\alpha\gamma}_{\gamma l} + (\lambda_\alpha F_{\gamma k} + \lambda_\gamma F_{\alpha k})\tilde{R}^{\alpha\gamma}_{il}\\ &- (\lambda_\alpha F_{il} + \lambda_i F_{\alpha l})\tilde{R}^{\alpha\gamma}_{\gamma k} - (\lambda_\alpha F_{\gamma l} + \lambda_\gamma F_{\alpha l})\tilde{R}^{\alpha\gamma}_{ik} + a_{\alpha(i}\tilde{R}^{\alpha\gamma}_{\gamma)[l,k]}\end{aligned}$$

or in view of (3.6)

$$(\mu_{,[l} + 2\mu q_{[l})F_{k]i} + \lambda_\alpha \tilde{T}^\alpha_{ikl} + a_{\alpha\beta}\tilde{T}^{\alpha\beta}_{ikl} = 0, \tag{3.7}$$

where

$$\begin{aligned}\tilde{T}^\alpha_{ikl} &= \tfrac{n}{2}\delta^\alpha_i q_{[k,l]} - \tfrac{n}{2}R^\alpha_{ikl} + \tilde{R}^{\alpha\gamma}_{\gamma[k}F_{l]i} + \tilde{R}^{\alpha\gamma}_{i[k}F_{l]\gamma} + \delta^\alpha_{(i}\tilde{R}^{\beta\gamma}_{\gamma)[k}F_{l]\beta},\\ \tilde{T}^{\alpha\beta}_{ikl} &= \delta^\beta_{(i}\big(\tilde{R}^{\alpha\gamma}_{\gamma)[l}q_{k]} + \tilde{R}^{\alpha\gamma}_{\gamma)[l,k]}\big).\end{aligned}$$

Comparing the result of cycling (3.7) by $i$, $k$, $l$ with the original relations, we get:

$$2\Big(\mu_{,i} + 2\mu q_i\Big)F_{lk} + \lambda_\alpha\Big(\tilde{T}^\alpha_{(ikl)} - 2\tilde{T}^\alpha_{ikl}\Big) + a_{\alpha\beta}\Big(\tilde{T}^{\alpha\beta}_{(ikl)} - 2\tilde{T}^{\alpha\beta}_{ikl}\Big) = 0.$$

From here, after contraction with $A^{kl}$ with respect to the indices $l$, $k$, we find

$$\mu_{,i} = a_{\alpha\beta}T^{\alpha\beta}_i + \lambda_\alpha T^\alpha_i - 2\mu q_i, \tag{3.8}$$

where

$$\begin{aligned}T^{\alpha\beta}_i &= \Big(\tilde{T}^{\alpha\beta}_{(i\sigma\gamma)} - 2\tilde{T}^{\alpha\beta}_{i\sigma\gamma}\Big)A^{\sigma\gamma},\\ T^\alpha_i &= \Big(\tilde{T}^\alpha_{(i\sigma\gamma)} - 2\tilde{T}^\alpha_{i\sigma\gamma}\Big)A^{\sigma\gamma}.\end{aligned}$$

Relations (2.8), (3.6) and (3.8) form a closed system of the first order partial differential equations of Cauchy type with respect to the unknown functions $a_{ij}$, $\lambda_i$, $\mu$. Let us denote it by $(B)$. In the theory of differential equations regular methods have been developed for such systems. Thus, we proved

**Theorem 3.2.1.** *A pseudo-Riemannian space $(V_n, g_{ij}(x), F^h_i(x))$ with an integrable recurrent-parabolic structure $F^h_i(x)$ admits a canonical QGM if and only if the system of differential equations $(B)$ has a non-trivial solution*

$$a_{ij}(x), \qquad \lambda_i(x) \not\equiv 0, \qquad \mu(x),$$

*satisfying the conditions*

$$a_{ij}(x) = a_{ji}(x), \quad \det\|a_{ij}(x)\| \neq 0, \quad a_{i\alpha}F^\alpha_j = -a_{j\alpha}F^\alpha_i, \quad \lambda_{\bar{i}}(x) = 0.$$

3.3. As is known from the theory of differential equations, system $(B)$ has at most one solution for each set of initial Cauchy values

$$a_{ij}(x_0) = \overset{\circ}{a}_{ij}, \qquad \lambda_i(x_0) = \overset{\circ}{\lambda}_i, \qquad \mu(x_0) = \overset{\circ}{\mu}.$$

However, this system is consistent if and only if the set of integrability conditions $(B)$ and their differential prolongations are consistent.

The integrability conditions for the first group of equations $(B)$, taking into account (3.1), (3.2), (3.5), can be represented in the form

$$a_{\alpha\beta} S^{\alpha\beta}_{ijkl} = 0, \tag{3.9}$$

where

$$S^{\alpha\beta}_{ijkl} = \delta^{\alpha}_{(i} R^{\beta}_{j)kl} + \tfrac{n}{2}\left(\delta^{\beta}_{i} \tilde{R}^{\alpha\gamma}_{\gamma[l} + \tilde{R}^{\alpha\beta}_{i[l}\right) F_{k]j} + \tfrac{n}{2}\left(\delta^{\beta}_{j} \tilde{R}^{\alpha\gamma}_{\gamma[l} + \tilde{R}^{\alpha\beta}_{j[l}\right) F_{k]i}.$$

The integrability conditions for the second group of equations $(B)$, taking into account (3.7), (3.8), have the form

$$a_{\alpha\beta} P^{\alpha\beta}_{ikl} + \lambda_{\alpha} P^{\alpha}_{ikl} = 0, \tag{3.10}$$

where

$$P^{\alpha}_{ikl} = \tilde{T}^{\alpha}_{ikl} + \left(\tilde{T}^{\alpha}_{(l\sigma\gamma)} - 2\tilde{T}^{\alpha}_{l\sigma\gamma}\right) A^{\sigma\gamma} F_{ki} - \left(\tilde{T}^{\alpha}_{(k\sigma\gamma)} - 2\tilde{T}^{\alpha}_{k\sigma\gamma}\right) A^{\sigma\gamma} F_{li},$$

$$P^{\alpha\beta}_{ikl} = \tilde{T}^{\alpha\beta}_{ikl} + \left(\tilde{T}^{\alpha\beta}_{(l\sigma\gamma)} - 2\tilde{T}^{\alpha\beta}_{l\sigma\gamma}\right) A^{\sigma\gamma} F_{ki} - \left(\tilde{T}^{\alpha\beta}_{(k\sigma\gamma)} - 2\tilde{T}^{\alpha\beta}_{k\sigma\gamma}\right) A^{\sigma\gamma} F_{li}.$$

Finally, the integrability conditions for the third group of equations $(B)$, taking into account (2.5), (3.6), (3.8), can be represented in the form:

$$a_{\alpha\beta} Q^{\alpha\beta}_{lk} + \lambda_{\alpha} Q^{\alpha}_{lk} + \mu Q_{lk} = 0, \tag{3.11}$$

$$\begin{aligned} Q^{\alpha\beta}_{lk} &= T^{\alpha\beta}_{[l,k]} + 2T^{\alpha\beta}_{[l} q_{k]} + \frac{2}{n} \delta^{\alpha}_{(\sigma} \tilde{R}^{\beta\gamma}_{\gamma[k} T^{\sigma}_{l]}, \\ Q^{\alpha}_{lk} &= T^{\alpha}_{[l,k]} + F_{\beta[k} T^{\alpha\beta}_{l]} - T^{\alpha}_{[k} q_{l]}, \\ Q_{lk} &= 2\left(q_{[k,l]} + \tfrac{1}{n} F_{\alpha[k} T^{\alpha}_{l]}\right). \end{aligned}$$

Denote the integrability conditions for the system $(B)$, (3.9)–(3.11) by $(B_0)$, and their differential prolongations by $(B_1)$, $(B_2)$, $(B_3), \ldots$ As we see, $(B_0)$, $(B_1)$, $(B_2)$, $(B_3), \ldots$ is a system of linear homogeneous algebraic equations for $a_{ij}(x)$, $\lambda_i(x) \not\equiv 0$, $\mu(x)$ with coefficients from $V_n$. Since the number of unknown functions is finite, there is a natural number $s$ such that $(B_s)$ and subsequent continuations will be consequences of $(B_0)$, $(B_1)$, $(B_2)$, $(B_3), \ldots, (B_{s-1})$. In accordance with the analytical theory of differential equations, the system $(B)$ has a non-trivial solution in the neighborhood of the point $M_0$ if and only if the system of equations $(B_0)$, $(B_1)$, $(B_2)$, $(B_3), \ldots, (B_{s-1})$ has a non-trivial solution at this point.

Hence, we have the following theorem.

**Theorem 3.3.1.** *A pseudo-Riemannian space $(V_n, g_{ij}(x), F_i^h(x))$ with an integrable recurrent-parabolic structure admits a canonical QGM if and only if the system of homogeneous algebraic equations $(B_0)$, $(B_1)$, $(B_2)$, $(B_3), \ldots, (B_{s-1})$ has a non-trivial solution in $(V_n, g_{ij}, F_i^h)$*

$$a_{ij}(x), \qquad \lambda_i(x) \not\equiv 0, \qquad \mu(x),$$

*satisfying the conditions*

$$a_{ij}(x) = a_{ji}(x), \quad \det \|a_{ij}(x)\| \neq 0, \quad a_{i\alpha}F_j^\alpha = -a_{j\alpha}F_i^\alpha, \quad \lambda_{\bar{i}}(x) = 0.$$

Theorems 3.2.1 and 3.3.1 together supply us with a regular method that enables us to decide effectively whether a recurrent-parabolic space $(V_n, g_{ij}, F_i^h)$ admits non-trivial canonical $QGM$ or not, and in the affirmative case, we are in principle able to find all recurrent-parabolic spaces $\left(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h\right)$ that can serve as images of $V_n$ under the mappings considered. Hence, Theorems 3.2.1 and 3.3.1 turn out to be the fundamental theorems of the theory of canonical $QGM$s.

## 4. Canonical $QGM$ of recurrent-parabolic spaces onto semisymmetric spaces

4.1. Let us assume that the recurrent-parabolic space $(V_n, g_{ij}, F_i^h)$ admits a non-trivial canonical $QGM$ onto semisymmetric space $(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$, that is, $R^h_{ijk,[lm]} = 0$. Based on the Ricci identity, this is equivalent to the following conditions

$$-\overline{R}^\alpha_{ijk}\overline{R}^h_{\alpha lm} + \overline{R}^h_{\alpha jk}\overline{R}^\alpha_{ilm} + \overline{R}^h_{i\alpha k}\overline{R}^\alpha_{jlm} + \overline{R}^h_{ij\alpha}\overline{R}^\alpha_{klm} = 0. \tag{4.1}$$

The dependence between the components of the Riemannian tensors of the recurrent-parabolic spaces $V_n$ and $\overline{V}_n$ under the canonical $QGM$ can be written as

$$\overline{R}^h_{ijk} = R^h_{ijk} + F^h_i\tilde{\phi}_{[kj]} + F^h_k\tilde{\phi}_{ij} - F^h_j\tilde{\phi}_{ik}, \tag{4.2}$$

where

$$\tilde{\phi}_{ij} = \phi_{i,j} + \phi_i q_j.$$

In view of (1.9), (2.2), (2.5) we have

$$\tilde{\phi}_{\bar{i}j} = 0.$$

Recall that in the recurrent-parabolic space $V_n$ for $n \neq 4$:

$$R_{\bar{i}j} = -R_{i\bar{j}}.$$

We substitute (4.2) into (4.1). In the obtained relations, we lower the index $h$ in $V_n$, symmetrize with respect to the indices $h$, $i$, then contract with $g^{kl}$ with respect to $k, l$, and symmetrize the result in $j$, $m$:

$$\begin{aligned} 2\big(&\tilde{\phi}_{j\bar{i}}\tilde{\phi}_{h\overline{m}} + \tilde{\phi}_{m\bar{i}}\tilde{\phi}_{h\bar{j}} + \tilde{\phi}_{m\bar{h}}\tilde{\phi}_{i\bar{j}} + \tilde{\phi}_{j\bar{h}}\tilde{\phi}_{i\overline{m}}\big)\\ &+ \tilde{\phi}_{ij}R_{\overline{h}m} + \tilde{\phi}_{im}R_{\overline{h}j} + \tilde{\phi}_{hj}R_{\bar{i}m} + \tilde{\phi}_{hm}R_{\bar{i}j}\\ &+ F_{hj}\big(\tilde{\phi}_{i\alpha}R^{\alpha}_{m} + \tilde{\phi}_{i\overline{\alpha}}\tilde{\phi}^{\alpha}_{m.} + \tilde{\phi}_{\alpha\beta}g^{\alpha\beta}\tilde{\phi}_{i\overline{m}}\big)\\ &+ F_{hm}\big(\tilde{\phi}_{i\alpha}R^{\alpha}_{j} + \tilde{\phi}_{i\overline{\alpha}}\tilde{\phi}^{\alpha}_{j.} + \tilde{\phi}_{\alpha\beta}g^{\alpha\beta}\tilde{\phi}_{i\bar{j}}\big)\\ &+ F_{ij}\big(\tilde{\phi}_{h\alpha}R^{\alpha}_{m} + \tilde{\phi}_{h\overline{\alpha}}\tilde{\phi}^{\alpha}_{m.} + \tilde{\phi}_{\alpha\beta}g^{\alpha\beta}\tilde{\phi}_{h\overline{m}}\big)\\ &+ F_{im}\big(\tilde{\phi}_{h\alpha}R^{\alpha}_{j} + \tilde{\phi}_{h\overline{\alpha}}\tilde{\phi}^{\alpha}_{j.} + \tilde{\phi}_{\alpha\beta}g^{\alpha\beta}\tilde{\phi}_{h\bar{j}}\big) = 0. \end{aligned} \tag{4.3}$$

Conjugation (4.3) with respect to the index $m$ gives us

$$\tilde{\phi}_{i\overline{m}}R_{\overline{h}j} + \tilde{\phi}_{h\overline{m}}R_{\bar{i}j} + F_{hj}\tilde{\phi}_{i\alpha}R^{\alpha}_{\overline{m}} + F_{ij}\tilde{\phi}_{h\alpha}R^{\alpha}_{\overline{m}} = 0. \tag{4.4}$$

4.2. If $\tilde{\phi}_{i\overline{m}} \neq 0$, then there are vectors $a^i$, $b^j$ such that

$$a^{\alpha}\tilde{\phi}_{\alpha\overline{m}} \neq 0 \qquad \text{and} \qquad a^{\alpha}b^{\beta}\tilde{\phi}_{\alpha\overline{\beta}} = 1.$$

Let's contract (4.4) with $a^i b^m$ in the indices $i$, $m$, and then with $a^h$ in $h$. As a result, we get

$$R_{\overline{h}j} + \xi F_{hj} + a_{\bar{j}}c_h = 0, \tag{4.5}$$

where

$$\xi = a^{\beta}b^{\gamma}\tilde{\phi}_{\beta\alpha}R^{\alpha}_{\overline{\gamma}},$$
$$c_h = b^{\gamma}\tilde{\phi}_{h\alpha}R^{\alpha}_{\overline{\gamma}} - \big(\tilde{\phi}_{h\overline{\beta}}b^{\beta}\big)\big(a^{\delta}b^{\gamma}\tilde{\phi}_{\delta\alpha}R^{\alpha}_{\overline{\gamma}}\big).$$

We symmetrize (4.5) with respect to the indices $h$, $j$

$$a_{\bar{j}}c_h + a_{\overline{h}}c_j = 0. \tag{4.6}$$

If $c_j \neq 0$, then there is a vector $d^h$ such that $d^{\alpha}c_{\alpha} = 1$. Then after contraction (4.6) with $d^h$ and then with $d^j$, we find

$$a_{\bar{j}} + a_{\overline{\alpha}}d^{\alpha}c_j = 0, \qquad a_{\overline{\alpha}}d^{\alpha} = 0.$$

Hence, $a_{\bar{j}} = 0$. Similarly, it can be proved that if $a_{\bar{j}} \neq 0$, then $c_j = 0$.

Now (4.5) takes the form:

$$R_{\overline{h}j} = -\xi F_{hj}.$$

Contraction this equality with $A^{jh}$ in the indices $h$, $j$ gives us $\xi = \frac{R}{n}$ and hence

$$R_{j\overline{h}} = \tfrac{R}{n}F_{jh}.$$

Here $R$ is the scalar curvature of $V_n$.

4.3. If $\tilde{\phi}_{i\bar{j}} = 0$, then from (4.3) we obtain

$$\tilde{\phi}_{i(j}R_{m)\bar{h}} + \tilde{\phi}_{h(j}R_{m)\bar{i}} - \tilde{\phi}_{i\alpha}R^{\alpha}_{(m}F_{j)h} - \tilde{\phi}_{h\alpha}R^{\alpha}_{(m}F_{j)i} = 0. \tag{4.7}$$

Cycling (4.7) by $h$, $j$, $m$ gives us:

$$\begin{aligned} &R_{\bar{i}m}\tilde{\phi}_{(hj)} + R_{\bar{i}h}\tilde{\phi}_{(mj)} + R_{\bar{i}j}\tilde{\phi}_{(hm)} \\ &\qquad + F_{ij}R^{\alpha}_{(h}\tilde{\phi}_{m)\alpha} + F_{ih}R^{\alpha}_{(m}\tilde{\phi}_{j)\alpha} + F_{im}R^{\alpha}_{(j}\tilde{\phi}_{h)\alpha} = 0. \end{aligned} \tag{4.8}$$

Suppose that

$$\tilde{\phi}_{ij} = -\tilde{\phi}_{ji}. \tag{4.9}$$

Then, after contraction (4.7) with $A^{ji}$ in the indices $i$, $j$, we get

$$\tilde{\phi}_{m\alpha}R^{\alpha}_{h} = -\tilde{\phi}_{h\alpha}R^{\alpha}_{m}.$$

In view of the obtained equality, the result of contraction (4.8) with $A^{ji}$ with respect to $i$, $j$ gives us

$$n\tilde{\phi}_{h\alpha}R^{\alpha}_{m} = 2\big(\tilde{\phi}_{\beta\alpha}A^{\beta\alpha}\big)\big(R_{\bar{h}m} + \tfrac{R}{n}F_{hm}\big) + R\tilde{\phi}_{hm}. \tag{4.10}$$

Based on (4.9), (4.10), we represent (4.8) in the form

$$a_{ij}b_{hm} + a_{hm}b_{ij} + a_{hj}b_{im} + a_{im}b_{hj} = 0. \tag{4.11}$$

Here

$$a_{ij} = n\tilde{\phi}_{ij} - \vartheta F_{ij}, \qquad \vartheta = 2\tilde{\phi}_{\alpha\beta}A^{\beta\alpha}, \qquad b_{ij} = nR_{\bar{i}j} + RF_{ij}.$$

From (4.11) it follows that either $a_{ij} = 0$ or $b_{ij} = 0$, i.e for $\tilde{\phi}_{ij} = -\tilde{\phi}_{ji}$ one of the following conditions holds:

$$a)\ \tilde{\phi}_{ij} = \tfrac{\vartheta}{n}F_{ij}; \qquad\qquad b)\ R_{j\bar{i}} = \tfrac{R}{n}F_{ji}.$$

4.4. Assume as before $\tilde{\phi}_{i\bar{j}} = 0$, but $\tilde{\phi}_{ij} \neq -\tilde{\phi}_{ji}$. Let us multiply equality (4.8) by $A^{\beta i}\tilde{\phi}_{(\beta k)}$, sum over the index $i$, and cycle the result in the indices $k$, $j$, $h$, $m$. As a result, we get

$$\begin{aligned} &\tilde{\phi}_{(hj)}c_{km} + \tilde{\phi}_{(hm)}c_{kj} + \tilde{\phi}_{(km)}c_{jh} \\ &\qquad + \tilde{\phi}_{(kj)}c_{hm} + \tilde{\phi}_{(jm)}c_{kh} + \tilde{\phi}_{(kh)}c_{jm} = 0, \end{aligned} \tag{4.12}$$

where

$$c_{km} = R^{\alpha}_{(k}\tilde{\phi}_{m)\alpha} - \tilde{\phi}_{\alpha(m}R^{\alpha}_{k)}.$$

Since $\tilde{\phi}_{(ij)} \neq 0$ holds, then there are vectors $a^i$, $b^i$ such that

$$\tilde{\phi}_{(\alpha i)}b^{\alpha} \neq 0, \qquad\qquad \tilde{\phi}_{(\alpha\beta)}a^{\alpha}b^{\beta} = 1.$$

From (4.12) by the suitable contraction with the vectors $a^h$, $b^j$ we obtain $c_{km} = 0$, i.e.

$$R^{\alpha}_{(k}\tilde{\phi}_{m)\alpha} = \tilde{\phi}_{\alpha(m}R^{\alpha}_{k)}.$$

Contracting (4.8) with $A^{mi}$ and then taking into account the last equality, we obtain

$$R^{\alpha}_{(j}\tilde{\phi}_{h)\alpha} = \tfrac{R}{n}\tilde{\phi}_{(hj)}.$$

Hence, equation (4.8) takes the form

$$b_{im}\tilde{\phi}_{(hj)} + b_{ih}\tilde{\phi}_{(mj)} + b_{ij}\tilde{\phi}_{(hm)} = 0, \tag{4.13}$$

where $b_{ij} = nR_{\bar{i}j} + RF_{ij}$.

Using the vectors $a^i$, $b^i$, it is easy to show that (4.13) implies

$$R_{j\bar{i}} = \tfrac{R}{n}F_{ji}.$$

Sections 4.2–4.4 imply the next result.

**Theorem 4.4.1.** *If a recurrent-parabolic space $(V_n, g_{ij}, F_i^h)$, $n \neq 4$ admits a non-trivial canonical $QGM$ on a semisymmetric space $\overline{V}_n$, then at least one of the following relations holds:*

$$a)\ \tilde{\phi}_{ij} = \tfrac{\vartheta}{n}F_{ij}; \qquad b)\ R_{j\bar{i}} = \tfrac{R}{n}F_{ji}.$$

## 5. Conclusion

The main problem in studying mappings $f\colon (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$ is to find out if the given space $(V_n, g_{ij}, F_i^h)$ admits the specified mapping.

The question of existence of a canonical $QGM$ of the recurrent-parabolic space $(V_n, g_{ij}, F_i^h)$ reduces to the study of differential equations (2.8) with respect to the tensor $a_{ij}$ and the vector $\lambda_i$, which satisfy conditions (2.10) and (2.11). Theorems 3.2.1 and 3.3.1 allow for any recurrent-parabolic space $(V_n, g_{ij}, F_i^h)$ either to find all spaces $(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$ on which $V_n$ admits a canonical quasi-geodesic mapping or prove that there are no such spaces. However, for large $n$, the direct solution of this problem is technically rather complicated.

In addition, many questions concerning the features of canonical $QGM$s cannot be solved using the fundamental Theorems 3.2.1 and 3.3.1 and require the use of other methods. As an example, we present the problem of existence of a canonical quasi-geodesic mapping of a recurrent parabolic space onto a semisymmetric space.

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