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Some remarks on a theorem of Green

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Abstract. The purpose of this paper is to study holomorphic curves f from $\mathbb{C}$ to $\mathbb{C}^3$ avoiding four complex hyperplanes and a real subspace of real dimension four in $\mathbb{C}^3$. We show that the projection of f into the complex projective space $\mathbb{CP}^2$ does not remain constant as in the complex case studied by Green, which indicates that the complex structure of the avoided hyperplanes is a necessary condition in the Green theorem.

Анотація. Метою роботи є дослідження голоморфних кривих f із $\mathbb{C}$ в $\mathbb{C}^3$, які лежать в доповненні до чотирьох комплексних гіперплощин і дійсного підпростору дійсної розмірності чотири в $\mathbb{C}^3$. Доведено, що проєкція f на комплексний проєктивний простір $\mathbb{CP}^2$ не є сталим відображенням, як у дослідженому Гріном комплексному випадку, що вказує на те, що комплексна структура гіперплощин є необхідною умовою в теоремі Гріна.

1. Introduction

In his famous paper, Green [3] (see also [2,6]) proved the following rather striking result: the complement of $2n+1$ complex hyperplanes in general position in the complex projective space $\mathbb{CP}^n$ is Kobayashi hyperbolic. Since $\mathbb{CP}^n$ is compact, according to Bordy [5] (see also [1,4]), this result is equivalent to the fact that any holomorphic curve $f\colon \mathbb{C} \to \mathbb{CP}^n$ whose image lies in the complement of $2n+1$ hyperplanes in general position in $\mathbb{CP}^n$, is constant. We recall that given complex hyperplanes $H_1, \dots, H_m$ in $\mathbb{CP}^n$, then they are said to be in general position if $m \geqslant n+1$ and any $(n+1)$ of these hyperplanes are linearly independent. Since Bloch and Cartan, the hyperbolicity of the complement of a family of projective lines in general position in the complex projective plane has been the subject of various

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Ключові слова: комплексний многовид, проєктивний простір, гіперболічність Кобаяші, голоморфні криві, голоморфні криві в доповненні до гіперплощини,

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studies for many years. Different results were obtained for some special cases, especially the following theorem due to Borel, stated by Cartan in the following form (see [1, 4, 5]):

Theorem 1.1 (Borel Theorem). *Let $(L_i)_{1\leqslant i\leqslant 4}$ be four projective lines in general position in $\mathbb{CP}^2$. We denote by S_k $(1 \leqslant k \leqslant 6)$ the six double points of the configuration $C = L_1 \cup L_2 \cup L_3 \cup L_4$. We call diagonals the three lines Δ_1, Δ_2 and Δ_3 each cutting C in two double points. Their union Δ is called the diagonal divisor. Then every non-constant entire curve $f\colon \mathbb{C} \to \mathbb{CP}^2 \setminus C$ degenerates in Δ (i.e. there exists $i \in \{1, 2, 3\}$ such that $f(\mathbb{C}) \subset \Delta_i$).*

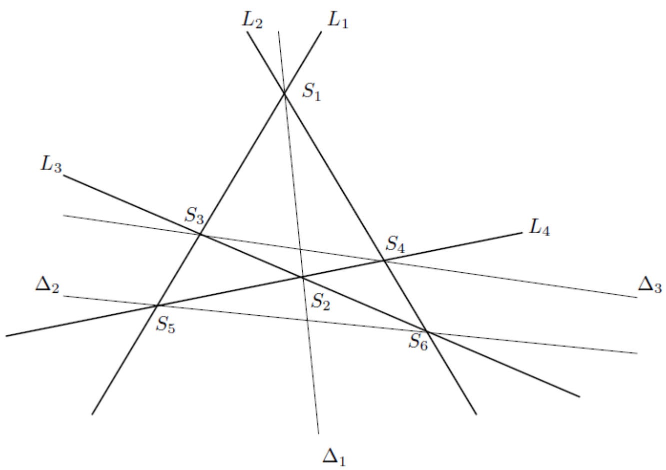

FIGURE 1.1. Configuration $\bigcup_{1\leqslant k\leqslant 4} L_k$ and its diagonals

The purpose of this paper is to give some remarks on the Green theorem [3] (in the case $n = 2$ for the moment). First we adapt the result of Green to show that the projection into the complex projective space $\mathbb{CP}^2$ of any holomorphic curve $f\colon \mathbb{C} \to \mathbb{C}^3$ that avoids five complex lines in general position, is constant. The second most interesting goal is to study the projection into $\mathbb{CP}^2$ of a holomorphic curves $f\colon \mathbb{C} \to \mathbb{C}^3$ avoiding four complex hyperplanes in general position in $\mathbb{C}^3$ and a real subspace H of real dimension *four* and we show that the projection of f does not remain constant as in the complex case studied in the first main results. This proves that the complex structure of the avoided complex lines is a necessary condition in the Green theorem.

2. Main results

Throughout the paper we identify $\mathbb{R}^6$, endowed with its standard complex structure J_{st}, with $\mathbb{C}^3$.

Definition 2.1. *Let $n \geqslant 3$ and $\mathcal{L} = (L_1, \ldots, L_n)$ be a family of real subspaces of $\mathbb{R}^6$ of real codimension 2. Then we say that $\mathcal{L}$ is in general position if for every 3-tuple (i, j, l) of distinct integers $i, j, l \in \{1, \ldots, n\}$,*

$$\mathrm{Span}_{\mathbb{R}}\big(L_i^{\perp}, L_j^{\perp}, L_l^{\perp}\big) = \mathbb{R}^6.$$

We note that if L is a real subspace in $\mathbb{R}^6$, then $L^{\perp}$ denotes the orthogonal complement of L.

Lemma 2.2. *Let $C = \bigcup_{i=1}^{5} L_i$ be a configuration of five complex hyperplanes in $\mathbb{C}^3$, then for all holomorphic curves $g\colon \mathbb{C} \to \mathbb{C}^3 \backslash C$, $\pi \circ g$ is constant (here π denotes the canonical projection from $\mathbb{C}^3 \backslash \{0\}$ into $\mathbb{CP}^2$ and $\pi(g) := \pi \circ g$).*

Proof. Notation: if $Z \in \mathbb{CP}^2$, we denote $[\alpha_1 : \alpha_2 : \alpha_3]$ its homogeneous coordinates, where $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$. For L a real subspace of $\mathbb{R}^6$, we denote by $L^{\star}$ the set $L \backslash \{0\}$.

First we show that if L is a complex hyperplane in $\mathbb{C}^3$ (real subspace of $\mathbb{R}^6$), then $\pi(L)$ is a complex projective hyperplane in $\mathbb{CP}^2$. In fact, we may assume that

$$L = \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 \mid b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 = 0\}$$

with $b_1, b_2, b_3 \in \mathbb{C}$, $b_3 \neq 0$. Hence,

$$\begin{aligned}\pi(L^{\star}) &= \Big\{\big[1 : \alpha_2 : \alpha_3\big] \in \mathbb{CP}^2 \mid b_1 + b_2\alpha_2 + b_3\alpha_3 = 0\Big\} \cup \Big\{\big[0 : 1 : -\tfrac{b_2}{b_3}\big]\Big\} \\ &= \Big\{\Big[1 : z : -\tfrac{b_1 + b_2 z}{b_3}\Big],\ z \in \mathbb{C}\Big\} \cup \Big\{\Big[0 : 1 : -\tfrac{b_2}{b_3}\Big]\Big\}.\end{aligned}$$

We remark that $\big[0 : 1 : -\frac{b_2}{b_3}\big]$ corresponds to $\big[\frac{1}{\infty} : 1 : -\frac{b_1 + b_2\infty}{b_3\infty}\big]$. Thus $\pi(L^{\star})$ is a projective complex line in $\mathbb{CP}^2$.

Now, to finish our proof we show that if $g\colon \mathbb{C} \to \mathbb{C}^3$ is holomorphic and L is a complex hyperplane in $\mathbb{C}^3$, then

$$g(\mathbb{C}) \cap L = \varnothing \quad \text{implies} \quad \pi(g)(\mathbb{C}) \cap \pi(L^{\star}) = \varnothing.$$

Indeed, we first notice that $\pi(g)$ is well-defined since by hypothesis $g(\mathbb{C}) \cap L = \varnothing$, which implies that $g(\mathbb{C}) \subset \mathbb{C}^3 \backslash \{0\}$. Presume now, to get a contradiction, that $\pi(g)(\mathbb{C})$ does not avoid $\pi(L^{\star})$. Then there are two cases.

Case 1. There exists $\alpha \in \mathbb{C}$ and there exists $z \in \mathbb{C}$ such that

$$\pi(g)(\alpha) = \left[1 : z : -\tfrac{b_1+b_2z}{b_3}\right].$$

Then, there exists $c_\alpha \in \mathbb{C}^*$ such that $g(\alpha) = \left(c_\alpha, zc_\alpha, -\frac{b_1+b_2z}{b_3}c_\alpha\right)$. In particular, $b_1g_1(\alpha) + b_2g_2(\alpha) + b_3g_3(\alpha) = 0$, where $g = (g_1, g_2, g_3)$.

Hence, $g(\alpha) \in L$. This is a contradiction.

Case 2. There exists $\alpha \in \mathbb{C}$ such that

$$\pi(g)(\alpha) = \left[0 : 1 : -\tfrac{b_2}{b_3}\right].$$

Then, there exists $c_\alpha \in \mathbb{C}^*$ such that $g(\alpha) = \left(0, c_\alpha, -\frac{b_2}{b_3}c_\alpha\right)$ and

$$b_1g_1(\alpha) + b_2g_2(\alpha) + b_3g_3(\alpha) = 0.$$

It implies that $g(\alpha) \in L$. This is also a contradiction.

Now since g avoids the configuration C of five complex hyperplanes $(L_i)_{1\leqslant i\leqslant 5}$ in $\mathbb{C}^3$, according to what precedes, $\pi(g) = \pi \circ g\colon \mathbb{C} \to \mathbb{CP}^2$ avoids $\pi(L_i)$ for all $1 \leqslant i \leqslant 5$ which are complex projective lines in $\mathbb{CP}^2$. Hence, by Green's theorem $\pi \circ g$ is constant. This proves the lemma. □

Following the previous lemma, it is tempting to study the importance of the complex structure of the avoided lines in $\mathbb{C}^3$. Does it affect the projection into the complex projective space? Does the projection remain constant as in Lemma 2.2?

Theorem 2.3. *Let L_1, L_2, L_3, L_4 be four complex hyperplanes in $\mathbb{C}^3$. Then there exists a real subspace L of $\mathbb{R}^6$, of real dimension four, such that (L, L_i, L_j) are in general position for all $j \neq i$ with $j, i \in \{1, \ldots, 4\}$, and there exists $g\colon \mathbb{C} \to \mathbb{C}^3$ non constant holomorphic curve, such that*

$$g(\mathbb{C}) \cap \Big(\bigcup_{i=1}^{4} L_i \cup L\Big) = \varnothing$$

and $\pi(g)$ is not constant.

Here π denotes the canonical projection from $\mathbb{C}^3\backslash\{0\}$ into $\mathbb{CP}^2$. Notice that $\pi(g)$ is well-defined since $g(\mathbb{C}) \subset \mathbb{C}^3\backslash\{0\}$.

Proof. We denote by $z = (\alpha_1, \alpha_2, \alpha_3)$ the coordinates in $\mathbb{C}^3$, where

$$\alpha_j = x_j + iy_j, \quad j = 1, 2, 3.$$

Hence $(x_1, y_1, x_2, y_2, x_3, y_3)$ denote the coordinates in $\mathbb{R}^6$.

Let L_1, L_2, L_3 and L_4 be four complex hyperplanes in general position in $\mathbb{C}^3$. We know that there is a linear change of coordinate such that L_1,

L_2, L_3 and L_4 are defined in the standard form by:

$$\begin{aligned} L_1 &= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 \mid \alpha_1 = 0\}, \\ L_2 &= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 \mid \alpha_2 = 0\}, \\ L_3 &= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 \mid \alpha_3 = 0\}, \\ L_4 &= \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 \mid \alpha_1 + \alpha_2 + \alpha_3 = 0\}. \end{aligned}$$

Hence, we get

$$\begin{aligned} L_1^{\perp} &= \mathrm{Span}_{\mathbb{R}}\big[(1,0,0,0,0,0);(0,1,0,0,0,0)\big], \\ L_2^{\perp} &= \mathrm{Span}_{\mathbb{R}}\big[(0,0,1,0,0,0);(0,0,0,1,0,0)\big], \\ L_3^{\perp} &= \mathrm{Span}_{\mathbb{R}}\big[(0,0,0,0,1,0);(0,0,0,0,0,1)\big], \\ L_4^{\perp} &= \mathrm{Span}_{\mathbb{R}}\big[(1,0,1,0,1,0);(0,1,0,1,0,1)\big]. \end{aligned}$$

We pose now

$$L = \begin{cases} -x_1 + 2x_2 + x_3 = 0 \\ 3y_1 - y_2 + y_3 = 0 \end{cases}$$

Then $L^{\perp} = \mathrm{Span}_{\mathbb{R}}\big[(-1,0,-2,0,1,0);(0,3,0,-1,0,1)\big]$, which of course satisfies the condition $\mathrm{Span}_{\mathbb{R}}(L^{\perp}, L_j^{\perp}, L_j^{\perp}) = \mathbb{R}^6$ for all $j \neq i$ from $\{1,\dots,4\}$.

Now, by definition of the hyperplanes L_1, L_2 and L_3, a function

$$g \colon \mathbb{C} \to \mathbb{C}^3$$

that avoids $\bigcup_{i=1}^{4} L_i$ admits the following form

$$g = (e^{g_1}, e^{g_2}, e^{g_3}).$$

where $g_i \colon \mathbb{C} \to \mathbb{C}$, $i = 1,2,3$ are non constant holomorphic applications. On the other hand, $h := \pi(g)$ satisfies $h(\mathbb{C}) \subset \mathbb{CP}^2 \setminus \bigcup_{j=1}^{4} \pi(L_j^{\star})$. Hence, h has the following form

$$h = \big[1 : e^{h_2} : e^{h_3}\big],$$

where $h_2 = g_2 - g_1$ and $h_3 = g_3 - g_1$. According to Theorem 1.1, there exists three diagonals $D_{12,34}, D_{13,24}, D_{14,23}$ such that $h = \pi(g(\mathbb{C}))$ is contained in one of these diagonals, where $D_{ij,kl}$ is the diagonal line passing through $\big(\pi(L_i^{\star}) \cap \pi(L_j^{\star})\big)$ and $\big(\pi(L_k^{\star}) \cap \pi(L_l^{\star})\big)$.

We recall that

$$\begin{aligned} \pi(L_i^{\star}) &= \{[\alpha_1 : \alpha_2 : \alpha_3] \in \mathbb{CP}^2 : \alpha_i = 0\} \text{ for } i = 1,2,3, \\ \pi(L_4^{\star}) &= \{[\alpha_1 : \alpha_2 : \alpha_3] \in \mathbb{CP}^2 : \alpha_1 + \alpha_2 + \alpha_3 = 0\}. \end{aligned}$$

Hence, $D_{12,34}$, $D_{13,24}$, $D_{14,23}$ are given by

$$D_{12,34} = \{[\alpha_1 : \alpha_2 : \alpha_3] \in \mathbb{CP}^2 : \alpha_1 + \alpha_2 = 0\},$$
$$D_{13,24} = \{[\alpha_1 : \alpha_2 : \alpha_3] \in \mathbb{CP}^2 : \alpha_2 + \alpha_3 = 0\},$$
$$D_{14,23} = \{[\alpha_1 : \alpha_2 : \alpha_3] \in \mathbb{CP}^2 : \alpha_1 + \alpha_3 = 0\}.$$

Suppose that $h(\mathbb{C})$ is contained in $D_{12,34}$, the cases $h(\mathbb{C}) \subset D_{13,24}$ or $h(\mathbb{C}) \subset D_{14,23}$ are being similar. Thus,

$$e^{h_2} + 1 = 0 \quad \Rightarrow \quad e^{h_2} = -1 \quad \Rightarrow \quad h = \left[1 : -1 : e^{h_3}\right],$$

where $h_3 = g_3 - g_1$. Hence, $g = (e^{g_1}, -e^{g_1}, e^{g_3})$.

On the other hand, $g(\mathbb{C}) \cap L = \varnothing$ means that for every $\alpha \in \mathbb{C}$ at least one of the following inequalities holds:

$$\left[\begin{aligned} &-3\Re(e^{g_1(\alpha)}) + \Re(e^{g_3(\alpha)}) \neq 0, \\ &4\Im(e^{g_1(\alpha)}) + \Im(e^{g_3(\alpha)}) \neq 0. \end{aligned}\right. \tag{2.1}$$

For simplicity, let $x = \Re(e^{g_1(\alpha)})$ and $y = \Im(e^{g_1(\alpha)})$.

Let pose $g_3 = 2g_1$. Then

$$\begin{cases} \Re(e^{g_3(\alpha)}) = 2xy, \\ \Im(e^{g_3(\alpha)}) = x^2 - y^2, \end{cases}$$

and the system (2.1) is equivalent to

$$\left[\begin{aligned} &x(2y-3) \neq 0, \\ &4y + x^2 - y^2 \neq 0. \end{aligned}\right.$$

Suppose that $x(2y-3) = 0$. Then $2y - 3 = 0$, which implies that $y = \frac{3}{2}$. But for $y = \frac{3}{2}$, the second equation takes the following form:

$$4y + x^2 - y^2 = 4 \cdot \tfrac{3}{2} + x^2 - \tfrac{9}{4} = x^2 + \tfrac{15}{4} \neq 0.$$

Hence $g = (e^{g_1}, -e^{g_1}, e^{2g_1})$ avoids L and $h = \pi(g) = [1 : -1 : e^{g_1}]$ is not constant. This concludes the proof of Theorem 2.3. □

Remark 2.4. In the equation $x(2y-3) = 0$, x cannot be zero because it implies that $e^{g_1} = 0$ (conditions of Cauchy–Riemann for holomorphic curves), and g_1 will be constant, which is impossible.

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