Some remarks on a theorem of Green

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Abstract. The purpose of this paper is to study holomorphic curves $f$ from $\mathbb{C}$ to $\mathbb{C}^3$ avoiding four complex hyperplanes and a real subspace of real dimension four in $\mathbb{C}^3$. We show that the projection of $f$ into the complex projective space $\mathbb{CP}^2$ does not remain constant as in the complex case studied by Green, which indicates that the complex structure of the avoided hyperplanes is a necessary condition in the Green theorem.

1. Introduction

In his famous paper, Green [3] (see also [2,6]) proved the following rather striking result: the complement of $2n + 1$ complex hyperplanes in general position in the complex projective space $\mathbb{CP}^n$ is Kobayashi hyperbolic. Since $\mathbb{CP}^n$ is compact, according to Bordy [5] (see also [1,4]), this result is equivalent to the fact that any holomorphic curve $f: \mathbb{C} \to \mathbb{CP}^n$ whose image lies in the complement of $2n + 1$ hyperplanes in general position in $\mathbb{CP}^n$, is constant. We recall that given complex hyperplanes $H_1, \ldots, H_m$ in $\mathbb{CP}^n$, then they are said to be in general position if $m \geq n + 1$ and any $(n + 1)$ of these hyperplanes are linearly independent. Since Bloch and Cartan, the hyperbolicity of the complement of a family of projective lines in general position in the complex projective plane has been the subject of various

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studies for many years. Different results were obtained for some special cases, especially the following theorem due to Borel, stated by Cartan in the following form (see [1,4,5]):

**Theorem 1.1** (Borel Theorem). Let \((L_i)_{1\leq i\leq 4}\) be four projective lines in general position in \(\mathbb{CP}^2\). We denote by \(S_k\) \((1 \leq k \leq 6)\) the six double points of the configuration \(C = L_1 \cup L_2 \cup L_3 \cup L_4\). We call diagonals the three lines \(\Delta_1, \Delta_2\) and \(\Delta_3\) each cutting \(C\) in two double points. Their union \(\Delta\) is called the diagonal divisor. Then every non-constant entire curve \(f: \mathbb{C} \to \mathbb{CP}^2 \setminus C\) degenerates in \(\Delta\) (i.e. there exists \(i \in \{1, 2, 3\}\) such that \(f(\mathbb{C}) \subset \Delta_i\)).

![Figure 1.1. Configuration \( \bigcup_{1\leq k\leq 4} L_k\) and its diagonals](image)

The purpose of this paper is to give some remarks on the Green theorem [3] (in the case \(n = 2\) for the moment). First we adapt the result of Green to show that the projection into the complex projective space \(\mathbb{CP}^2\) of any holomorphic curve \(f: \mathbb{C} \to \mathbb{C}^3\) that avoids five complex lines in general position, is constant. The second most interesting goal is to study the projection into \(\mathbb{CP}^2\) of a holomorphic curves \(f: \mathbb{C} \to \mathbb{C}^3\) avoiding four complex hyperplanes in general position in \(\mathbb{C}^3\) and a real subspace \(H\) of real dimension four and we show that the projection of \(f\) does not remain constant as in the complex case studied in the first main results. This proves that the complex structure of the avoided complex lines is a necessary condition in the Green theorem.
2. MAIN RESULTS

Throughout the paper we identify $\mathbb{R}^6$, endowed with its standard complex structure $J_{st}$, with $\mathbb{C}^3$.

**Definition 2.1.** Let $n \geq 3$ and $\mathcal{L} = (L_1, \ldots, L_n)$ be a family of real subspaces of $\mathbb{R}^6$ of real codimension 2. Then we say that $\mathcal{L}$ is in general position if for every 3-tuple $(i, j, l)$ of distinct integers $i, j, l \in \{1, \ldots, n\}$,

$$\text{Span}_{\mathbb{R}}(L_i^\perp, L_j^\perp, L_l^\perp) = \mathbb{R}^6.$$ 

We note that if $L$ is a real subspace in $\mathbb{R}^6$, then $L^\perp$ denotes the orthogonal complement of $L$.

**Lemma 2.2.** Let $C = \bigcup_{i=1}^{5} L_i$ be a configuration of five complex hyperplanes in $\mathbb{C}^3$, then for all holomorphic curves $g: \mathbb{C} \to \mathbb{C}^3 \setminus C$, $\pi \circ g$ is constant (here $\pi$ denotes the canonical projection from $\mathbb{C}^3 \setminus \{0\}$ into $\mathbb{CP}^2$ and $\pi(g) := \pi \circ g$).

**Proof.** Notation: if $Z \subset \mathbb{CP}^2$, we denote $[\alpha_1: \alpha_2: \alpha_3]$ its homogeneous coordinates, where $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$. For $L$ a real subspace of $\mathbb{R}^6$, we denote by $L^\perp$ the set $L \setminus \{0\}$.

We remark that $[0: 1: -\frac{b_2}{b_3}]$ corresponds to $[\frac{1}{b_2}: 1: -\frac{b_1 + b_2}{b_3}]$. Thus $\pi(L^\ast)$ is a projective complex line in $\mathbb{CP}^2$.

Now, to finish our proof we show that if $L$ is a complex projective hyperplane in $\mathbb{CP}^2$. In fact, we may assume that

$$L = \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 \mid b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 = 0\}$$

with $b_1, b_2, b_3 \in \mathbb{C}$, $b_3 \neq 0$. Hence,

$$\pi(L^\ast) = \left\{[1: \alpha_2: \alpha_3] \in \mathbb{CP}^2 \mid b_1 + b_2\alpha_2 + b_3\alpha_3 = 0\right\} \cup \left\{[0: 1: -\frac{b_2}{b_3}]\right\}$$

$$= \left\{[1: z: -\frac{b_1 + b_2z}{b_3}, z \in \mathbb{C}\right\} \cup \left\{[0: 1: -\frac{b_2}{b_3}]\right\}.$$ 

We remark that $[0: 1: -\frac{b_2}{b_3}]$ corresponds to $[\frac{1}{b_2}: 1: -\frac{b_1 + b_2}{b_3}]$. Thus $\pi(L^\ast)$ is a projective complex line in $\mathbb{CP}^2$.

Now, to finish our proof we show that if $g: \mathbb{C} \to \mathbb{C}^3$ is holomorphic and $L$ is a complex hyperplane in $\mathbb{C}^3$, then

$$g(\mathbb{C}) \cap L = \emptyset \text{ implies } \pi(g)(\mathbb{C}) \cap \pi(L^\ast) = \emptyset.$$ 

Indeed, we first notice that $\pi(g)$ is well-defined since by hypothesis $g(\mathbb{C}) \cap L = \emptyset$, which implies that $g(\mathbb{C}) \subset \mathbb{C}^3 \setminus \{0\}$. Presume now, to get a contradiction, that $\pi(g)(\mathbb{C})$ does not avoid $\pi(L^\ast)$. Then there are two cases.
Case 1. There exists $\alpha \in \mathbb{C}$ and there exists $z \in \mathbb{C}$ such that
$$\pi(g)(\alpha) = \left[1: z: \frac{-b_1 + b_2 + z b_3}{b_3}\right].$$

Then, there exists $c_\alpha \in \mathbb{C}^*$ such that $g(\alpha) = (c_\alpha, z c_\alpha, \frac{-b_1 + b_2 + z b_3}{b_3} c_\alpha)$. In particular, $b_1 g_1(\alpha) + b_2 g_2(\alpha) + b_3 g_3(\alpha) = 0$, where $g = (g_1, g_2, g_3)$. Hence, $g(\alpha) \in L$. This is a contradiction.

Case 2. There exists $\alpha \in \mathbb{C}$ such that
$$\pi(g)(\alpha) = \left[0: 1: \frac{-b_2}{b_3}\right].$$

Then, there exists $c_\alpha \in \mathbb{C}^*$ such that $g(\alpha) = (0, c_\alpha, \frac{-b_2}{b_3} c_\alpha)$ and
$$b_1 g_1(\alpha) + b_2 g_2(\alpha) + b_3 g_3(\alpha) = 0.$$ It implies that $g(\alpha) \in L$. This is also a contradiction.

Now since $g$ avoids the configuration $C$ of five complex hyperplanes $(L_i)_{1 \leq i \leq 5}$ in $\mathbb{C}^3$, according to what precedes, $\pi(g) = \pi \circ g : \mathbb{C} \to \mathbb{CP}^2$ avoids $\pi(L_i)$ for all $1 \leq i \leq 5$ which are complex projective lines in $\mathbb{CP}^2$. Hence, by Green’s theorem $\pi \circ g$ is constant. This proves the lemma. □

Following the previous lemma, it is tempting to study the importance of the complex structure of the avoided lines in $\mathbb{C}^3$. Does it affect the projection into the complex projective space? Does the projection remain constant as in Lemma 2.2?

**Theorem 2.3.** Let $L_1, L_2, L_3, L_4$ be four complex hyperplanes in $\mathbb{C}^3$. Then there exists a real subspace $L$ of $\mathbb{R}^6$, of real dimension four, such that $(L, L_i, L_j)$ are in general position for all $j \neq i$ with $j, i \in \{1, \ldots, 4\}$, and there exists $g : \mathbb{C} \to \mathbb{C}^3$ non constant holomorphic curve, such that
$$g(\mathbb{C}) \cap \left(\bigcup_{i=1}^{4} L_i \cup L\right) = \emptyset$$

and $\pi(g)$ is not constant.

Here $\pi$ denotes the canonical projection from $\mathbb{C}^3 \setminus \{0\}$ into $\mathbb{CP}^2$. Notice that $\pi(g)$ is well-defined since $g(\mathbb{C}) \subset \mathbb{C}^3 \setminus \{0\}$.

**Proof.** We denote by $z = (\alpha_1, \alpha_2, \alpha_3)$ the coordinates in $\mathbb{C}^3$, where
$$\alpha_j = x_j + iy_j, \quad j = 1, 2, 3.$$ Hence $(x_1, y_1, x_2, y_2, x_3, y_3)$ denote the coordinates in $\mathbb{R}^6$.

Let $L_1, L_2, L_3$ and $L_4$ be four complex hyperplanes in general position in $\mathbb{C}^3$. We know that there is a linear change of coordinate such that $L_1$,
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$L_2, L_3$ and $L_4$ are defined in the standard form by:

\[ L_1 = \{ (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 \mid \alpha_1 = 0 \}, \]
\[ L_2 = \{ (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 \mid \alpha_2 = 0 \}, \]
\[ L_3 = \{ (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 \mid \alpha_3 = 0 \}, \]
\[ L_4 = \{ (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 \mid \alpha_1 + \alpha_2 + \alpha_3 = 0 \}. \]

Hence, we get

\[ L_i^\perp = \text{Span}_{\mathbb{R}}[(1, 0, 0, 0, 0, 0); (0, 1, 0, 0, 0, 0)], \]
\[ L_j^\perp = \text{Span}_{\mathbb{R}}[(0, 0, 1, 0, 0, 0); (0, 0, 1, 0, 0, 0)], \]
\[ L_k^\perp = \text{Span}_{\mathbb{R}}[(0, 0, 0, 1, 0, 0); (0, 0, 0, 0, 1, 0)], \]
\[ L_l^\perp = \text{Span}_{\mathbb{R}}[(1, 0, 1, 0, 1, 0); (0, 1, 0, 1, 0, 1)]. \]

We pose now

\[ L = \begin{cases} -x_1 + 2x_2 + x_3 = 0 \\ 3y_1 - y_2 + y_3 = 0 \end{cases} \]

Then $L^\perp = \text{Span}_{\mathbb{R}}[(-1, 0, -2, 0, 1, 0); (0, 3, 0, -1, 0, 1)]$, which of course satisfies the condition $\text{Span}_{\mathbb{R}}(L_i^\perp, L_j^\perp, L_k^\perp) = \mathbb{R}^6$ for all $j \neq i$ from $\{1, \ldots, 4\}$.

Now, by definition of the hyperplanes $L_1, L_2$ and $L_3$, a function

\[ g : \mathbb{C} \to \mathbb{C}^3 \]

that avoids $\bigcup_{i=1}^{4} L_i$ admits the following form

\[ g = (e^{g_1}, e^{g_2}, e^{g_3}). \]

where $g_i : \mathbb{C} \to \mathbb{C}$, $i = 1, 2, 3$ are non constant holomorphic applications.

On the other hand, $h := \pi(g)$ satisfies $h(\mathbb{C}) \subset \mathbb{C}P^2 \setminus \bigcup_{j=1}^{4} \pi(L_j^\ast)$. Hence, $h$ has the following form

\[ h = [1 : e^{h_2} : e^{h_3}], \]

where $h_2 = g_2 - g_1$ and $h_3 = g_3 - g_1$. According to Theorem 1.1, there exists three diagonals $D_{12,34}, D_{13,24}, D_{14,23}$ such that $h = \pi(g(\mathbb{C}))$ is contained in one of these diagonals, where $D_{ij,kl}$ is the diagonal line passing through $(\pi(L_i^\ast) \cap \pi(L_j^\ast))$ and $(\pi(L_k^\ast) \cap \pi(L_l^\ast))$.

We recall that

\[ \pi(L_i^\ast) = \{ [\alpha_1 : \alpha_2 : \alpha_3] \in \mathbb{C}P^2 : \alpha_i = 0 \} \text{ for } i = 1, 2, 3, \]
\[ \pi(L_4^\ast) = \{ [\alpha_1 : \alpha_2 : \alpha_3] \in \mathbb{C}P^2 : \alpha_1 + \alpha_2 + \alpha_3 = 0 \}. \]
Hence, $D_{12,34}, D_{13,24}, D_{14,23}$ are given by

\[D_{12,34} = \{[\alpha_1 : \alpha_2 : \alpha_3] \in \mathbb{CP}^2 : \alpha_1 + \alpha_2 = 0\},\]
\[D_{13,24} = \{[\alpha_1 : \alpha_2 : \alpha_3] \in \mathbb{CP}^2 : \alpha_2 + \alpha_3 = 0\},\]
\[D_{14,23} = \{[\alpha_1 : \alpha_2 : \alpha_3] \in \mathbb{CP}^2 : \alpha_1 + \alpha_3 = 0\}.
\]

Suppose that $h(\mathbb{C})$ is contained in $D_{12,34}$, the cases $h(\mathbb{C}) \subset D_{13,24}$ or $h(\mathbb{C}) \subset D_{14,23}$ are being similar. Thus,

\[e^{h_2} + 1 = 0 \implies e^{h_2} = -1 \implies h = [1 : -1 : e^{h_3}],\]

where $h_3 = g_3 - g_1$. Hence, $g = (e^{g_1}, -e^{g_1}, e^{g_3})$.

On the other hand, $g(\mathbb{C}) \cap L = \emptyset$ means that for every $\alpha \in \mathbb{C}$ at least one of the following inequalities holds:

\[\begin{bmatrix}
-3 \Re(e^{g_1(\alpha)}) + \Re(e^{g_3(\alpha)}) \\
4 \Im(e^{g_1(\alpha)}) + \Im(e^{g_3(\alpha)})
\end{bmatrix} \neq 0, \tag{2.1}
\]

For simplicity, let $x = \Re(e^{g_1(\alpha)})$ and $y = \Im(e^{g_1(\alpha)})$.

Let pose $g_3 = 2g_1$. Then

\[\begin{cases}
\Re(e^{g_3(\alpha)}) = 2xy, \\
\Im(e^{g_3(\alpha)}) = x^2 - y^2,
\end{cases}
\]

and the system (2.1) is equivalent to

\[\begin{align*}
x(2y - 3) &\neq 0, \\
4y + x^2 - y^2 &\neq 0.
\end{align*}
\]

Suppose that $x(2y - 3) = 0$. Then $2y - 3 = 0$, which implies that $y = \frac{3}{2}$. But for $y = \frac{3}{2}$, the second equation takes the following form:

\[4y + x^2 - y^2 = 4 \cdot \frac{3}{2} + x^2 - \frac{9}{4} = x^2 + \frac{15}{4} \neq 0.
\]

Hence $g = (e^{g_1}, -e^{g_1}, e^{2g_1})$ avoids $L$ and $h = \pi(g) = [1 : -1 : e^{g_1}]$ is not constant. This concludes the proof of Theorem 2.3. \hfill \Box

**Remark 2.4.** In the equation $x(2y - 3) = 0$, $x$ cannot be zero because it implies that $e^{g_1} = 0$ (conditions of Cauchy–Riemann for holomorphic curves), and $g_1$ will be constant, which is impossible.

**References**


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