Proceedings of the
International Geometry Center
Vol. 16, no. 3-4 (2023) pp. 192–216

On transversely holomorphic foliations with homogeneous transverse structure

Liliana Jurado, Bruno Scardua

Abstract. In this paper we study transversely holomorphic foliations of complex codimension one with a transversely homogeneous complex transverse structure. We prove that the only cases are the transversely additive, affine and projective cases. We shall focus on the transversely affine case and describe the holonomy of a leaf which is "at the infinity" with respect to this structure and prove this is a solvable group. Using this we are able to prove linearization results for the foliation under the assumption of existence of some hyperbolic map in the holonomy group. Such foliations will then be given by simple-poles closed transversely meromorphic one-forms.

Анотація. В роботі вивчаються трансверсально-голоморфні шарування комплексного ковиміру 1 з трансверсально-однорідною комплексною трансверсальною структурою. Показано, такі шарування розбиваються на такі три класи: трансверсально-адитивні, трансверсально-афінні та трансверсально-проективні. Для трансверсально-афінного випадку описано групи голономій листів, які знаходяться «на нескінченності» відносно цієї структури, і показано, що для цих листів їх групи голономій є розв'язними. В якості наслідку отримано результати лінеаризації для шарування за умови існування деякого гіперболічного відображення в групі голономії. Виявилось, що такі шарування задаються простими полюсами замкнутих трансверсально мероморфних 1-форм.

1. Introduction

Transversely homogeneous foliations have been studied by Blumenthal in his pioneering work [3]. Since then this has proved to be a valuable

---

The authors are grateful to IMCA-Peru for its warm hospitality and support during the preparation of this work. We are deeply indebted with the referee for the careful reading and many suggestions that helped us improving the original manuscript.

2020 Mathematics Subject Classification: Primary 57R30, 58E05, 37F75, 57R30; Secondary 32M25, 32S65..

Keywords: foliation, transversely holomorphic, holonomy, transverse structure..

DOI: http://dx.doi.org/10.15673/pigc.v16i3.2304

notion used to describe a number of cases where a foliation can be well-described. In this paper we study transversely holomorphic foliations of complex codimension one with some hypothesis on the transverse structure.

A real codimension two smooth foliation $\mathcal{F}$ of a differentiable manifold $M^{\ell+2}$ is *transversely holomorphic* of (complex) codimension one if its holonomy pseudogroup is given by biholomorphic maps between open subsets of $\mathbb{C}$ (see [13]). In [5] and [9] one finds the complete classification of transversely holomorphic flows on closed 3-manifolds. This paper deals with the case $\ell \geqslant 2$, i.e., when the leaves of $\mathcal{F}$ have real dimension $\ell \geqslant 2$. Due to the lack of some ingredients that play a fundamental role in the case of flows, as harmonic time parametrizations and classification of compact complex surfaces, we make additional hypothesis on the foliation $\mathcal{F}$. The idea is to classify, at a first moment, the simplest transversely holomorphic foliations of codimension one. From the structural point of view the simplest foliations are those with a homogeneous transverse structure compatible with the transversely holomorphic structure. These will be called $\mathbb{C}$-*transversely homogeneous foliations*. Examples are given by foliations with $\mathbb{C}$-additive, affine or projective transverse structure. We shall prove that these are the only cases (cf. Proposition 3.4). Examples are constructed, and we introduce a notion of *Riccati foliation* compatible with our framework. In these examples the transverse structure is defined in the complement of some compact invariant set $\Lambda \subset M$.

We shall focus on the $\mathbb{C}$-affine case. We then describe the holonomy of a leaf which is at the infinity with respect to this structure, i.e., of a leaf $L \subset \Lambda$. For this we assume that there is a hyperbolic map in the holonomy group and obtain:

Theorem 1.1. *Let $\mathcal{F}$ be a transversely holomorphic foliation of codimension one on M given by a transversely holomorphic integrable* 1*-form Ω. Assume that $\mathcal{F}$ is $\mathbb{C}$-transversely affine on $M \backslash L_o$ for a compact leaf $L_o \subset M$ such that:*

(i) *L_o contains a hyperbolic map in its holonomy group;*

(ii) *the leaf L_o is given by some equation $L_o\colon \{f = 0\}$, where $f\colon M \to \mathbb{C}$ is transversely holomorphic with isolated singularities;*

(iii) *any closed transversely holomorphic one-form in M is exact.*

Then the holonomy group of L_o is abelian. In particular $\mathcal{F}$ is logarithmic near L_o, i.e., given by a closed transversely meromorphic 1*-form ξ with simple poles in a neighborhood of L_o in M.*

We recall that $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ is the group of nonzero complex numbers with usual multiplication.

For our next result we shall recall the notion of codimension one holomorphic foliation with singularities we refer to. Given a complex manifold $\tilde{M}$ by a *codimension one holomorphic foliation with singularities on* $\tilde{M}$ we shall mean a pair $\tilde{\mathcal{F}} = (\mathcal{F}_o, \tilde{\mathcal{S}})$ where $\tilde{\mathcal{S}} \subset \tilde{M}$ is a codimension $\geqslant 2$ analytic subset, $\mathcal{F}_o$ is a codimension one holomorphic foliation (in the usual sense) in the open subset $\tilde{M} \backslash \tilde{\mathcal{S}}$.

We shall refer to $\tilde{\mathcal{S}}$ as the *singular set* of $\tilde{\mathcal{F}}$ and write $sing(\tilde{\mathcal{F}}) = \tilde{\mathcal{S}}$. The *leaves* of the foliation $\tilde{\mathcal{F}}$ are defined as the leaves of the (nonsingular) foliation $\tilde{\mathcal{F}}_o$ on $\tilde{M} \backslash \operatorname{sing}(\tilde{\mathcal{F}})$. The *holonomy group* of a leaf $\tilde{L}_o$ of $\tilde{\mathcal{F}}$ is defined as the holonomy group of the corresponding leaf of $\mathcal{F}_o$.

As an application of our techniques we prove:

Theorem 1.2. *Let $\tilde{\mathcal{F}}$ be a codimension one holomorphic foliation with singularities in a complex Stein manifold $\tilde{M}$ of dimension $n \geqslant 2$. Suppose that $\tilde{\mathcal{F}}$ has an affine transverse structure in $\tilde{M} \backslash \tilde{\Lambda}$ where $\tilde{\Lambda} \subset \tilde{M}$ is an irreducible analytic invariant subset of codimension one. Let $A \subset \tilde{M}$ be a relatively compact open subset with smooth simply-connected boundary $M = \partial A$ transverse to $\tilde{\mathcal{F}}$. Also assume that some leaf $L_o \subset \Lambda := \tilde{\Lambda} \cap M$ of the restriction $\mathcal{F} = \tilde{\mathcal{F}}|_M$ contains a hyperbolic map in its holonomy group. Then the holonomy group of the leaf $\tilde{L}_o = \tilde{\Lambda} \backslash \operatorname{sing}(\tilde{\mathcal{F}})$ of $\tilde{\mathcal{F}}$ is abelian linearizable. In particular, $\tilde{\mathcal{F}}$ is locally logarithmic, i.e., it is given by a closed meromorphic one-form with simple poles, in a neighborhood of $\overline{A} \cup \tilde{L}_o$ in $\tilde{M}$.*

In Theorem 1.2 we are assuming that the affine structure exists in the complement of an invariant irreducible analytic subset of codimension one $\tilde{\Lambda} \subset \tilde{M}$. Notice that the intersection $\Lambda = \tilde{\Lambda} \cap M$ may contain several connected components, i.e., it may consist of several compact leaves of the induced foliation $\mathcal{F} = \tilde{\mathcal{F}}|_M$ on M. Nevertheless, by the irreducibility of $\tilde{\Lambda}$ the complement $\tilde{M} \backslash \operatorname{sing}(\tilde{\mathcal{F}})$ is a leaf of $\tilde{\mathcal{F}}$. So, all these leaves in $\Lambda \subset M$ are related in the sense that a global closed meromorphic one-form will have same residues at each of these leaves.

We may partially reduce our hypotheses by assuming the existence of structure only for the restriction $\mathcal{F}$ in $M \backslash \Lambda$. Nevertheless, the price we must pay is to assume that Λ is connected, i.e., it is a single compact leaf of $\mathcal{F}$. We have the following result:

Theorem 1.3. *Let $\tilde{\mathcal{F}}$ be a codimension one holomorphic foliation with singularities in a complex Stein manifold $\tilde{M}$ of dimension $n \geqslant 2$. Let $A \subset \tilde{M}$ be a relatively compact open subset with smooth simply-connected boundary $M = \partial A$ transverse to $\tilde{\mathcal{F}}$. Suppose that the induced foliation*

$\mathcal{F} = \tilde{\mathcal{F}}|_M$ has a $\mathbb{C}$-affine transverse structure in $M \backslash L_o$ where $L_o \subset M$ is a compact leaf of the form $L_o = \tilde{\Lambda} \cap M$ and $\tilde{\Lambda} \subset \tilde{M}$ is analytic of codimension one and invariant by $\tilde{\mathcal{F}}$. Also assume that the leaf L_o of $\mathcal{F}$ contains a hyperbolic map in its holonomy group. Then $\tilde{\mathcal{F}}$ is logarithmic in a neighborhood of $\overline{A}$ in $\tilde{M}$.

2. FIXED POINTS, NONSOLVABLE AND ABELIAN SUBGROUPS OF Diff($\mathbb{C}$, 0)

We shall denote by $\mathrm{Diff}(\mathbb{C}, 0)$ the group of germs of complex diffeomorphisms f fixing $0 \in \mathbb{C}$, say

$$f(z) = \lambda z + \sum_{j=1}^{\infty} a_j z^j, \quad \lambda \neq 0.$$

Let $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ be a subgroup. Denote by $[G, G]$ the subgroup of G generated by the commutators $[f, g] = f^{-1} \circ g^{-1} \circ f \circ g$ of elements $f, g \in G$. We recall that G is solvable if the sequence

$$G^1 := [G, G], \qquad\qquad G^{n+1} := [G^n, G^n],\ n \geqslant 2,$$

is trivial for $n \in \mathbb{N}$ big enough. According to the well-known literature ([8, 15] for instance) we have:

Theorem 2.1 ([8, 15, 18]). *Let $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ be a finitely generated subgroup. Then the following conditions are equivalent:*

(i) *G is solvable;*
(ii) *the group of commutators $G_1 = [G, G]$ is abelian;*
(iii) *the subgroup $G_1 \subset G$ of elements tangent to the identity (i.e, elements $f \in G$ with $f'(0) = 1$) is abelian;*
(iv) *there is a unique $k \in \mathbb{N}$ such that every map $f \in G_1 \backslash \{\mathrm{Id}\}$ tangent to but not equal to the identity, writes as $f = z + \sum\limits_{j=k+1}^{\infty} a_j z^j$ with $a_{k+1} \neq 0$.*

In other words, all nontrivial elements tangent to the identity have the same order of tangency. If G is solvable and nonabelian then G admits a formal embedding into one group

$$\mathbb{H}_k = \left\{ \varphi(z) = \frac{\lambda z}{(1 + \mu z^k)^{1/k}}, \lambda \in \mathbb{C}^*, \mu \in \mathbb{C} \right\}.$$

This embedding converges except if G is *exceptional* which means that its group of commutators is cyclic. A group containing a hyperbolic map is not exceptional.

Regarding the dynamics of nonsolvable groups in $\mathrm{Diff}(\mathbb{C}, 0)$ we have:

Theorem 2.2 ([2, 16, 20, 23]). *Suppose G is non-solvable.*

(i) *The basin of attraction of (the pseudo-orbits of) G is an open neighborhood of the origin $0 \in \Omega$,* (*cf.* [16, Theorem 1, page 570]).

(ii) *Either G has dense pseudo-orbits in some neighborhood V of the origin or there exists an invariant germ of analytic curve $\tilde{\Lambda}$* (*equivalent to $\Im z^k = 0$ for some $k \in \mathbb{N}$*) *where G has dense pseudo-orbits and also G has dense pseudo-orbits in each component of $V \backslash \tilde{\Lambda}$* (*cf.* [16, Theorem 3 page 571]).

(iii) *There exists a neighborhood $0 \in V \subset \Omega$, where G has a dense set of hyperbolic fixed points* (*cf.* [2, Theorem 1] *or* [23, Theorem 1]).

Thus, according to this result, a subgroup $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ having discrete pseudo-orbits outside the origin or without many fixed points close to the origin must be solvable. Combining this with [15] we obtain:

Proposition 2.3. *If $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ is solvable non-abelian and has discrete set of fixed points off the origin, then G must have discrete pseudo-orbits and is either formally conjugate to some group*

$$G_\nu^2 := \left\langle z \mapsto az,\ z \mapsto z/(1+z^\nu)^{\frac{1}{\nu}} \right\rangle$$

where a^ν has order 2; or it is analytically conjugate to some group

$$G_{\nu,\tau}^2 := \left\langle z \mapsto az,\ z \mapsto z/(1+z^\nu)^{\frac{1}{\nu}},\ z \mapsto z/(1+\tau z^\nu)^{\frac{1}{\nu}} \right\rangle$$

where a^ν has order 2 and $\tau \in \mathbb{C} \backslash \mathbb{R}$; or finally it is analytically conjugate to some group

$$G_\nu^n := \left\langle z \mapsto az,\ z \mapsto z/(1+z^\nu)^{\frac{1}{\nu}} \right\rangle$$

where a^ν has order $n \in \{3, 4, 6\}$.

Finally, we shall make use of the following lemma:

Lemma 2.4 (Linearization of abelian groups)**.** *Let $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ be an abelian subgroup. If G contains some hyperbolic element $f \in G$ such that $|f'(0)| \neq 1$, then G is analytically linearizable, i.e., there exits an analytic diffeomorphism $\varphi \in \mathrm{Diff}(\mathbb{C}, 0)$ conjugating G to a subgroup of the linear group $\mathbb{C}^*$.*

Proof. Given a hyperbolic map $f \in G$ according to Poincaré-Lyapunov linearization theorem we may an analytic coordinate in which f is linear. Let us then assume that $f(z) = \lambda z$. Given a map $g \in \mathrm{Diff}(\mathbb{C}, 0)$ that commutes with f, we write $g(z) = \sum\limits_{j=1}^{\infty} g_j z^j$. From $g \circ f = f \circ g$ we obtain

$$g_j \lambda^j = \lambda g_j, \quad \forall j \in \mathbb{N}.$$

Since $\lambda^j \neq \lambda, \forall j \geqslant 2$, we conclude that $g_j = 0$, $\forall j \geqslant 2$. ☐

3. TRANSVERSELY HOLOMORPHIC FOLIATIONS WITH HOMOGENEOUS TRANSVERSE STRUCTURE

Let $\mathcal{F}$ be a transversely holomorphic foliation of codimension one on $M^{\ell+2}$. Thus, any point $p \in M$ has an open neighborhood $p \in U \subset M$ where we have local coordinates $(x, z) \in \mathbb{R}^\ell \times \mathbb{C}$ in which $\mathcal{F}$ is given by $z = c \in \mathbb{C}$. We can, as in [5], introduce the sheaf $\mathcal{O}(\mathcal{F})$ (respectively $\mathcal{M}(\mathcal{F})$) of *transversely holomorphic*(resp. *meromorphic*) functions on M as given by the functions defined on open subsets of M which are locally constant along the leaves of $\mathcal{F}$ and transversely holomorphic. This means that such a function $f(x, z)$ does not depend on x, and the induced function $\hat{f}(z) = f(x, z)$ is holomorphic.

Similarly, we can introduce the sheaf $\Omega^1(\mathcal{F})$ (respectively, $\Omega^1_m(\mathcal{F})$) of transversely holomorphic/meromorphic one-forms on M.

Definition 3.1. A smooth foliation $\mathcal{F}$ of real codimension two has a *holomorphic homogeneous transverse structure* if there exist a complex Lie group G and a connected closed complex subgroup $H < G$ such that $\mathcal{F}$ admits an atlas of submersions $y_j\colon U_j \subset M \to G/H$ satisfying $y_i = g_{ij} \circ y_j$ for some locally constant map $g_{ij}\colon U_i \cap U_j \to G$ for each $U_i \cap U_j \neq \emptyset$.

In other words, $\mathcal{F}$ has a transversely holomorphic atlas of submersions whose transition maps are given by left translations on G and submersions taking values on the homogeneous space G/H. In particular $\mathcal{F}$ is transversely holomorphic. We shall say that $\mathcal{F}$ is $\mathbb{C}$-transversely homogeneous *of model G/H*.

The foliation is $\mathbb{C}$-*transversely additive* whenever there are maps g_{ij} in the definition of holomorphic homogeneous transverse structure which are of the form $g_{ij}(z) = z + b_{ij}$, $b_{ij} \in \mathbb{C}$ locally constant in $U_i \cap U_j$. If $g_{ij}(z) = a_{ij}\, z + b_{ij}$, for locally constant $a_{ij} \in \mathbb{C}^*$ and $b_{ij} \in \mathbb{C}$ we say that $\mathcal{F}$ is $\mathbb{C}$-*transversely affine* and it is $\mathbb{C}$-*transversely projective* if $g_{ij}(z) = \frac{a_{ij} z + b_{ij}}{c_{ij} z + d_{ij}}$ with locally constant $\left(\begin{smallmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{smallmatrix}\right) \in SL(2, \mathbb{C})$.

Next we give a couple of examples of these structures.

Example 3.2. We will define a $\mathbb{C}$-transversely affine (transversely holomorphic) foliation on a compact manifold. This will be a non-singular foliation with dense leaves which are biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ or cylinders $(\mathbb{C}^*/\mathbb{Z}) \times \mathbb{C}^*$.

We begin with a general construction inspired in [21]. Let M be a compact differentiable manifold of real dimension n, equipped with a nonsingular smooth closed one-form ω. Define a one-form Ω on $M \times \mathbb{C}^*$ by

$$\Omega(x, t) = t\omega(x),\ x \in M, t \in \mathbb{C}^*.$$

Then we have $d\Omega = \eta \wedge \Omega$, where $\eta(x,t)$ is defined by $\eta(x,t) = \frac{dt}{t}$. This shows that $\Omega \wedge d\Omega = 0$, i.e., Ω is integrable. The one-form Ω defines a codimension one transversely holomorphic foliation $\tilde{\mathcal{F}}$ in the product $M \times \mathbb{C}^*$. Since $d\eta = 0$ and η is transversely holomorphic, the foliation $\tilde{\mathcal{F}}$ is $\mathbb{C}$-transversely affine as a consequence of Proposition 5.1.

Assume now that we have a smooth diffeomorphism $f \colon M \to M$ such that $f^*\omega = \lambda\omega$ for some $\lambda \in \mathbb{R}^*$ with $|\lambda| \neq 1$. In this case we consider the action

$$\begin{aligned} \Phi : \mathbb{Z} \times (M \times \mathbb{C}) &\longrightarrow M \times \mathbb{C}^* \\ n \cdot (x,t) &\longmapsto \big(f^n(x), \lambda^{-n} t\big) . \end{aligned}$$

This is a locally free action generated by the transversely holomorphic diffeomorphism

$$\varphi \colon M \times \mathbb{C}^* \to M \times \mathbb{C}^*, \qquad \varphi(x,t) = (f(x), \lambda^{-1}\, t).$$

Notice that the action of φ preserves $\tilde{\mathcal{F}}$ as well as its $\mathbb{C}$-affine transverse structure. Indeed, we have

$$\varphi^*\, \Omega(x,t) = \lambda^{-1}\, t\lambda\, \omega(x) = \Omega(x,t)$$

and $\varphi^*\eta = \eta$. In particular, $\tilde{\mathcal{F}}$ induces a codimension one transversely holomorphic foliation $\mathcal{F}$ on the quotient manifold $V = (M \times \mathbb{C}^*)/\mathbb{Z}$. The foliation $\mathcal{F}$ inherits the $\mathbb{C}$-affine transverse structure induced by the pair (Ω, η). This is a pretty general construction. Let us give a more concrete one.

We consider a variant of the Furness example (see also [19]). Consider the unimodular map

$$U = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2; \quad U(x,y) = (x+y, x+2y).$$

This map induces a biholomorphism $f \colon T^2 \to T^2$, where $T^2 = S^1 \times S^1$. Let $\tilde{\tilde{\omega}} := (1+\sqrt{5})dx - 2dy$ in $\mathbb{R}^2$. Then $U^*\tilde{\tilde{\omega}} = \lambda\tilde{\tilde{\omega}}$, where $\lambda = \frac{3-\sqrt{5}}{2}$ and U is $\mathbb{Z} \times \mathbb{Z}$ invariant ($\mathbb{Z} \times \mathbb{Z}$ acts on $\mathbb{R}^2$ by the natural product action) so that it induces a one-form ω in the torus T^2. The one-form ω satisfies $f^*\omega = \lambda\omega$, and the foliation induced on

$$V = (M \times \mathbb{C}^*)/\mathbb{Z} = (T^2 \times \mathbb{C}^*)/\mathbb{Z}$$

is $\mathbb{C}$-transversely affine.

Example 3.3 (Riccati and Bernoulli foliations)**.** We consider a *Riccati foliation* on $M^3 = S^1 \times S^2$ as follows. Recall that

$$S^1 = \mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}$$

and

$$S^2 = \mathbb{C}P^1 = \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

Given affine coordinates $x \in \mathbb{R} \subset S^1$ and $z \in \mathbb{C} \subset S^2$, consider the differential equation

$$\dot{x} = p(x),\ \dot{z} = a(x)z^2 + b(x)z + c(x)$$

where $p(x), a(x), b(x), c(x) \in \mathbb{R}[x]$ are polynomials. In real coordinates we may write $z = z_1 + iz_2$, where $z_1, z_2 \in \mathbb{R}$ and $i^2 = -1$, and rewrite the above ODE as follows:

$$\begin{aligned}\dot{x} &= p(x),\\ \dot{z}_1 &= a(x)(z_1^2 - z_2^2) + b(x)z_1 + c(x),\\ \dot{z}_2 &= 2a(x)z_1z_2 + b(x)z_2,\end{aligned}$$

which defines a polynomial vector field $\mathcal{Z}$ on $\mathbb{R}\times\mathbb{R}^2 \subset M^3$. This polynomial vector field induces a transversely holomorphic foliation $\mathcal{F}(\mathcal{Z})$ on $M^3\backslash S$ for a finite singular set $S \subset M^3$ given in the affine space by the pairs (x_j, z_j) such that $p(x_j) = 0$ and $a(x_j)z_j^2 + b(x_j)z_j + c(x_j) = 0$.

The foliation $\mathcal{F}(\mathcal{Z})$ has the following property: $\mathcal{F}(\mathcal{Z})$ is transverse to the fibers of the fibration

$$\begin{gathered}S^1 \times S^3 \to S^1,\\ (x, z) \mapsto x\end{gathered}$$

except for a finite number of such fibers (those given in the affine part by $p(x) = 0$ and perhaps the fiber $x = \infty$).

This implies, together with Ehresmann theorem on fibrations [10] that $\mathcal{F}(\mathcal{Z})$ is transverse to the fibers of the fiber bundle $S^1 \times S^2 \to S^1$ except for those mentioned fibers. All these non-transverse fibers are invariant by $\mathcal{F}(\mathcal{Z})$. In particular, if $\Lambda \subset M^3$ denotes the union of these invariant fibers, then the restriction $\mathcal{F}(\mathcal{Z})|_{M^3\backslash\Lambda}$ is conjugate to the suspension of a certain representation

$$\varphi\colon \pi_1(S^1\backslash\sigma) \to \mathrm{Diff}(S^2).$$

Since the foliation is clearly transversely holomorphic, in a way compatible with the holomorphic structure of the fibers of the bundle $M^3 \to S^1$, we conclude that the image $\varphi(\pi_1(S^1\backslash\sigma)) \subset \mathrm{Diff}(S^2)$ consists of holomorphic diffeomorphisms of the Riemann sphere, i.e., we have a suspension of a group of Moebius maps $G \subset SL(2,\mathbb{C})$. This shows that $\mathcal{F}(\mathcal{Z})$ is indeed, transversely projective in $M\backslash\Lambda$.

Now assume that $c(x) = 0$. Then we have a foliation with a $\mathbb{C}$-affine transverse structure out of the set

$$\Lambda \cup \overline{(z = 0)} \subset M.$$

Indeed, a subgroup of $SL(2,\mathbb{C})$ with a common fixed point is affine. More generally, we can consider pull-backs of such Riccati foliations by maps of the form

$$(x,z) \mapsto (x, z^{k+1})$$

and obtain *Bernoulli foliations* which are given by expressions of the form

$$\begin{aligned} \dot{x} &= p(x), \\ \dot{z} &= a(x)z^{k+1} + b(x)z. \end{aligned}$$

These are foliations with a $\mathbb{C}$-affine transverse structure out of a set of the form $\Lambda \cup \overline{(z=0)} \subset M$ as above.

The proof of the next proposition follows the one in [19].

Proposition 3.4. *Let $\mathcal{F}$ be a codimension one transversely holomorphic foliation on M. If $\mathcal{F}$ is holomorphically transversely homogeneous, then $\mathcal{F}$ is holomorphically transversely additive, affine or projective.*

Proof. By the hypotheses the quotient $R = G/H$ is a Riemann surface. We may assume that R is simply-connected (otherwise we consider the universal covering of R and lift the submersions to this space).

By the Riemann-Koebe Uniformization theorem we have a conformal equivalence $R \equiv \overline{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$ the unitary disc. This implies that G is a subgroup of one of the following groups:

$$Aut(\overline{\mathbb{C}}) = \mathbb{P}SL(2,\mathbb{C}), \qquad Aut(\mathbb{C}) = \mathrm{Aff}(\mathbb{C}), \qquad Aut(\mathbb{D}) \cong \mathbb{P}SL(2,\mathbb{R}).$$

The proposition follows. □

4. THE TRANSVERSELY ADDITIVE CASE

In what follows we study the following situation: $\mathcal{F}$ has a $\mathbb{C}$-additive transverse structure on $M \backslash \Lambda$, where $\Lambda \subset M$ is a finite union of compact leaves of $\mathcal{F}$. This means that there is an open cover of $M \backslash \Lambda$ by open sets U_j, where are defined submersions $y_j \colon U_j \to \mathbb{C}$ such that

- $\mathcal{F}|_{U_j}$ is given by $dy_j = 0$,
- and for each intersection $U_i \cap U_j$ we have that $y_i = y_j + a_{ij}$ for some locally constant a_{ij}.

Taking the differential $dy_i = dy_j$ we conclude that there exists a closed transversely holomorphic one-form ω on $M \backslash \Lambda$ such that $\mathcal{F}|_{M\backslash\Lambda}$ is given by $\omega = 0$ and, by construction, $\omega|_{U_j} = dy_j$. Thus, ω is invariant with respect to the holonomy pseudogroup of $\mathcal{F}$ in $M \backslash \Lambda$. Our aim is to study the holonomy group of a leaf $L \in \mathcal{F}$, $L \subset \Lambda$.

Proposition 4.1. *The holonomy group of each leaf $L \subset \Lambda$ is solvable.*

Proof. Given a transverse disc

$$D \subset M, \qquad D \cap \Lambda = D \cap L = \{p\}, \qquad D \pitchfork \mathcal{F},$$

consider the holonomy representation

$$\operatorname{Hol}(\mathcal{F}, L, D, p) \cong G < \operatorname{Diff}(\mathbb{C}, 0)$$

as a subgroup of the group of germs of holomorphic diffeomorphisms fixing the origin $0 \in \mathbb{C}$. Assume by contradiction that $\operatorname{Hol}(\mathcal{F}, L, D, p)$ is not solvable.

By [15, 16] we may find in D a dense subset of points $q \in D$ being hyperbolic fixed points for some elements $f_q \in \operatorname{Hol}(\mathcal{F}, L, D, p)$. Given such a fixed point $f_q(q) = q$, choose a local chart $z_q\colon V \subset D \to \mathbb{C}$, taking q to $0 \in \mathbb{C}$ and such that $f_q(z_q) = \lambda z_q$ with $|\lambda| \neq 1$, i.e., f_q is linear. The leaf $L_q \in \mathcal{F}$ containing q has therefore the homotopy class $\gamma \in \pi_1(L_q, q)$ that originates the holonomy map

$$f_\gamma = f_q|_V \in \operatorname{Hol}(\mathcal{F}, L_q, V, q).$$

As we have seen, there exists a closed transversely holomorphic one-form ω on $M \backslash \Lambda$ such that $\mathcal{F}|_{M\backslash\Lambda}$ is given by $\omega = 0$ and, by construction, ω is invariant by the holonomy pseudogroup of $\mathcal{F}$ in $M\backslash\Lambda$. In particular, we must have that $f_\gamma^*(\omega|_V) = \omega|_V$. Write $\omega|_V = g(z_q)\, dz_q$ to obtain

$$f_\gamma^*(\omega|_V) = \omega|_V \quad \Rightarrow \quad \lambda g(\lambda z_q) = g(z_q) \quad \Rightarrow \quad g(z_q) = \frac{1}{z_q} g_{-1} \tag{4.1}$$

for some constant $g_{-1} \in \mathbb{C}^*$: indeed, write $g(z_q) = \sum\limits_{j=0}^{\infty} g_j\, z_q^j$ in Laurent series, then the middle equation in (4.1) is equivalent to

$$\sum_{j=-\infty}^{\infty} g_j\, \lambda^{j+1} z_q^j = \sum_{j=-\infty}^{\infty} g_j z_q^j \quad \Rightarrow \quad (\lambda^{j+1} - 1) g_j = 0 \quad \forall\, z_q \in V.$$

Since $|\lambda| \neq 1$, this implies $g_j = 0, \quad \forall\, j \neq -1$. Thus, $\omega|_V = g_{-1} \frac{dz_q}{z_q}$. Therefore, either $g_{-1} = 0$, or $\omega|_V$ has a pole of order one at $z_q = 0$, i.e., at q. Since the set of poles of the restriction $\omega|_V$ is discrete and by Nakai's density theorem the set of hyperbolic fixed points $q \in D$ of the holonomy group is dense in D (assuming that this group is nonsolvable), we must have $g_{-1} = 0$ and $\omega|_V \equiv 0$. This implies $\omega \equiv 0$ in $M\backslash\Lambda$, which gives a contradiction. ☐

If we add a dynamical hypothesis, then more can be said:

Proposition 4.2. *Let $\mathcal{F}$ be a transversely holomorphic codimension one foliation on M and suppose that $\mathcal{F}$ is holomorphically transversely additive* *in $M\backslash\Lambda$, where $\Lambda\subset M$ is a finite union of compact leaves. Then $\mathcal{F}$ is given by a closed transversely meromorphic 1-form ω with simple poles and polar divisor $(\omega)_\infty=\Lambda$ provided that each leaf $L\subset\Lambda$ contains a hyperbolic map in its holonomy group.*

The proof is a straightforward consequence of the following lemma:

Lemma 4.3 (extension lemma)**.** *If ω is a closed transversely holomorphic 1-form defining $\mathcal{F}$ in $W^*=W\backslash\Lambda$, then ω extends to a transversely meromorphic 1-form in W provided that the holonomy group* $\operatorname{Hol}(\mathcal{F},L)$ *contains a hyperbolic map for each leaf $L\subset\Lambda$.*

Proof. We may assume that $\Lambda=L$ is a single compact leaf of $\mathcal{F}$. Choose a C^∞ tubular neighborhood $\mathcal{U}$ of Λ fibered by holonomy holomorphic discs, we may also assume that $\mathcal{U}=W$; the extension is a local problem around Λ. Let $\pi\colon\tilde{W}^*\to W^*$ be the universal covering of W^* with transversely holomorphic projection. Lift ω to a Δ-form $\tilde\omega=\pi^*(\omega)$ in $\tilde{W}^*$. Then $\tilde\omega$ is closed, transversely holomorphic, and therefore $\tilde\omega=d\tilde F$ for some function $\tilde F\colon\tilde W^*\to\mathbb{C}$ which is transversely holomorphic.

Thus, $\omega=dF$ for a *multivalued* transversely holomorphic function F in W^* (notice that F is not actually a function in W^*). Given a point $p\in\Lambda$, consider the corresponding disc $D_p\ni p$ and the corresponding punctured disc $D_p^*=D_p\backslash\{p\}\subset W^*$. Then

$$D_p^*\simeq\mathbb{D}^*=\{z\in\mathbb{C};0<|z|<1\}$$

by conformal equivalence. Therefore, the restriction

$$\pi|_{\pi^{-1}(D_p^*)}\colon\pi^{-1}(D_p^*)\to D_p^*$$

is a holomorphic covering of the punctured unit disc $\mathbb{D}^*$. We may therefore consider a holomorphic covering $\Pi\colon\mathbb{D}\to\mathbb{D}^*$, a lift $\tilde f\colon\mathbb{D}\to\mathbb{D}$ of the restriction to $\mathbb{D}^*$ of the hyperbolic holonomy diffeomorphism

$$f\colon\mathbb{D}\to\mathbb{D},\qquad f(z)=\lambda z,$$

the restrictions $\omega|_{\mathbb{D}^*}$, and $\tilde\omega|_{\mathbb{D}}=d(\tilde F|_{\mathbb{D}})$.

The lift $\tilde f$ preserves the foliation $\tilde{\mathcal{F}}=\pi^*(\mathcal{F})$ and therefore it satisfies

$$\tilde F(\tilde f(\tilde z))=\tilde F(\tilde z),\quad\forall\tilde z\in\mathbb{D}.$$

We have $\Pi(\tilde z)=e^{2\pi i\tilde z}=z$ so that if $\lambda=e^{2\pi i\mu}$ then $\tilde f(\tilde z)=\mu+\tilde z$ we may assume. Therefore, $\tilde F(\tilde z)$ satisfies

$$(\star)\quad\begin{cases}d\tilde F(\tilde z+1)=d\tilde F(\tilde z)\\ \tilde F(\tilde z+\mu)=\tilde F(\tilde z)\end{cases}$$

Since $|\lambda| \neq 1$, we have $\mu \in \mathbb{R}$ and therefore $d\tilde{F}(\tilde{z})$ gives a holomorphic differential one-form in the complex 1-torus $\mathbb{C}\big/(\mathbb{Z} \oplus \mu\mathbb{Z})$. Hence, $d\tilde{F}(\tilde{z})$ must be linear, i.e., $d\tilde{F}(\tilde{z}) = \alpha\, d\tilde{z}$ for some $\alpha \in \mathbb{C}^*$.

This implies that

$$\tilde{\omega} = \alpha\,\frac{d(\log z)}{2\pi i} = \frac{\alpha}{2\pi i}\frac{dz}{z}$$

and therefore $\omega(z) = c\frac{dz}{z}$ for some constant $c \in \mathbb{C}^*$.

This proves the existence of an extension of $\omega|_{\mathbb{D}^*}$ to $\mathbb{D}$. By Hartogs' Extension Theorem ω extends to a transversely meromorphic one-form on W with polar set $(\omega)_\infty = \Lambda$ of order one. ☐

Lemma 4.3 will be useful also in the $\mathbb{C}$-affine case. For the moment we state a natural consequence of Proposition 5.9 above:

Corollary 4.4. *If $\mathcal{F}$ is transversely holomorphically additive in $M\backslash\Lambda$ as in Proposition 4.2 and each $L \subset \Lambda$ contains some hyperbolic map in its holonomy group, then* $\operatorname{Hol}(\mathcal{F}, L)$ *is abelian linearizable, i.e., it is analytically conjugate to a subgroup of the multiplicative group $\mathbb{C}^*$.*

Proof. The main point is that the foliation is defined by a simple poles closed meromorphic 1-form ω in M, having polar set $(\omega)_\infty = \Lambda$. For our purposes we may assume that Λ consists of a single compact leaf and $\dim M = 2$. Then there is an open cover $\bigcup\limits_{j\in J} U_j$ of Λ in M by connected open sets U_j such that on each U_j we have local coordinates

$$(x_j, y_j)\colon U_j \to \mathbb{C}^2$$

in which $\Lambda \cap U_j = \{y_j = 0\}$ and $\omega|_{U_j} = a\frac{dy_j}{y_j}$ for some $a \in \mathbb{C}^*$ (the residue of ω in Λ). In each nonempty intersection $U_i \cap U_j \neq \varnothing$ we have $\frac{dy_i}{y_i} = \frac{dy_j}{y_j}$ which implies that y_i/y_j is a locally constant function. This proves the statement. ☐

Notice that this is not clear at first sight and shows that an additive transverse holomorphic structure can "degenerate" into a multiplicative structure.

5. The transversely affine case

Let $\mathcal{F}$ be a real codimension 2 foliation on $M^{\ell+2}$. The foliation $\mathcal{F}$ is $\mathbb{C}$-*transversely affine* if there exists a transversely holomorphic atlas of submersions $y_j\colon U_j \to \mathbb{C}$ for $\mathcal{F}$ such that if $U_i \cap U_j \neq \varnothing$ then $y_i = a_{ij}\, y_j + b_{ij}$ for locally constant maps $a_{ij}, b_{ij}\colon U_i \cap U_j \to \mathbb{C}$.

The following proposition is a characterization of the existence of such structure in terms of differential forms.

Proposition 5.1. (i) *The possible* $\mathbb{C}$*-affine transverse structures for* $\mathcal{F}$ *in* M *are classified by the collections* (Ω_j, η_j) *of differential one-forms defined in the open sets* $U_j \subset M$ *such that:*

- Ω_j *and* η_j *are transversely holomorphic,*
- Ω_j *is integrable and defines* $\mathcal{F}$ *in* U_j*,*
- $d\Omega_j = \eta_j \wedge \Omega_j$ *and* $d\eta_j = 0$ *in* U_j*,*
- *if* $U_i \cap U_j \neq \emptyset$*, then* $\Omega_i = g_{ij}\,\Omega_j$ *and* $\eta_i = \eta_j + \frac{dg_{ij}}{g_{ij}}$ *for non-vanishing transversely holomorphic function* $g_{ij}\colon U_i \cap U_j \to \mathbb{C}^*$*.*

(ii) *Two such collections* (Ω_j, η_j) *and* (Ω'_j, η'_j) *define the same affine transverse structure for* $\mathcal{F}$ *in* M *if and only if* $\Omega'_j = g_j\,\Omega_j$ *and* $\eta'_j = \eta_j + \frac{dg_j}{g_j}$ *for some transversely holomorphic non-vanishing functions* $g_j\colon U_j \to \mathbb{C}^*$*.*

Proof. First we prove (i). Assume that $\mathcal{F}$ is $\mathbb{C}$-transversely affine with transversely holomorphic atlas of submersions $y_j\colon U_j \to \mathbb{C}$. Given any transversely holomorphic non-singular one-form Ω_j defining $\mathcal{F}$ in U_j we have $\Omega_j = g_j\,dy_j$ for some transversely holomorphic function $g_j\colon U_j \to \mathbb{C}^*$ and we define $\eta_j = \frac{dg_j}{g_j}\cdot$ If $U_i \cap U_j \neq \emptyset$ then $\Omega_i = g_{ij}\,\Omega_j$ and $y_i = a_{ij}\,y_j + b_{ij}$ imply $dy_i = a_{ij}\,dy_j$ and therefore $a_{ij}\,g_i = g_j\,g_{ij}$. Thus

$$\frac{dg_i}{g_i} = \frac{dg_j}{g_j} + \frac{dg_{ij}}{g_{ij}}$$

in $U_i \cap U_j$. Clearly $d\eta_j = 0$, $d\Omega_j = \eta_j \wedge \Omega_j$ and $\eta_i = \eta_j + \frac{dg_{ij}}{g_{ij}}\cdot$ This proves (i). Item (ii) is proved similarly and we refer to [19]. ☐

As an immediate corollary we have:

Proposition 5.2. *Let* $\mathcal{F}$ *be a transversely holomorphic foliation of codimension one on* M *given by a transversely holomorphic integrable* 1*-form* Ω*. Then* $\mathcal{F}$ *is* $\mathbb{C}$*-transversely affine on* M *if and only if there exists a transversely meromorphic one-form* η *in* M *which is closed and satisfies the equation* $d\Omega = \eta \wedge \Omega$*.*

Holonomy. From now on in this section we consider the following situation. We have a transversely holomorphic foliation $\mathcal{F}$ on M having a $\mathbb{C}$-affine transverse structure in $M\backslash\Lambda$ for some compact analytic set of dimension one $\Lambda \subset M$. We shall study the holonomy groups of leaves $L \subset \Lambda$.

As we have shown, in the $\mathbb{C}$-additive case, the holonomy groups are solvable. A similar result holds for the $\mathbb{C}$-affine case. Nevertheless, the proof requires more technical features as we will see below.

Theorem 5.3. *Let $\mathcal{F}$ be a transversely holomorphic foliation of codimension one on $M^{\ell+2}$ given by a transversely holomorphic integrable* 1*-form Ω. Assume that $\mathcal{F}$ has a $\mathbb{C}$-transversely affine structure on $M^{\ell+2}\backslash\Lambda$ for a finite union of compact leaves $\Lambda \subset M$. Given any leaf $L \subset \Lambda$ the holonomy group of L is solvable.*

Proof. According to Proposition 5.2 there exists a closed transversely holomorphic one-form η in $M\backslash\Lambda$ such that $d\Omega = \eta \wedge \Omega$ in $M\backslash\Lambda$. Given a point $p \in M\backslash\Lambda$, we may consider a small open simply-connected neighborhood $U_p \ni p$ in $M\backslash\Lambda$ where we can write $\eta|_{U_p} = \frac{dg_p}{g_p}$ for some transversely holomorphic non-vanishing function $g_p\colon U_p \to \mathbb{C}^*$. Then

$$0 = d\left(\frac{\Omega}{e^{\int\eta}}\right) = d\left(\frac{\Omega}{g_p}\right)$$

in U_p implies $\Omega = g_p\, dF_p$ for some transversely holomorphic function

$$F_p\colon U_p \to \mathbb{C},$$

which is a local first integral for $\mathcal{F}$ in U_p. If we choose another

$$\overline{p} \in U_{\overline{p}} \subset M\backslash\Lambda$$

then in case $U_p \cap U_{\overline{p}} \neq 0$ we have $F_{\overline{p}} = \alpha\, F_p + \beta$ for some affine mapping $(z \mapsto \alpha z + \beta)$. We introduce therefore the multiform function F on $M\backslash\Lambda$ by writing $F = \displaystyle\int \frac{\Omega}{e^{\int\eta}}$ or also $dF = \frac{\Omega}{e^{\int\eta}}$. This is a locally well-defined transversely holomorphic function $\big(F|_{U_p} = F_p$ as above$\big)$ which lifts to a well-defined transversely holomorphic function on the universal covering $\widetilde{M\backslash\Lambda}$ of $M\backslash\Lambda$.

Fixing a point $p \in M\backslash\Lambda$, any local determination F_p of F, and a path $\gamma\colon [0,1] \to M\backslash\Lambda$, we may consider the transversely analytic continuation $F_{p,\gamma}$ of F_p along γ. If γ is closed, i.e., $\gamma(0) = \gamma(1)$, then $F_{p,\gamma}$ depends only on the homotopy class $[\gamma] \in \pi_1(M\backslash\Lambda; p)$. Let

$$A(F) := \{F_{p,[\gamma]}; [\gamma] \in \pi_1(M\backslash\Lambda; p)\}$$

be the set of such transversely analytic continuations and

$$P^{-1}(p) = \{\text{determinations } F_p \text{ on } F \text{ at } p\}.$$

Then $\pi_1(M\backslash\Lambda; p)$ acts transitively on $P^{-1}(p)$. There exists therefore a regular covering $A(F) \xrightarrow[P]{} M\backslash\Lambda$ with total space $A(F)$ such that fiber over

p equal to $P^{-1}(p)$ and covering automorphisms group is isomorphic to

$$\pi_1(M\backslash\Lambda;p)/P_{\#}(\pi_1(A(F),F_p) =: \mathcal{M}(F).$$

We will call $\mathcal{M}(F)$ the *monodromy group* of F.

The canonical projection $\mu\colon \pi_1(M\backslash\Lambda;p) \to \mathcal{M}(F)$ is called the *monodromy map* of F and associates to each homotopy class $[\gamma] \in \pi_1(M\backslash\Lambda;p)$ the corresponding determination $F_{p,[\gamma]}$ at the point $\gamma(1) = p$.

Given any leaf $L_0 \subset \Lambda$, we fix a point $p_0 \in L_0$ and a small holomorphic holonomy disc D_0 such that $D_0 \pitchfork \mathcal{F}$ and $D_0 \cap L_0 = \{p_0\}$ so we have a holonomy representation

$$\operatorname{Hol}(\mathcal{F}, L_0, D_0, p_0) \subseteq \operatorname{Diff}(D_0;p_0).$$

Given a C^∞ tubular neighborhood $r\colon N \to L_0$ fibered by holomorphic holonomy discs $D_q = r^{-1}(q)$, $q \in L_0$ with $r^{-1}(p_0) = D_0$, and

$$N^* = N\backslash L = N\backslash\Lambda,$$

we obtain, by restriction a C^∞ fibration $r^*\colon N^* \to L_0$ fiber, a punctured disc $\mathbb{D}^* = \mathbb{D} - \{0\}$. The homotopy sequence for this fibration gives the exact sequence below:

$$\begin{tikzcd}
0 \arrow[r] \arrow[dr] & \mathbb{Z} \arrow[r] & \pi_1(N^*,\tilde{p}_0) \arrow[r] & \pi_1(L_0;p) \arrow[r] & 0 \\
& \pi_1(\mathbb{D}^*) \arrow[ur] & & &
\end{tikzcd}$$

where $\tilde{p}_0 \in N^*$ is a fixed base point with $r(\tilde{p}_0) = p_0$.

Denote by $A(F)|_{N^*} \xrightarrow[P|_{N^*}]{} N^*$ the natural restriction of $A(F) \xrightarrow[P]{} M\backslash\Lambda$ and by $A(F)|_{N^*}$ the connected component which contains the local determination $F_{\tilde{p}_0}$. As above, we denote by

$$\mu\colon \pi_1(N^*,\tilde{p}_0) \to \mathcal{M}(F)(N^*) \cong \frac{\pi_1(N^*,\tilde{p}_0)}{P_{\#}(\pi_1(A(F)|_{N^*};F_{\tilde{p}_0})}$$

the monodromy map. We shall use the following lemma:

Lemma 5.4 ([17])**.** *There exists a unique morphism* (μ) *which makes commutative the following diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \pi_1(N^*,\tilde{p}_0) \arrow[r] \arrow[d, "\mu"] & \pi_1(L_0,p_0) \arrow[r] \arrow[d, "(\mu)"] & 0 \\
& & \mathcal{M}(F;N^*) \arrow[r] & \frac{\mathcal{M}(F;N^*)}{\mathbb{Z}} \arrow[r] & 0
\end{tikzcd}$$

We finally define $\mu(F; L_0) := \frac{\mathcal{M}(F;N^*)}{\mathbb{Z}}$ as the monodromy of F associated to L_0 and call the morphism

$$(\mu)\colon \pi_1(L_0; p) \to \mathcal{M}(F; L_0)$$

the monodromy mapping of F relatively to L_0. Now we use

Lemma 5.5 ([17]). *There exists a surjective morphism α which makes commutative the following diagram*

$$\begin{tikzcd} & \pi_1(L_0;p_0) \arrow[dl, "\mathrm{Hol}"'] \arrow[dr, "(\mu)"] & \\ \mathrm{Hol}(\mathcal{F}, L_0, D_0) \arrow[rr, "\alpha"] & & \mathcal{M}(F; L_0) \end{tikzcd}$$

Given any local determination $\ell(y)$ of $F|_{D_0^*}$, where y is a holomorphic coordinate on D_0, denote by $\ell(y)_{[\gamma]}$ its analytic continuation along $[\gamma] \in \pi_1(L_0; p_0)$. Then we get an element $\ell(y)_{[\gamma]} = (\alpha \circ \mathrm{Hol})([\gamma])$ of the monodromy group $\mathcal{M}(F; L_0)$. Also, we need:

Lemma 5.6 ([17]). *For each $[\gamma] \in \pi_1(L_0, p_0)$ we have $\ell(y)_{[\gamma]} = a_{[\gamma]}y + b_{[\gamma]}$ for some affine mapping $(z \mapsto a_{[\gamma]}z + b_{[\gamma]})$. In particular $\mathcal{M}(F_0; L_0)$ is solvable.*

Since the exact sequence

$$0 \to \ker(\alpha) \to \mathrm{Hol}(\mathcal{F}, L_0, D_0) \xrightarrow{\alpha} \mathcal{M}(F; D_0^*) \to 0$$

has $\mathcal{M}(F; D_0^*)$ solvable and $\ker(\alpha)$ abelian, we conclude that $\mathrm{Hol}(\mathcal{F}, L_0, D_0)$ is solvable. ☐

Extension. We keep on considering the situation of the previous paragraph. Moreover, let $\mathcal{F}$ be given in M by a transversely holomorphic integrable 1-form Ω. Our first step is the following:

Proposition 5.7. *Let $L \subset \Lambda$ a leaf whose holonomy group contains a hyperbolic map. Then we may find a fibered neighborhood W of L by holonomy discs D as above and a transversely meromorphic 1-form η_L in W such that:*

- *$\eta_L|_D$ is a meromorphic 1-form written as $\eta_L|_D(y) = \frac{a\,dy}{y} + \frac{dg}{g}$, $a \in \mathbb{C}$, in a suitable holonomy holomorphic coordinate y in D for which $\Omega = g\,dy$;*

If L has abelian holonomy we may take $a = 0$ and if L is non abelian then $a = k + 1$ for some $k \in \mathbb{N}$, and we have an analytic imbedding of $\mathrm{Hol}(\mathcal{F}, L, D)$ into the group $\mathbb{H}_k = \left\{\varphi(z) = \frac{\lambda z}{\sqrt[k]{(1+\mu z^k)}},\ \lambda \in \mathbb{C}^*, \mu \in \mathbb{C}\right\}$.

Proof. We already know from Theorem 5.3 that the holonomy group of each leaf $L \subset \Lambda$ is solvable. Given a point $p \in L$ consider a transverse disc $\Sigma \subset M$ to $\mathcal{F}$ with $\Sigma \cap \Lambda = \Sigma \cap L = \{p\}$.

We put $G = \operatorname{Hol}(\mathcal{F}, L, \Sigma, p)$. By means of a local holomorphic coordinate $z \in \Sigma$ we may consider G as a subgroup of $\operatorname{Diff}(\mathbb{C}, 0)$.

Remark 5.8. There is a germ of a holomorphic vector field $\mathcal{X}(z)$ in Σ such that for any $g \in G$ we have $g_*\mathcal{X} = c_g\tilde{\mathcal{M}}$ for some constant $c_g \in \mathbb{C}^* = \mathbb{C}^*$.

Indeed, it is well-known that a solvable subgroup $G < \operatorname{Diff}(\mathbb{C}, 0)$ admits such a formal vector field which is projectively invariant. Moreover, this vector field is convergent in the case the group contains some nonresonant (hyperbolic for instance) map [8], [15]). Now we may write, up to an analytic change of coordinates in Σ, $\mathcal{X}(z) = \frac{z^{k+1}}{1+az^k}\frac{d}{dz}$. Then for any $g \in G$ we have

$$g_*\left(\frac{z^{k+1}}{1+az^k}\frac{d}{dz}\right) = c_g\left(\frac{z^{k+1}}{1+az^k}\frac{d}{dz}\right)$$

for some constant $c_g \in \mathbb{C}^* = \mathbb{C}^*$. Consider two cases.

G **is abelian**. In this case, because it contains a hyperbolic (analytically linearizable) map, the group G is analytically linearizable. We may therefore construct a closed meromorphic one-form ω with simple poles, $(\omega)_\infty = L$, in a neighborhood W of L in M. The form ω is given in any linearizing transverse coordinate z by $\omega(z) = \frac{dz}{z}$. In this case we have $\Omega = g\omega$ for some transversely meromorphic one-form g in W. Since Ω/g is closed, we have that $\eta_L := \frac{dg}{g}$ satisfies $d\Omega = \eta \wedge \Omega$.

G **is solvable non-abelian**. If we have $c_g = 1$ for every element $g \in G$ then the vector field

$$\mathcal{X}(z) = \frac{z^{k+1}}{1+az^k}\frac{d}{dz}$$

is invariant under the action of G. As it is well-known this implies that G is abelian (see for instance the survey part in [18]). Hence, there must be some $g \in G$ with $c_g \neq 1$. This together with $g_*\mathcal{X}(z) = c_g\mathcal{X}(z)$ implies by straight computation that $a = 0$, i.e., $\mathcal{X}(z) = z^{k+1}\frac{d}{dz}$. Integrating now the equation $g_*(z^{k+1}\frac{d}{dz}) = c_g(z^{k+1}\frac{d}{dz})$ we obtain an analytic embedding

$$G \hookrightarrow \mathbb{H}_k = \left\{\left(z \mapsto \frac{\lambda z}{\sqrt[k]{1+\mu z^k}}\right)\right\},$$

(compare with Theorem 2.1, see [18] for a more detailed exposition). Using this we cover a neighborhood of $L \subset M$ by local charts $(x, z) \in \mathbb{R}^\ell \times \mathbb{C}$, where $\mathcal{F}$ is given by $dz = 0$, and the coordinate z gives the embedding of the holonomy group as a subgroup of $\mathbb{H}_k$. If we write on each chart

$\Omega(x,z) = gdz$, then let us put

$$\eta(x,z) := \frac{dg}{g} + (k+1)\frac{dz}{z}.$$

On each intersection of two such coordinate charts (x,z) and $(\tilde{x},\tilde{z})$, we have:

(i) $\Omega = gdz = \tilde{g}d\tilde{z}$

(ii) $\tilde{z} = \frac{\lambda z}{\sqrt[k]{1+\mu z^k}}$

Taking derivatives in the second equation we have

$$\frac{d\tilde{z}}{\tilde{z}^{k+1}} = \frac{dz}{\lambda^k z^{k+1}}.$$

Replacing this in the first equation we conclude that $\tilde{g}\tilde{z}^{k+1} = \lambda^k g z^{k+1}$. Hence, $d\ln\tilde{g}\tilde{z}^{k+1} = d\ln g z^{k+1}$, i.e.,

$$(k+1)\frac{d\tilde{z}}{\tilde{z}} + \frac{d\tilde{g}}{\tilde{g}} = (k+1)\frac{dz}{z} + \frac{dg}{g}.$$

This defines the one-form η_L on each coordinate system by

$$\eta_L(x,z) := (k+1)\frac{dz}{z} + \frac{dg}{g}$$

and that one-form satisfies the equation $d\Omega = \eta_L \wedge \Omega$. □

Now we prove the following extension result:

Proposition 5.9. *Let $\mathcal{F}$ be $\mathbb{C}$-transversely affine on $M\backslash\Lambda$ and given in M by Ω. Suppose also that each leaf $L \subset \Lambda$ contains a hyperbolic map in its holonomy group. Then there is a closed transversely meromorphic 1-form η in M with polar set $(\eta)_\infty = \Lambda$ of order one such that $d\Omega = \eta \wedge \Omega$. Moreover, for any leaf $L \subset \Lambda$ we have that*

(i) *if $Res_L\eta = a \notin \{2,3,4,\dots\}$, then $\mathrm{Hol}(\mathcal{F},L)$ is abelian linearizable.*

(ii) *if $\mathrm{Hol}(\mathcal{F},L)$ is not abelian linearizable, then $Res_L\eta = k+1$ with $k \in \mathbb{N}$ and $\mathrm{Hol}(\mathcal{F},L)$ embeds analytically into*

$$\mathbb{H}_k = \left\{\left(z \mapsto \frac{\lambda z}{\sqrt[k]{1+\mu z^k}}\right)\right\}.$$

Proof. The existence of a $\mathbb{C}$-affine structure in $M\backslash\Lambda$ gives us (via Proposition 5.1) a form η defined in $M\backslash\Lambda$ with the properties announced. The problem is to show the existence of an extension of η to Λ. For this sake we consider the restriction of η to $W\backslash\Lambda = W\backslash L =: W^*$. In W^* we have

$$d\Omega = \eta \wedge \Omega = \eta_L \wedge \Omega \qquad \Rightarrow \qquad \eta - \eta_L = h\Omega$$

for some transversely meromorphic function h in W^* such that $d(h\Omega) = 0$.

If $\eta - \eta_L \not\equiv 0$, then $\Omega|_{W^*}$ admits h as an integrating factor. Let us prove the existence of an extension of η to L. We have the following two possibilities.

$1^{\mathbf{st}}$**.** $\mathrm{Hol}(\mathcal{F}, L)$ is abelian. In this case, since it contains a linearizable hyperbolic map, $\mathrm{Hol}(\mathcal{F}, L)$ is abelian linearizable (Lemma 2.4) and $\mathcal{F}$ is given by a closed transversely meromorphic one-form ξ in a fibered neighborhood W of L in M as above. Moreover, $\xi|_D$ writes as $\xi|_D(z) = \frac{dz}{z}$ in suitable holonomy holomorphic coordinates z in D. Since $\mathrm{Hol}(\mathcal{F}, L)$ contains a hyperbolic element, there exists no meromorphic first integral for $\mathcal{F}$ in W. Therefore, we must have ξ unique, up to multiplicative constants. Hence, $\omega(z) = \varphi_{-1}\frac{dz}{z}$ and

$$\eta|_D(z) = \eta_L|_D(z) + \varphi_{-1}\frac{dz}{z} = \eta_L|_D(z) + \varphi_{-1}\,\xi|_D(z).$$

Therefore, $\eta = \eta_L + c\xi$ in W for some constant $c \in \mathbb{C}$.

$2^{\mathbf{nd}}$**.** $\mathrm{Hol}(\mathcal{F}, L)$ is solvable but not abelian. In this case there exists a holomorphic imbedding $\mathrm{Hol}(\mathcal{F}, L, D) \hookrightarrow \mathbb{H}_k$ for a unique $k \in \mathbb{N}$ and $\mathcal{F}|_W$ cannot be given by a closed transversely meromorphic one-form. Therefore, $\omega \equiv 0$ and $\eta = \eta_L$ in W. □

6. PROOF OF THEOREM 1.1

Before proving Theorem 1.1 we make some general considerations. Let $\mathcal{F}$ be a transversely holomorphic foliation of codimension one on M given by a transversely holomorphic integrable 1-form Ω. Assume that $\mathcal{F}$ is $\mathbb{C}$-transversely affine on $M\backslash\Lambda$ for some compact leaf L_o of $\mathcal{F}$. Also assume that:

(i) The holonomy group of the leaf L_o contains a hyperbolic map.
(ii) Each closed transversely one-form in M is exact.
(iii) The leaf L_o is given by some equation $L_o\colon \{f = 0\}$ where $f\colon M \to \mathbb{C}$ is transversely holomorphic and with isolated singularities.

By Propositions 5.2 and 5.9 there is a closed transversely meromorphic 1-form η in M, with polar set $(\eta)_\infty = \Lambda$ of order one, such that $d\Omega = \eta \wedge \Omega$.

We consider the above situation in the case when $\Lambda = L_o$ is a single leaf of $\mathcal{F}$. Let $a \in \mathbb{C}$ be the residue $Res_{L_o}\eta$ of η in L_o. Let us prove that the holonomy group $\mathrm{Hol}(\mathcal{F}, L_o)$ is abelian, which will imply Theorem 1.1. If this is not the case, then, according to Proposition 5.9(i), we must have that $a = k + 1$ for some $k \in \mathbb{N}$. This case is dealt with via the following lemma:

Lemma 6.1. *Suppose that any closed transversely holomorphic* 1*-form on* M *is exact and* $\Lambda = L_o = \{f = 0\}$ *for some transversely holomorphic* *function* $f\colon M \to \mathbb{C}$. *Then* $\mathcal{F}$ *is given in a neighborhood of* Λ *by a closed transversely meromorphic* 1*-form having only simple poles* ω *and defined in a neighborhood* $U \supset \Lambda$ *of* Λ *in* M.

Proof. Since $\operatorname{Res}_{L_o}\eta = k+1$, we consider the one-form $\Theta := \eta - (k+1)\frac{df}{f}$ in M. This is a closed transversely holomorphic one-form with empty polar set.

By hypothesis, we must have then $\Theta = d\varphi$ for some transversely holomorphic function $\varphi\colon M \to \mathbb{C}$. Hence, $\eta = (k+1)\,\frac{df}{f} + d\varphi$ for some transversely holomorphic function $\varphi\colon M \to \mathbb{C}$. The equation $d\Omega = \eta \wedge \Omega$ implies

$$d\left(\frac{\Omega}{f^{k+1}e^{\varphi}}\right) = 0.$$

Thus, $\omega = \frac{\Omega}{f^{k+1}e^{\varphi}}$ is closed transversely meromorphic and defined $\mathcal{F}$ in M with polar set $(\omega)_\infty = L$. But this implies, as in the case of holomorphic foliations given by closed meromorphic one-forms ([18]), that $\operatorname{Hol}(\mathcal{F}, L_o)$ is abelian. Since this holonomy group contains a hyperbolic map, it is therefore analytically linearizable (Lemma 2.4). Using well-known techniques from [1,6,7] one can construct a transversely meromorphic one-form ω, with order one polar set $(\omega)_\infty = \Lambda$, in a neighborhood U of Λ in M, such that $\mathcal{F}|_U$ is given by ω. Namely, given any holonomy disc D with $D \cap \Lambda \neq \emptyset$ and any holomorphic coordinate $z \in D$ that linearizes $\operatorname{Hol}(\mathcal{F}, \Lambda, D)$, we define $\omega|_D := \frac{dz}{z}$ and extend it constant along the leaves of the foliation. □

Proof of Theorem 1.1. Theorem 1.1 is now a consequence of the above argumentation. Recall that we are assuming that $\mathcal{F}$ is $\mathbb{C}$-transversely affine in the complement $M \backslash L_o$ of a compact leaf admitting a hyperbolic holonomy map. □

7. APPLICATIONS: PROOF OF THEOREMS 1.2 AND 1.3

In this section we prove Theorems 1.2 and 1.3 as applications of the techniques we have introduced. We start with the following situation: $\tilde{\mathcal{F}}$ is a holomorphic foliation of codimension one and with singularities on a complex manifold $\tilde{M}$ of dimension $n \geqslant 1$. Assume that $\mathcal{G}$ is defined by an integrable one-form $\tilde{\Omega}$ in $\tilde{M}$, having singular set of codimension $\geqslant 2$. Suppose also that $\tilde{\mathcal{F}}$ is transversely affine (as a singular holomorphic foliation) (see [1, 19]) on $\tilde{M}\backslash\tilde{\Lambda}$ for some irreducible analytic invariant codimension one subset $\tilde{\Lambda} \subset \tilde{M}$. According to [19] (or Proposition 5.2) there exists a closed holomorphic one-form $\tilde{\eta}$ defined in $\tilde{M}\backslash\tilde{\Lambda}$ such that $d\tilde{\Omega} = \tilde{\eta} \wedge \tilde{\Omega}$.

Let $M \subset \tilde{M}$ be a real closed hypersurface that we suppose to be transverse to $\tilde{\mathcal{F}}$ (see [14]). Then the induced foliation $\mathcal{F} = \tilde{\mathcal{F}}|_M$ is transversely holomorphic of codimension one with a holomorphic affine transverse structure on $M\backslash\Lambda$, and $\Lambda := \tilde{\Lambda} \cap M$ is a finite union of compact leaves of $\mathcal{F}$ (recall that M is compact).

Lemma 7.1. *In the above situation for $\mathcal{F}$, Λ, $\tilde{\mathcal{F}}$, $\tilde{\Lambda}$, M, and $\tilde{M}$, suppose that some leaf $L_o \subset \Lambda$ of the induced foliation $\mathcal{F}$ contains a hyperbolic map in its holonomy group. Then $\tilde{\eta}$ extends to a meromorphic form in $\tilde{M}$, having simple poles and the polar set $(\tilde{\eta})_\infty = \tilde{\Lambda}$.*

Proof. We put $\Omega = \tilde{\Omega}|_M$ and $\eta = \tilde{\eta}|_M$. Since L_o is an isolated irreducible component of the intersection $\tilde{\Lambda} \cap M$, there is a neighborhood W of L_o in $\tilde{M}$ such that $\tilde{\Lambda} \cap W \cap M = L_0$. Let us now observe that, since $\tilde{\eta}$ is already defined in $\tilde{M}\backslash\tilde{\Lambda}$, it is therefore defined in $W\backslash L_0$. Then, by Proposition 5.9, the one-form η admits an extension to L_0. This implies that $\tilde{\eta}$ extends to W and $W \cap \tilde{\Lambda} \neq \emptyset$. By Levi's extension theorem ([22]) this implies (recall that $\tilde{\Lambda} \subset \tilde{M}$ is irreducible) that $\tilde{\eta}$ extends to $\tilde{\Lambda}$ proving the lemma. □

Proof of Theorem 1.2. We proceed under the additional assumption that M is simply-connected. Recall that since $\tilde{M}$ is Stein, the irreducible analytic subset $\tilde{\Lambda} \subset \tilde{M}$ is given by a global equation, i.e., $\tilde{\Lambda} : (\tilde{f} = 0)$ for some holomorphic function $\tilde{f} \colon \tilde{M} \to \mathbb{C}$.

Remark 7.2. The holonomy group $\operatorname{Hol}(\tilde{\mathcal{F}}, \tilde{L})$ of the leaf $\tilde{L}_o = \tilde{\Lambda}\backslash\operatorname{sing}(\tilde{\mathcal{F}})$ of $\tilde{\mathcal{F}}$ is abelian linearizable.

Proof. We have $(\tilde{\eta})_\infty = (\tilde{f} = 0)$ of order one. Let $a = \operatorname{Res}_{\tilde{\Lambda}}\tilde{\eta} \in \mathbb{C}^*$ (this residue is well-defined since $\tilde{\Lambda}$ is irreducible). Thus, $\tilde{\Theta} := \tilde{\eta} - a\,\frac{d\tilde{f}}{\tilde{f}}$ is holomorphic and closed in $\tilde{M}$. Since by hypothesis M is simply-connected, we may obtain a holomorphic function $\tilde{h} \colon W(M) \to \mathbb{C}$ in a tubular neighborhood $W(M)$ of M in $\tilde{M}$ such that $\tilde{\Theta} = d\tilde{h}$ in this neighborhood. By Levi's Extension Theorem [22] (recall that $\tilde{M}$ is a Stein manifold) the function $\tilde{h}$ extends to a holomorphic function to $W \cup \overline{A}$, and we have $\tilde{\varphi} = d\tilde{h}$ on $W \cup \overline{A}$.

Thus, $\tilde{\eta} - a\,\frac{d\tilde{f}}{\tilde{f}} = d\tilde{h}$ and therefore

$$\tilde{\eta} = a\,\frac{d\tilde{f}}{\tilde{f}} + \frac{d(e^{\tilde{h}})}{e^{\tilde{h}}} = a\,\frac{d(\tilde{f}\,e^{\frac{\tilde{h}}{a}})}{(\tilde{f}\,e^{\frac{\tilde{h}}{a}})}\cdot$$

We may therefore assume that $\tilde{\eta} = a\,\frac{d\tilde{f}}{\tilde{f}}$ in $W \cup \overline{A}$.

$1^{\textbf{st}}$ **case**. $2 \leqslant a = k+1$ for some $k \in \mathbb{N}$. In this case

$$d\tilde{\Omega} = \tilde{\eta} \wedge \tilde{\Omega} \qquad \Rightarrow \qquad d\left(\frac{\tilde{\Omega}}{\tilde{f}^{k+1}}\right) = 0.$$

Thus, $\tilde{\mathcal{F}}$ is given by a closed meromorphic one-form on $W \cup \overline{A}$. But, since $\frac{\tilde{\Omega}}{\tilde{f}^{k+1}}$ has polar set $\{\tilde{f} = 0\}$ and of order $k+1 \geqslant 2$, we must integrate it (as above for $\tilde{\Theta}$) and obtain $\frac{\tilde{\Omega}}{\tilde{f}^{k+1}} = d\tilde{F}$, where $\tilde{F}\colon W \cup \overline{A} \to \overline{\mathbb{C}}$ is meromorphic with polar set $(\tilde{f})_\infty = \{\tilde{f} = 0\}$ of order k. Thus, actually we have $\tilde{F} = \frac{\tilde{G}}{\tilde{f}^k}$ for some holomorphic function $\tilde{G}\colon W \cup \overline{A} \to \mathbb{C}$. This gives

$$\tilde{\Omega} = f^{k+1}\, d\left(\frac{\tilde{G}}{\tilde{f}^k}\right) = \frac{\tilde{f}^{k+1}(\tilde{f}^k d\tilde{G} - \tilde{G}k\tilde{f}^{k-1}d\tilde{f}^k)}{\tilde{f}^{2k}},$$

whence

$$\tilde{\Omega} = \tilde{f}\, d\tilde{G} - k\tilde{G}d\tilde{f}.$$

Since $\tilde{\Omega}$ is the pull-back of a linear 2-dimensional foliation, the holonomy of its analytic leaves, and in particular of the leaf $\{\tilde{f} = 0\}$, is abelian linearizable. Alternatively, observe that since the holonomy group of the leaf $\tilde{L}_o = \tilde{\Lambda} \backslash \operatorname{sing}(\tilde{\mathcal{F}})$ contains by hypothesis some hyperbolic element, this is not compatible with the model $\tilde{\Omega} = \tilde{f}\, d\tilde{G} - k\tilde{G}d\tilde{f}$ because this later admits a meromorphic first integral. Hence, we have concluded that this case does not occur.

$2^{\textbf{nd}}$ **case**. $a - 1 \notin \mathbb{N} = \{1, 2, 3, \dots\}$.

In this case by [19, Section 3] the holonomy group of the $\{\tilde{f} = 0\}$ is abelian linearizable. Indeed, we may find local coordinates $(x_j, y_j) \in U_j$ covering a neighborhood of $\{\tilde{f} = 0\}$ in $W \cup \overline{A}$ such that:

$$\{\tilde{f} = 0\} \cap U_j : \{y_j = 0\}, \qquad \tilde{\mathcal{F}}|_{U_j} : dy_j = 0,$$

$$\tilde{\Omega}(x_j, y_j) = g_j dy_j, \qquad \tilde{\eta}(x_j, y_j) = \frac{dg_j}{g_j} + a\,\frac{dy_j}{y_j}\,.$$

Since in each intersection $U_i \cap U_j \neq \varnothing$ of two such charts we have

$$\begin{cases} g_i\, dy_i = g_j\, dy_j \\ a\,\frac{dy_i}{y_i} + \frac{dg_i}{g_i} = a\,\frac{dy_j}{y_j} + \frac{dg_j}{g_j} \end{cases}$$

we must also have (for $a - 1 \notin \mathbb{N}$) that $y_i = c_{ij}\, y_j$ for some locally constant c_{ij} in $U_i \cap U_j$ ([19]). The claim is proved. □

It remains to prove the existence of a logarithmic one-form $\tilde{\xi}$ defining $\tilde{\mathcal{F}}$ in a neighborhood of $\overline{A} \cup \tilde{L}_o$ where $\tilde{L}_o$ is the leaf of $\tilde{\mathcal{F}}$ contained in $\tilde{\Lambda}$. Indeed, using Claim 7.2 and the construction in [6, § 2] (see for instance the proof of Proposition 1 therein) we can construct a closed meromorphic one-form $\tilde{\xi}$ defined $\tilde{\mathcal{F}}$ in a neighborhood $W(\tilde{L}_o)$ of $\tilde{L}_o$ in $\tilde{M}$, such that $\tilde{\xi}$ defines $\tilde{\mathcal{F}}$ outside the polar set $(\tilde{\xi})_\infty = \tilde{L}_o$ which is of order one. Let us now check that we may extend this one-form $\tilde{\xi}$ to a neighborhood of $\overline{A}$ in $\tilde{M}$.

Notice that, given a non-singular point $p \in \tilde{\Lambda}$ and a transverse disc Σ to $\tilde{\Lambda}$ at $p = \tilde{\Lambda} \cap \Sigma$, we may choose a holomorphic coordinate $z \in \Sigma$ such that the holonomy group $\mathrm{Hol}(\tilde{\mathcal{F}}, \tilde{L}, \Sigma, p)$ is linearized by the coordinate $z\colon (\Sigma, p) \to (\mathbb{C}, 0)$. Then, we have $\tilde{\xi}|_\Sigma = \frac{dz}{z}$. Now, the dynamics of $\tilde{\mathcal{F}}$ near M allows to extend $\tilde{\xi}$ to a neighborhood of M. Indeed, note that the holonomy of any leaf of $\mathcal{F}$ not contained in Λ is abelian linearizable (a subgroup of $Aff(\mathbb{C})$ with a fixed point at $0 \in \mathbb{C}$). Therefore, we may write locally $\tilde{\xi} = \frac{d\tilde{y}}{\tilde{y}}$ and extend the function y to a "multivalued" holomorphic function constant on the leaves of $\mathcal{F}$. Since the only nontrivial dynamics of $\mathcal{F}$ is concentrated on Λ we conclude that this extension is coherent. Hence, we may take $\tilde{\xi}$ as $\frac{d\tilde{y}}{\tilde{y}}$ for such an extension which gives the desired extension of $\tilde{\xi}$ (see [4] for a similar argumentation on the extension).

Finally, $\tilde{\xi}$ extends by transversality to a neighborhood W of M in $\tilde{M}$ and by Levi's Extension Theorem it extends to $W \cup \overline{A}$. Thus, we have obtained a closed meromorphic one-form $\tilde{\xi}$ in $W(\tilde{L}_o) \cup W \cup \overline{A}$ with simple poles so that $(\tilde{\xi})_\infty = \tilde{L}_o$. ☐

Now we complete our study with the proof of Theorem 1.3.

Proof of Theorem 1.3. The situation is similar to the one addressed in Theorem 1.2. The main difference is that we only assume an existence of an affine transverse structure for the restriction $\mathcal{F} = \tilde{\mathcal{F}}|_M$. On the other hand, we compensate by assuming that the intersection $L_o = \tilde{\Lambda} \cap M$ consists of a single leaf of $\mathcal{F}$. Preserve the notation in the proof of Theorem 1.2 above. Then we are in the above situation for $\mathcal{F}$, M, $\mathcal{G}$, $\tilde{\Lambda}$ and $\tilde{M}$, and assume that the leaf L_o of $\mathcal{F}$ contains a hyperbolic map in its holonomy group. Put $\Omega = \tilde{\Omega}|_M$ and consider η as given by the $\mathbb{C}$-affine transverse structure of $\mathcal{F}$ in $M \backslash L_o$. By Proposition 5.9 the one-form L_o extends to M.

Remark 7.3. There is an extension $\tilde{\eta}$ of η to a neighborhood $W(L_o)$ of L_o in $\tilde{M}$. This extension is closed meromorphic, satisfies $d\tilde{\Omega} = \tilde{\eta} \wedge \tilde{\Omega}$, and also has simple poles and the polar set $(\tilde{\eta})_\infty = \tilde{\Lambda}$.

Proof. For each regular point $q \in M \backslash sing\mathcal{F}$ we have that $\Omega = gdf$ for some meromorphic function g and some holomorphic function f, both defined in a neighborhood U of q in the complex manifold $\tilde{M}$. Furthermore, in $U \cap M$ we have

$$\eta = a\frac{df|_M}{f|_M} + \frac{dg|_M}{g|_M} + d\varphi,$$

for some transversely holomorphic function $\varphi : U \cap M \to \mathbb{C}$. From the above writing and from $d\Omega|_M = \eta \wedge \Omega|_M$ we get that $d\varphi \wedge \Omega\big|_M = 0$. Using a local flow box for $\mathcal{F}$ around q we can extend φ and therefore η to U. Indeed, φ is extended as a holomorphic first integral for $\mathcal{G}$ in U. Two such extensions, η in U and η' in U', are related by $\eta - \eta' = h\Omega$, where h is holomorphic in $U \cap U'$ and $d(h\Omega) = 0$. Therefore, $h\Omega = d\psi$ for some holomorphic first integral ψ for $\mathcal{G}$ in $U \cap U'$, and since $(\eta - \eta')|_{U \cap U' \cap M} \equiv 0$, it follows that ψ is constant in $U \cap U' \cap M$. Since M is transverse to $\mathcal{G}\big|_{U \cap U'}$, it follows that $d\psi \equiv 0$ and therefore $\eta \equiv \eta'$ in $U \cap U'$. Thus, we can extend η to a neighborhood of $M \backslash (sing\mathcal{F} \cap M)$. Using now classical Hartogs' extension theorem ([11, 12]) we can extend η to a neighborhood $W' \subset W$ of M in $\tilde{M}$ (recall that $sing\mathcal{F}$ has codimension $\geqslant 2$ in M). $\square$

The proof of Theorem 1.3 is then completed by applying Theorem 1.1 and the arguments in the final part of the proof of Theorem 1.2. $\square$

REFERENCES

[1] B. Azevedo Scárdua. Integration of complex differential equations. *J. Dynam. Control Systems*, 5(1):1–50, 1999. `doi:10.1023/A:1021740700501`.

[2] Michel Belliart, Isabelle Liousse, and Frank Loray. Sur l'existence de points fixes attractifs pour les sous-groupes de $\mathrm{Aut}(\mathbf{C}_{,0})$. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(4):443–446, 1997. `doi:10.1016/S0764-4442(97)80083-3`.

[3] Robert A. Blumenthal. Transversely homogeneous foliations. *Ann. Inst. Fourier (Grenoble)*, 29(4):vii, 143–158, 1979. `doi:10.5802/aif.771`.

[4] Marco Brunella. On holomorphic foliations in complex surfaces transverse to a sphere. *Bol. Soc. Brasil. Mat. (N.S.)*, 26(2):117–128, 1995. `doi:10.1007/BF01236988`.

[5] Marco Brunella. On transversely holomorphic flows. I. *Invent. Math.*, 126(2):265–279, 1996. `doi:10.1007/s002220050098`.

[6] C. Camacho, A. Lins Neto, and P. Sad. Foliations with algebraic limit sets. *Ann. of Math. (2)*, 136(2):429–446, 1992. `doi:10.2307/2946610`.

[7] D. Cerveau and J.-F. Mattei. *Formes intégrables holomorphes singulières*, volume 97 of *Astérisque*. Société Mathématique de France, Paris, 1982. URL: http://www.numdam.org/issues/AST_1982__97__1_0.

[8] D. Cerveau and R. Moussu. Groupes d'automorphismes de $(\mathbf{C}, 0)$ et équations différentielles $ydy + \cdots = 0$. *Bull. Soc. Math. France*, 116(4):459–488 (1989), 1988. `doi:10.24033/bsmf.2108`.

[9] Étienne Ghys. On transversely holomorphic flows. II. *Invent. Math.*, 126(2):281–286, 1996. `doi:10.1007/s002220050099`.

[10] Claude Godbillon. *Feuilletages*, volume 98 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1991. Études géométriques.

[11] Robert C. Gunning. *Introduction to holomorphic functions of several variables. Vol. I.* The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1990. Function theory. `doi:10.1201/9780203750063`.

[12] Robert C. Gunning. *Introduction to holomorphic functions of several variables. Vol. II.* The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole

Advanced Books & Software, Monterey, CA, 1990. Local theory. `doi:10.1201/9780203750056`.

[13] A. Haefliger and D. Sundararaman. Complexifications of transversely holomorphic foliations. *Math. Ann.*, 272(1):23–27, 1985. `doi:10.1007/BF01455925`.

[14] Toshikazu Ito and Bruno Scárdua. On the geometry of holomorphic flows and foliations having transverse sections. *Geom. Dedicata*, 111:23–42, 2005. `doi:10.1007/s10711-004-1553-5`.

[15] F. Loray. Feuilletages holomorphes à holonomie résoluble, 1994. PhD These. Université de Rennes I.

[16] Isao Nakai. Separatrices for nonsolvable dynamics on $\mathbf{C}, 0$. *Ann. Inst. Fourier (Grenoble)*, 44(2):569–599, 1994. `doi:10.5802/aif.1410`.

[17] M. Nicolau and E. Paul. A geometric proof of a galois differential theory theorems. pages 277–290. J. Mozo Fernandez, ed., Universidad de Valladolid, 1997.

[18] E. Paul. Feuilletages holomorphes singuliers à holonomie résoluble. *J. Reine Angew. Math.*, 514:9–70, 1999. `doi:10.1515/crll.1999.074`.

[19] B. Azevedo Scárdua. Transversely affine and transversely projective holomorphic foliations. *Ann. Sci. École Norm. Sup. (4)*, 30(2):169–204, 1997. `doi:10.1016/S0012-9593(97)89918-1`.

[20] A. Scherbakov. On the density of an orbit of a pseudogroup of conformal mappings. *Proceedings of the Petrovskii Seminar*, 10:170–192, 1984.

[21] Bobo Seke. Sur les structures transversalement affines des feuilletages de codimension un. *Ann. Inst. Fourier (Grenoble)*, 30(1):ix, 1–29, 1980. `doi:10.5802/aif.773`.

[22] Yum Tong Siu. *Techniques of extension of analytic objects*, volume Vol. 8 of *Lecture Notes in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1974.

[23] Bruno Wirtz. Fixed points and entropy for nonsolvable dynamics on $\mathbf{C}, 0$. *Bol. Soc. Brasil. Mat. (N.S.)*, 29(1):53–74, 1998. `doi:10.1007/BF01245868`.

Received: April 7, 2022, accepted: 30 June, 2023.

Liliana Jurado
IMCA, Inst. Mat. Ciencias Afines, Calle los Biologos 245, La Molina, Lima, Peru
Email: `lilianajurado1125@gmail.com`

Bruno Scardua
Instituto de Matematica, UFRJ, CP 68530. Rio de Janeiro, RJ, 21945-970, Brazil
Email: `bruno.scardua@gmail.com`
ORCID: `0000-0001-8280-1669`