

On diffeological principal bundles of non-formal pseudo-differential operators over formal ones

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Abstract. We describe the structure of diffeological bundle of non formal classical pseudo-differential operators over formal ones, and its structure group. For this, we give results on diffeological principal bundles with (a priori) no local trivialization including an Ambrose-Singer theorem, use the smoothing connections already exhibited by the author in previous works, and finish with open questions.

Анотація. В роботі описано структуру диференціальних розшарувань неформальних класичних псевдо-диференціальних операторів над формальними, та досліджено відповідні структурні групи. Для цього використовується техніка згладжувальних зв'язностей, розвинена в попередніх роботах автора, а також доведено декілька результатів про дифеологічні головні розшарування, які не обов'язково мають локальні тривіалізації, і, зокрема, встановлено аналог теореми Амброза-Зінгера. В кінці статті обговорюються деякі відкриті питання.

1. INTRODUCTION

The correspondence between formal and non-formal pseudo-differential operators plays a highly important role in analysis. Over the full interest of non-formal pseudo-differential operators for spectral analysis, theory of partial differential equations and differential operators, integration on infinite dimensional spaces, renormalization among other themes, over the crucial place of formal pseudo-differential operators for integrable systems, theory of r -matrices, representation theory, deformation quantization among other fields of studies, it often happens that technical steps can be overcome by passing from a family of non-formal pseudo-differential operators to its formal counterpart, and by interpreting the results. The reverse procedure

2020 Mathematics Subject Classification: 53C05, 57R55, 58B05, 58B10, 58J40.

Keywords: diffeology, principal bundle, holonomy, pseudo-differential operator, smoothing operator, index theory

DOI: <http://dx.doi.org/10.15673/pigc.v16i2.2298>

also exists, but less often because there is no canonical way to associate a non-formal pseudo-differential operator to a formal one.

One geometric description of this correspondence between non-formal and formal pseudo-differential operators is the aim of this work. In particular, we are pleased to produce a geometric interpretation of the lack of canonical way to associate a non-formal operator to a formal one. For this, and in order to circumvent a lack of differential geometric framework, we use diffeologies, a generalized setting for differential geometry first describe by Chen and Souriau. In this setting, we describe a structure analog to the one of principal bundles, with mild weaker conditions that are described. On this very general structure, we prove a reduction theorem and an Ambrose-Singer theorem for a given connection, when the structure group is fully regular, and show that the existence of horizontal lifts of the paths of the base not only depends on the properties of the structure group, but also on the diffeological properties of the bundle projection.

Our motivating example for this generalized setting relies on groups of unbounded, classical pseudo-differential operators. For this principal bundle and in an infinite dimensional geometric vocabulary, we review old and recent results. In this principal bundle, the question of the existence of local sections is addressed as deeply related to the problem of finding a canonical way to associate a non-formal operator to a formal one. This leads to open questions that may help to clarify, for example, the results on the index of pseudo-differential operators that actually obtained through arguments coming from K-theory.

2. ON DIFFEOLOGICAL BUNDLES

2.1. Preliminary notions. The main reference for a comprehensive exposition on diffeologies is [10]. This reference will be completed all along the text for specific concerns, and we already quote the review [8] that is centered on infinite dimensional examples, that may be useful for readers non-familiar with infinite dimensional topics.

Definition 2.2 (Diffeology). Let X be a set. A *parametrization* of X is a map of sets $p: U \rightarrow X$ where U is an open subset of Euclidean space (no fixed dimension). A *diffeology* \mathcal{P} on X is a set of parametrizations satisfying the following three conditions:

- (i) (Covering) $\forall x \in X, \forall n \in \mathbb{N}$, the constant function $p: \mathbb{R}^n \rightarrow \{x\} \subset X$ is in \mathcal{P} .
- (ii) (Locality) Let $p: U \rightarrow X$ be a parametrization such that for every $u \in U$ there exists an open neighbourhood $V \subset U$ of u satisfying $p|_V \in \mathcal{P}$. Then $p \in \mathcal{P}$.

- (iii) (Smooth Compatibility) Let $(p: U \rightarrow X) \in \mathcal{P}$. Then for every n , every open subset $V \subset \mathbb{R}^n$, and every smooth map $F: V \rightarrow U$, we have $p \circ F \in \mathcal{P}$.

A set X equipped with a diffeology \mathcal{P} is called a *diffeological space*, and the parametrizations $p \in \mathcal{P}$ are called *plots*.

Let (X, \mathcal{P}) and (Y, \mathcal{P}') be two diffeological space. Then $f: X \rightarrow Y$ is smooth if $f(\mathcal{P}) \subset \mathcal{P}'$.

Definition 2.3. Let X and Y be diffeological spaces. The *functional diffeology* on the set $\mathcal{C}^\infty(X, Y)$ of all smooth maps from X to Y is given by the following condition: A parametrization $Q: V \rightarrow \mathcal{C}^\infty(X, Y)$ is a plot for the functional diffeology if and only if for every plot $P: U \rightarrow X$, the parametrization $Q \odot P: V \times U \rightarrow Y$ with $(Q \odot P)(r, s) = Q(r)(P(s))$ is a plot in Y .

Indeed, the functional diffeology on $\mathcal{C}^\infty(X, Y)$ is the coarsest diffeology for which $\text{ev}: \mathcal{C}^\infty(X, Y) \times X \rightarrow Y$ defined by $\text{ev}(f, x)$ to $f(x)$ is smooth.

At this step of the exposition, and for the clarity of developments above, we have to precise some technical features and necessary notations. Any finite or infinite dimensional manifold M has a diffeology $\mathcal{P}_\infty(M)$ called *nebulae*, constructed as follows: let O be an open subset of a euclidean space and let us note $\mathcal{P}_\infty(M)_O = \mathcal{C}^\infty(O, M)$ (in the usual sense). The nebulae diffeology $\mathcal{P}_\infty(M)$ is

$$\mathcal{P}_\infty(M) = \bigcup_O \mathcal{P}_\infty(M)_O$$

where the latter union is extended over all open sets $O \subset \mathbb{R}^n$ for $n \in \mathbb{N}$.

We choose to note by $\mathcal{P}(X)$ the diffeology of a diffeological space X . From this first diffeology, one can produce many others, and especially two of them of technical interest, the second one generalizing the nebulae diffeology in a way new to our knowledge:

- The 1-dimensional diffeology $\mathcal{P}_1(X)$ made of plots $p \in \mathcal{P}(X)$ that read locally as $p|_O = \gamma \circ f$ where $\gamma \in \mathcal{C}^\infty(\mathbb{R}, X)$, O is an open subset of an Euclidian space and $f \in \mathcal{C}^\infty(O, \mathbb{R})$. This diffeology is called *spaghetti diffeology* in [5] and *1-dimensional diffeology* in [19].
- The *nebulae* diffeology, which definition which follows refines the one given independently in [18, 19] and [30], and extends the definition of the nebulae diffeology of a manifold. Let O be an open subset of a euclidean space equipped with its \mathcal{P}_1 -diffeology;

$$\mathcal{P}_\infty(\mathcal{F})_O = \{f \in \mathcal{C}^\infty(O, X)\}$$

in the sense of $\mathcal{P}_1(0)$ and $\mathcal{P}(X)$, and

$$\mathcal{P}_\infty(\mathcal{F}) = \bigcup_O \mathcal{P}_\infty(\mathcal{F})_O$$

where the latter union is extended over all open sets $O \subset \mathbb{R}^n$ for $n \in \mathbb{N}$. This notion refines the notion of nebulae diffeology in the following sense: there exists some diffeological space X which have very few smooth maps from X to \mathbb{R} , and the definition of the nebulae diffeology presented in e.g. [18, 19] is based on the existence of "enough" smooth functions in $\mathcal{C}^\infty(X, \mathbb{R})$. Therefore, this "new" notion of nebulae diffeology seems more intrinsic to us, and not exactly the same as the one presented in the existing litterature.

A group G is a *diffeological group* if multiplication and inversion are smooth for the underlying diffeology. Following [10, 14], a diffeological group has a *kinematic tangent space at identity*, that is, a vector space of tangent germs of paths [14], but the Lie bracket may not exist. If there is a smooth Lie bracket, G is called *diffeological Lie group*. If moreover there is an exponential map from $\mathcal{C}^\infty(\mathbb{R}, \mathfrak{g})$ to $\mathcal{C}^\infty(\mathbb{R}, G)$, that is, a smooth map \exp such that $\exp(v)(t) = g(t)$ if and only if g is the unique solution of the (right logarithmic) differential equation

$$\partial_t g \cdot g^{-1} = v$$

with initial value

$$g(0) = 1_G$$

then G is called *regular* [14]. If \mathfrak{g} is regular as an abelian diffeological group (vector space) we say that G is *fully regular*. All these notions coincide with more classical notions of regular Lie groups (see e.g. [13, 24]) and are reviewed in [21, 22].

2.4. On the way to diffeological principal bundles. We describe here a class of objects that generalize principal bundles. There are actually many generalizations of the notion of principal bundle to a diffeological setting, with more or less assumptions, depending on the results that we have to produce. To our best knowledge, if the fibers are assumed to be isomorphic to the structure group G , the largest existing generalization is the first the has been produced, in [29]. In this technically difficult setting, many subclasses have been considered and since then, to our knowledge, most results are stated for diffeological principal bundles *with local trivializations* while there exists examples where we do not know whether a system of local trivializations exists or not, and certainly examples where such local trivializations do not exist. Since non-existence is often more difficult to

prove than existence, we feel as normal the actual situation where there exists some examples where one can only say that there is an actual technical impossibility to prove existence or non-existence of local trivializations.

Example 2.5. The non-commutative torus [9] is a diffeological space over which it seems to exist no locally trivial and non-trivial principal bundle, at least with subordinate partition of the unit. But again, we state here our own technical incapacity and not a non existence theorem.

In view of the example that is developed in next section, we need to consider the following definition:

Definition 2.6. Let G be a diffeological Lie group acting, smoothly and freely, on the right on a diffeological (total) space P . Then we get a *diffeological principal bundle* P with structure group G over P/G . Here, P/G is equipped with any fixed diffeology for which the canonical projection $\pi : P \rightarrow P/G$ is smooth.

Remark 2.7. This definition is even weaker than the definition of "structure quantique" by Souriau in [29] where the diffeology of the base is fixed assuming the projection π to be a subduction. In our approach, the action of G on P is not assumed to be proper, and π is only assumed to be smooth.

Remark 2.8. This is always possible to find at least two diffeologies on P/G (one if they coincide) for which the canonical projection $\pi : P \rightarrow P/G$ is smooth along the lines of last definition: the quotient diffeology (which is the minimal one for inclusion) and the discrete diffeology (which is maximal for inclusion).

Classically, connections are 1-forms $\theta \in \Omega^1(P, \mathfrak{g})$ (see [10, 19] for a comprehensive definition of differential forms, de Rham differential and wedge product) that are covariant under the right action of G , that is, denoting the right action of $g \in G$ by R_g , we have:

$$(R_g)_*\theta = Ad_{g^{-1}}\theta.$$

We keep this definition for a connection 1-form in the considered class of diffeological principal bundles, mimicking what [19] did in a more specific context, even if the existence of connection 1-forms remains to be checked on the examples considered (see e.g. [11] for a slightly different framework in which similar technical issues are analyzed). Once this setting assumed, one can define the curvature $F(\theta)$ of a connection θ by

$$F(\theta) = d\theta - \frac{1}{2}\theta \wedge \theta.$$

In [10], another notion of connection is defined, developed independently in [19] under the terminology of horizontal lifts, which consists in lifting smooth paths of the base to smooth horizontal paths on the total space. Here again, we have to investigate deeply this relationship, already established in a more restricted framework [19].

Theorem 2.9. *If G is fully regular, then any connection 1-form θ defines a horizontal path projection, i.e. a smooth map*

$$H: \mathcal{C}^\infty(\mathbb{R}, P) \rightarrow \mathcal{C}^\infty(\mathbb{R}, P)$$

such that $H \circ H = H$, $H\gamma(0) = \gamma(0)$ and $\theta(\partial_t H\gamma) = 0$, and which is G -equivariant.

Conversely, if there exists such a horizontal path projection H , there exists only one connection 1-form θ which defines H .

Proof. Let $\gamma \in \mathcal{C}^\infty(\mathbb{R}, P)$. Consider the principal bundle γ^*P over \mathbb{R} and with structure group the fully regular group G . Then the differential equation

$$\partial_t g \cdot g^{-1} = -\theta(\partial_t \gamma(t))$$

has a unique solution. We define $H(\gamma)(t) = \gamma(t) \cdot g(t)$ which is by construction a smooth map, and a direct computation shows that

$$\theta(\partial_t H\gamma) = 0$$

and that it is G -equivariant. The property $H \circ H = H$ is obvious. The converse is noticed in [10]. \square

Definition 2.10. Let X be a diffeological spaces. Let $O \subset X$ be a non-empty, path connected set and let $x_0 \in O$. the set O is *weakly star-shaped* (for its subset diffeology) if there exists a smooth map

$$Rad_{x_0}: O \rightarrow \mathcal{C}^\infty([0, 1], O)$$

such that

- $\forall x \in O, Rad_{x_0}(x)(0) = x_0,$
- $\forall x \in O, Rad_{x_0}(x)(1) = x.$

Remark 2.11. Trivially, a star-shaped set is weakly star-shaped.

Remark 2.12. By its smoothness, the map Rad_{x_0} defines a smooth contraction from O to $\{x_0\}$, which shows that a weakly star-shaped set needs to be smoothly contractible.

Now, let us turn to holonomy. Let $p \in P$ and γ a smooth path in P starting at p , defined on $[0; 1]$. We have proved that horizontal lifts are well-defined, as in the case of manifolds, when G is fully regular. Moreover,

the map $H(\cdot)$ defines trivially a path-lifting along the lines of [19]. This enables us to consider the holonomy group of the connection. Notice that a straightforward adaptation of the arguments of [18] shows that the holonomy group is invariant (up to conjugation) under the choice of the base point p . We note it now \mathcal{H} , omitting the base point p and the connection θ in our notations since they are assumed fixed. We note by H -path the set of horizontal paths in P starting at p . We now turn to reduction of the structure group, adapting [16, 19]. For this, we also consider

$$P\mathcal{H} = \{q \in P \mid \exists \gamma \in H\text{-path}, \gamma(1) = q\}.$$

Theorem 2.13. *We assume that P is connected and that G_1 and G are fully regular diffeological groups with Lie algebras \mathfrak{g}_1 and \mathfrak{g} , respectively. Let $\rho: G_1 \mapsto G$ be an injective morphism of diffeological Lie groups. If there exists a connection θ on P , with curvature Ω , such that the restriction of Ω to $P\mathcal{H}$ is a smooth \mathfrak{g}_1 -valued 2-form and if M is simply connected, then the structure group G of P reduces to \tilde{G}_1 , a fully regular diffeological Lie group which has the same connected component of the neutral element as G_1 , and the connection θ also reduces to P_1 , a \tilde{G}_1 -principal bundle such that*

$$P\mathcal{H} \subset P_1 \subset P$$

with smooth inclusion.

Proof. Let us fix a plot of $h: U \rightarrow P\mathcal{H}$ that contains p , for which we assume that U is weakly starshaped and that $H\partial_t \text{Rad}_p(x) = 0$. Then, following the proof of the claim of [13, Theorem 39.2] with a connection with non vanishing curvature, see e.g. [16], we have:

$$\begin{aligned} \partial_s(h^*\tilde{\theta})(\partial_t) &= \partial_t(h^*\tilde{\theta})(\partial_s) - d(h^*\tilde{\theta})(\partial_t, \partial_s) - (h^*\tilde{\theta})([\partial_t, \partial_s]) \\ &= \partial_t(h^*\tilde{\theta})(\partial_s) - d(h^*\tilde{\theta})(\partial_t, \partial_s) \\ &= \partial_t(h^*\tilde{\theta})(\partial_s) + ad_{(h^*\tilde{\theta})(\partial_t)}((h^*\tilde{\theta})(\partial_s)) - (\Psi^*\Omega)(h_*(\partial_s), h_*(\partial_t)) \\ &= \partial_t(h^*\tilde{\theta})(\partial_s) - (h^*\Omega)(h_*(\partial_s), h_*(\partial_t)) \quad \text{since } (h^*\tilde{\theta})(\partial_t) = 0. \end{aligned}$$

Therefore, applying the same arguments as in [19],

$$\theta(\partial_s(\psi \circ c)) = \int_0^1 \left(\Omega(h_*(\partial_s), h_*(\partial_t))(t) \right) dt. \tag{2.1}$$

Since G_1 is fully regular, \mathfrak{g}_1 is regular (as a Lie algebra), this integral exists and belongs to \mathfrak{g}_1 .

Using this first calculation, we can divide the image of any horizontal path starting at p as a finite number of domains which are weakly star-shaped. Therefore, evaluating the holonomy of any path in $P\mathcal{H}$ which lies in G_1 , we define the principal bundle $P_1 = P\mathcal{H}.G_1$. We define a smooth

diffeology on P_1 which is generated by the product maps $h.g$, where h is a plot of the subset diffeology on $P\mathcal{H}$ and g is a plot for the diffeology on G_1 . With this diffeology:

- the inclusion map $P_1 \rightarrow P$ is smooth,
- the horizontal lift map $\alpha \mapsto H\alpha$ is a smooth map,
- the connection θ restricts to a smooth \mathfrak{g}_1 -valued form.

This ends the proof of the reduction theorem, since G_1 -right invariant trivially. \square

Lemma 2.14. *If G_1 and G_2 are fully regular diffeological Lie groups, then $G_1 \cap G_2$ is a fully regular diffeological Lie group.*

Proof. The proof is straightforward because the logarithmic derivative equation

$$g^{-1}dg = v$$

has an unique solution in G_1 , in G_2 , therefore, if v is a smooth path in $\mathfrak{g}_1 \cap \mathfrak{g}_2$, the integral path is with values in $G_1 \cap G_2$ and smooth for the intersection of the diffeologies. The same arguments hold to prove that the Lie algebra $\mathfrak{g}_1 \cap \mathfrak{g}_2$ is regular. \square

As a consequence, we get the following Ambrose-Singer theorem, which ends the investigations successively exposed in [16, 18, 19]:

Theorem 2.15. *Given a connection θ on P , there is a unique fully regular diffeological Lie group to which θ reduces, called fully regular holonomy group, which Lie algebra is spanned by the curvature elements of θ on $P\mathcal{H}$.*

Proof. By last Lemma, Zorn's Lemma applies to the set of subgroups G_1 of G , that are fully regular and for which inclusion in G is a smooth map. Therefore, its minimal element is the fully regular holonomy group of θ , defined for a fixed base point p and up to conjugation classes. The same holds for the Lie algebra. \square

We have now to investigate the lifts of paths on the base P/G to horizontal paths on the total space P .

Theorem 2.16. *Let θ be a connexion 1-form on P . If G is fully regular, then there exists a horizontal lift*

$$L: \mathcal{C}^\infty(\mathbb{R}, P/G) \rightarrow \mathcal{C}^\infty(\mathbb{R}, P)$$

such that

$$H = L \circ \pi$$

if and only if, for the corresponding 1-dimensional (spaghetti) diffeologies,

$$\mathcal{P}_1(P/G) = \pi_*\mathcal{P}_1(P).$$

Moreover, L is smooth if and only if

$$\mathcal{P}(P/G) = \pi_*\mathcal{P}(P).$$

Proof. Since H exists, L exists if and only if any smooth path on P/G is locally the projection of a smooth path of P , which can be formulated as $\mathcal{P}_1(P/G) = \pi_*(\mathcal{P}_1(P))$.

Since H is smooth, analysing the canonical family of plots generating the functional diffeologies of $\mathcal{C}^\infty(\mathbb{R}, P)$ and $\mathcal{C}^\infty(\mathbb{R}, P/G)$, we obtain the condition announced $\mathcal{P}(P/G) = \pi_*\mathcal{P}(P)$. \square

Theorem 2.17. *Let θ be a connection on P . If*

- G is fully regular,
- L is smooth,
- the base P/G can be covered by open subsets that are weakly star-shaped,

then P is locally trivial.

Proof. If $O \subset P/G$ is open and weakly star-shaped, then there exists a base point $x_0 \in O$ and radial paths starting at x_0 and covering O , that is, a smooth map

$$Rad : O \rightarrow \mathcal{C}^\infty([0, 1], O)$$

defined as before. Therefore, the map

$$\Phi : (x, g) \in O \times G \mapsto (L(Rad_x)(1)) .g$$

furnishes the desired local trivializations. \square

3. APPLICATION TO GROUPS OF PSEUDO-DIFFERENTIAL OPERATORS

3.1. Preliminaries on classical pseudo-differential operators. We introduce groups and algebras of non-formal pseudo-differential operators. Basic definitions are valid for real or complex finite-dimensional vector bundles E over a compact manifold M without boundary whose typical fiber is a finite-dimensional real or complex vector space V . We begin with the following definition after [2, Section 2.1].

Definition 3.1.1. The graded algebra of differential operators acting on the space of smooth sections $\mathcal{C}^\infty(M, E)$ is the algebra $DO(E)$ generated by:

- Elements of $\text{End}(E)$, the group of smooth maps $E \rightarrow E$ leaving each fibre globally invariant and which restrict to linear maps on each fibre. This group acts on sections of E via (matrix) multiplication.

- The differentiation operators

$$\nabla_X : g \in \mathcal{C}^\infty(M, E) \mapsto \nabla_X g$$

where ∇ is a connection on E and X is a vector field on M .

Multiplication operators are operators of order 0; differentiation operators and vector fields are operators of order 1. In local coordinates, a differential operator of order k has the form

$$P(u)(x) = \sum p_{i_1 \dots i_r}(x) \nabla_{X_{i_1}} \cdots \nabla_{X_{i_r}} u(x), \quad r \leq k$$

in which u is a (local) section and the coefficients $p_{i_1 \dots i_r}$ can be matrix-valued. The algebra $DO(M, E)$ is filtered by order: we note by $DO^k(M, E)$, $k \geq 0$, the differential operators of order less or equal than k .

Now we embed $DO(M, E)$ into the algebra of classical pseudo-differential operators. We need to assume that the reader is familiar with the basic facts on pseudo-differential operators defined on a vector bundle $E \rightarrow M$; these facts can be found for instance in [6], in the review [26, Section 3.3], and in the papers [3] and [31] in which the authors construct a global symbolic calculus for pseudo-differential operators showing, for instance, how the geometry of the base manifold M furnishes an obstruction to generalizing local formulas of composition and inversion of symbols.

Notations. We denote by $PDO(M, E)$ the space of pseudo-differential operators on smooth sections of E , see [26, p. 91]; by $PDO^o(M, E)$ the space of pseudo-differential operators of order o ; and by $Cl(M, E)$ the space of classical pseudo-differential operators acting on smooth sections of E , see [26, pp. 89-91]. We also denote by

$$Cl^o(M, E) = PDO^o(M, E) \cap Cl(M, E)$$

the space of classical pseudo-differential operators of order o , and by the symbol $Cl^*(M, E)$ the group of units of $Cl(M, E)$. In the sequel, the mixed notation $Cl^{o,*}(M, E)$ for $Cl^o(M, E) \cap Cl^*(M, E)$ will be extensively used.

A topology on spaces of classical pseudo-differential operators has been described implicitly by Kontsevich and Vishik in [12]: it is a Fréchet topology (and therefore it equips $Cl(M, E)$ with a smooth structure) such that each space $Cl^o(M, E)$ is closed in $Cl(M, E)$. This topology is discussed in [26, pp. 92-93], see also [4, 25, 28] for more explicit and rigorous descriptions. We will refer to it in this work under the terminology of *Kontsevich-Vishik topology*.

We set

$$PDO^{-\infty}(M, E) = \bigcap_{o \in \mathbb{Z}} PDO^o(M, E)$$

It is well-known that $PDO^{-\infty}(M, E)$ is a two-sided ideal of $Cl(M, E)$, closed for the Kontsevich-Vishik topology, see [6] and also [28] for topological aspects. This fact allows us to define the quotients

$$\mathcal{F}Cl(M, E) = Cl(M, E)/PDO^{-\infty}(M, E)$$

and

$$\mathcal{F}Cl^o(M, E) = Cl^o(M, E)/PDO^{-\infty}(M, E).$$

The script font \mathcal{F} stands for formal pseudo-differential operators. The quotient $\mathcal{F}PDO(M, E)$ is an algebra isomorphic to the space of formal symbols, see [3], and the identification is a morphism of \mathbb{C} -algebras for the usual multiplication on formal symbols (appearing for instance in [6, Lemma 1.2.3] and [26, p. 89], and in [28, Section 1.5.2, Equation (1.5.2.3)] for the particular case of classical symbols). Sets of formal classical pseudo-differential operators have been equipped with a topology, before and independently of the works of Kontsevich and Vishik, in [1].

Theorem 3.1.2. *The groups $Cl^{0,*}(M, E)$ and $\mathcal{F}Cl^{0,*}(M, E)$, in which group $\mathcal{F}Cl^{0,*}(M, E)$ is the group of units of the algebra $\mathcal{F}Cl^0(M, E)$, are regular Fréchet Lie groups equipped with smooth exponential maps. Their Lie algebras are $Cl^0(M, E)$ and $\mathcal{F}Cl^0(M, E)$ respectively.*

Regularity is reviewed in [22] and also in Paycha’s lectures, see [26, p. 95]. The Lie group structure of $Cl^{0,*}(M, E)$ is discussed in [26, Proposition 4]. Theorem 3.1.2 is essentially proven in [17]: it is noted in this reference that the results of [7] imply that the group $Cl^{0,*}(M, E)$ (resp. $\mathcal{F}Cl^{0,*}(M, E)$) is open in $Cl^0(M, E)$ (resp. $\mathcal{F}Cl^0(M, E)$) and that therefore it is a regular Fréchet Lie group.

3.2. Diffeologies, topologies and quotients by smoothing operators. We equip the ideal of smoothing operators $PDO^{-\infty}$ with its Fréchet topology on smooth kernels, which can be understood as the subset topology for the Kontsevich-Vishik topology. The algebras $PDO(M, E)$, $Cl(M, E)$ and $\mathcal{F}Cl(M, E)$ are equipped with diffeologies such that:

- (1) diffeologies make addition, multiplication and inversion smooth (i.e. they are diffeological algebras), and the L^2 -adjoint operation $(\cdot)^*$ is smooth.
- (2) The subset diffeology on $PDO^{-\infty}(M, E)$ is the nebulae diffeology on smooth kernels,
- (3) the quotient maps

$$PDO(M, E) \rightarrow \mathcal{F}PDO(M, E) \quad \text{and} \quad Cl(M, E) \rightarrow \mathcal{F}Cl(M, E)$$

are smooth.

We have to notice that

- if $Cl(M, E)$ equipped with the Kontsevich-Vishik topology,
- if $\mathcal{F}Cl(M, E)$ is equipped with the Adams, Ratiu and Schmid topology,

then their nebulae diffeology fulfill these assumptions. But we wish to avoid non necessary restrictions of frameworks in this work. Indeed, these topologies are actually adapted to spectral and microlocal analysis in the actual state of the art, and have never been questioned to our knowledge. This is not the place here to make this investigation, but in order to leave this question open, we only choose to assume assumptions (1)-(3), keeping in mind the two topologies mentioned before (and their underlying nebulae diffeologies) as a field of application.

For the sake of clarity for the reader, after these long topological and diffeological precisions, we feel the need to recall that $Cl^*(M, E)$ (resp. $\mathcal{F}Cl^*(M, E)$), the group of invertible operators of $Cl(M, E)$ (and respectively $\mathcal{F}Cl(M, E)$), is equipped with its subset diffeology in Cl (and respectively $\mathcal{F}Cl(M, E)$). Notice that the Lie brackets exist in these algebras. Let us first give an algebraic construction, which will be discussed from the diffeological viewpoint in next section.

Proposition 3.2.1. *If $G = \{Id + R \in Cl^* \mid R \in PDO^{-\infty}\}$, then $\mathcal{F}Cl^*$, as an algebra, is isomorphic to the quotient Cl^*/G .*

Proof. We have directly that $Cl^*(M, E)/G \subset \mathcal{F}Cl^*(M, E)$.

Now, let $A \in Cl(M, E)$ be such that its associated formal operator a is invertible in $\mathcal{F}Cl(M, E)$. Then a^{-1} is the formal operator associated to some operators $B \in Cl(M, E)$ (resp. $PDO(M, E)$) such that $AB = Id + R$, where $R \in PDO^{-\infty}(M, E)$.

Let us first assume that A (and a) are of order 0. Then the partial symbol of order 0 (the principal symbol) is invertible and hence the operator A is Fredholm. Setting p_K the L^2 orthogonal projection on its kernel, and p_I the orthogonal projection on the orthogonal complement of the image of A . These two subspaces are finite dimensional and made of smooth sections. Therefore, one can find $(\lambda, \mu) \in \mathbb{R}^2$ such that

$$A' = A + \lambda P_I + \mu P_K$$

and such that $A' \in Cl^{0,*}(M, E)$. By construction, the formal symbol associated to A' is a , and we get

$$\begin{aligned} R &= (A')^{-1}A - Id \\ &= (A')^{-1}(A' - \lambda P_I - \mu P_K) - Id \\ &= (A')^{-1}(-\lambda P_I - \mu P_K) \in PDO^{-\infty}(M, E). \end{aligned}$$

Now, $\text{ord}(A) = o \in \mathbb{Z}$. Let Δ be a (positive) Laplacian on $\mathcal{C}^\infty(M, E)$. Then $(Id + \Delta)^{-o/2}A$ is of order 0, to which one can apply the last construction. We finish the proof by remarking that smoothing operators are not invertible in $Cl(M, E)$. \square

3.3. Diffeological short exact sequences and principal bundles with connections. The construction of $\mathcal{F}Cl(M, E)$ can be rephrased into a short exact sequence

$$0 \rightarrow PDO^{-\infty}(M, E) \rightarrow Cl(M, E) \rightarrow \mathcal{F}Cl(M, E) \rightarrow 0$$

which is *diffeological* with respect to the diffeological assumptions (1)-(3) of section 3.2, that is, each arrow is smooth in the diffeological sense. According to Proposition 3.2.1 and under the same diffeological conditions, we get a short diffeological exact sequence of groups

$$1 \rightarrow G \rightarrow Cl^*(M, E) \rightarrow \mathcal{F}Cl^*(M, E) \rightarrow 1 \tag{3.1}$$

Unfortunately, Proposition 3.2.1 does not assert that groups $\mathcal{F}Cl^*(M, E)$ and $Cl^*(M, E)/G$ are diffeomorphic because we cannot decide actually if the diffeology of $\mathcal{F}Cl^*(M, E)$ as the group of units of $\mathcal{F}Cl(M, E)$ is the same as the quotient diffeology. We get analogous properties for diffeological exact sequences of bounded operators

$$0 \rightarrow PDO^{-\infty}(M, E) \rightarrow Cl^0(M, E) \rightarrow \mathcal{F}Cl^0(M, E) \rightarrow 0$$

and

$$1 \rightarrow G \rightarrow Cl^{0,*}(M, E) \rightarrow \mathcal{F}Cl^{0,*}(M, E) \rightarrow 1.$$

These exact sequences have actually no local slice $\mathcal{F}Cl^*(M, E) \rightarrow Cl^*(M, E)$ or $\mathcal{F}Cl^{0,*}(M, E) \rightarrow Cl^{0,*}(M, E)$, in other words the diffeological G -principal bundle $Cl^{0,*}$ over $\mathcal{F}Cl^{0,*}$ has actually no local trivialization (according to the Adams, Ratiu and Schmid topology which is the only one studied explicitly).

This is at this point that we carry now a new element, that generalize the constructions given in [20] which treats the case $M = S^1$.

Theorem 3.3.1. *There exists $PDO^{-\infty}$ -valued connections on the principal bundle $Cl^*(M, E)$ over $\mathcal{F}Cl^*(M, E)$. In particular, we define, for $s \in Cl^{-\infty}(M, E)$ and $(a, b) \in (Cl(M, E))^2$, three such classes of connections:*

$$\begin{aligned} \Theta_a^{s,l}b &= sas^*b, \\ \Theta_a^{s,r}b &= bsas^*, \\ \Theta_a^{s,\square}b &= [sas^*, b], \end{aligned}$$

and extend them by right-invariance on the full space $TCl^*(M, E)$.

Proof. Recall that, for the diffeologies under consideration, multiplication, adjoint and addition are smooth. Therefore, the right invariant vector fields on $TCl^*(M, E)$ defined by

$$\begin{aligned} g \in Cl^*(M, E) &\mapsto g^{-1}ag, \\ g \in Cl^*(M, E) &\mapsto g^{-1}bg, \\ g \in Cl^*(M, E) &\mapsto g^{-1}sg, \\ g \in Cl^*(M, E) &\mapsto g^{-1}s^*g, \end{aligned}$$

are smooth vector fields, and for the same reasons, when evaluation on right-invariant vector fields above, the map

$$(s, a, b) \mapsto (\Theta_a^{s,l}b, \Theta_a^{s,r}b, \Theta_a^{s,\square}b)$$

is smooth. By direct computation, each connection form is G -covariant. \square

Remark 3.3.2. By restriction, these constructions also show that there exists $PDO^{-\infty}$ -valued connections on the principal bundle $Cl^{0,*}(M, E)$ over $\mathcal{F}Cl^{0,*}(M, E)$.

Therefore, we have the following:

Remark 3.3.3. If there exists diffeologies on $Cl^*(M, E)$ and $\mathcal{F}Cl^*(M, E)$ such that Theorem 2.17 applies to at least one $PDO^{-\infty}(M, E)$ -valued connection, then there exists local trivializations for the diffeological principal bundle $Cl^*(M, E)$ over $\mathcal{F}Cl^*(M, E)$ and hence local slices to the exact sequence (3.1).

3.4. Yet another class of connections and the Schwinger cocycle when $M = S^1$. There is another class of smoothing connections defined by

$$\Theta_a^{\epsilon(D)}b = b[a, \epsilon(D)]$$

where $\epsilon(D) = D \cdot |D|^{-1}$, and $D = -i \frac{d}{dx}$. This connection fulfills also the requirements of Theorem 3.3.1 by the same arguments, see e.g. [20]. If the diffeology on Cl^* has *enough* 2-dimensional plots, one can define the curvature of the connection [19].

The same property of the curvature holds for a second class of connections, studied in [15]:

$$\Theta_a^+b = \frac{1}{2}b(1 + \epsilon(D))a$$

Remark 3.4.1. We have here to notice that Θ^+ is not $PDO^{-\infty}(M, E)$ -valued, but its curvature is.

Then we get, following [15] and denoting by Ω the curvature of Θ^+ :

Theorem 3.4.2. $tr(\Omega(a, b)(1 + \epsilon))$ is cohomologous to the Schwinger cocycle.

4. OPEN PROBLEMS AND PERSPECTIVES

The principal bundle construction for Cl^* gives new geometric lights about the correspondence between non-formal and formal pseudo-differential operators. To our knowledge, this correspondence is only treated actually from the (algebraic) viewpoint of K-theory. This approach raises the following two questions.

- (i) The question of the adequate topologies for such groups of operators. This question is difficult due to the hard technicalities that one can meet when dealing with such objects. But the question of local maps from $\mathcal{F}Cl^*(M, E)$ to $Cl^*(M, E)$ has its own interest. One can ask if the statement of homotopy equivalence between $Cl^*(M, E)$ and $\mathcal{F}Cl^*(M, E)$ can be obtained through a weak contractibility of G . The necessary results actually do not exist in the context of diffeological principal bundles in the class that we have to consider.
- (ii) This first question is related to index theory, actually again only related through K-theory, or in a slightly different approach through the restricted linear group following [27] for $M = S^1$. Indeed, for bounded pseudo-differential operators that are Fredholm, K-theory produces a link between homotopy of operators, their index, and the homotopy of their principal symbols. The same picture appears here in a heuristic way: our group G , which contains all finite rank projectors, is topologically contractible. This leads to a second question: in our framework, the index cocycles appear as quite natural differential geometric objects, even for unbounded operators. Indeed, the results by [23] may be applied in order to give new lights to homotopy problems in pseudo-differential operators, for groups of bounded or unbounded operators, from an easy differential geometric viewpoint if the necessary assumptions were fulfilled, but this is actually for us an open question.

Therefore, a key problem remains in the investigation of the topologies of interest for $Cl(M, E)$ and $\mathcal{F}Cl(M, E)$ and their underlying nebulae diffeologies. We have at hand an example of such topologies, but the key properties for the application of our results still need to be investigated.

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Received: June 28, 2022, accepted: May 15, 2023.

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