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The flow-curvature of spacelike parametrized curves in the Lorentz plane

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Abstract. We introduce and study a new frame and a new curvature function for a fixed parametrization of a spacelike curve in the Lorentz plane. This new frame is called flow-frame since it involves the time-dependent rotation of the usual Frenet flow.

Анотація. Ми вводимо та досліджуємо новий репер та нову функцію кривини для фіксованої параметризації просторовоподібної кривої на площині Лоренца. Цей новий репер називано потоковим, оскільки він включає залежне від часу обертання звичайного потоку Френе.

1. Preliminaries

The theory of geometric flows is a new, fascinating field of research in geometric analysis. The most simple of them is the *curve shortening flow* and already the excellent survey [4] is twenty years old. Recall that the main geometric tool in this last flow is the well-known curvature of plane curves. Hence, to give a re-start to this problem seams to search for variants of the curvature, or in terms of [8], *deformations* of the usual curvature. The goal of this short note is to propose such a deformation, not in the plane Euclidean geometry, but in its Lorentzian counterpart.

The setting of this paper is the Lorentz plane $\mathbb{R}^2_1 := (\mathbb{R}^2, \langle\cdot,\cdot\rangle_L)$:

$$\begin{cases}\langle u, v\rangle_L = -x^1y^1 + x^2y^2,\\ u = (x^1, y^1) \in \mathbb{R}^2, \qquad v = (x^2, y^2) \in \mathbb{R}^2.\end{cases}$$

Keywords: Lorentz plane, spacelike curve, flow-frame, flow-curvature

Ключові слова: Площина Лоренца, просторово-подібна крива, потоковий репер, потокова кривина

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The Lorentzian norm is defined through $\|u\|_L^2 := |\langle u, u\rangle_L|$ and then a vector is called *spacelike* if $\|u\|_L^2 > 0$. The infinitesimal generator of the Lorentz rotations in $\mathbb{R}^2_1$ is the linear vector field:

$$\xi_L(x, y) := y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

The first integrals of ξ_L are the functions: $f_\alpha(x, y) = \alpha(x^2 - y^2)$, $\alpha \in \mathbb{R}$. For an arbitrary vector field $X = A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}$ its Lie bracket with ξ is:

$$[X, \xi_L] = (A - xA_x - yA_y)\frac{\partial}{\partial x} + (B - xB_x - yB_y)\frac{\partial}{\partial y},$$

where the subscript denotes the variable corresponding to the partial derivative. For example, ξ_L commutes with *the radial* (or Euler) vector field $E(x, y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, which is also a complete vector field having as integral curves the Euclidean homotheties $\gamma^E_{u_0}(t) = e^t u_0$ for all $t \in \mathbb{R}$ and $u_0 \in \mathbb{R}^2$.

Note also that ξ_L can be expressed in another terms. Let $(\mathbb{R}^2, \cdot, j)$ be the two-dimensional paracomplex algebra, [5]. By definition, it has the basis $\langle 1, j\rangle$ such that $j^2 = 1$. In particular, the multiplication in that algebra is given by $(a + bj) \cdot (c + jd) := (ac + bd) + j(ad + bc)$ for all $a, b, c, d \in \mathbb{R}$, so j can be regarded as the linear isomorphism $j : \mathbb{R}^2 \to \mathbb{R}^2$, $j(x, y) = (y, x)$, exchanging coordinates. Then one can write

$$\xi_L(x, y) = j \cdot E(x, y).$$

Let us remark that our framework can be found in the literature with the some alternative names: Lorentz-Minkowski or simple Minkowski plane; see our references.

Now, fix an open interval $I \subseteq \mathbb{R}$ and consider a spacelike parametrized curve $C \subset \mathbb{R}^2$ given by the equation:

$$\begin{cases} C : r(t) = (x(t), y(t)) = x(t)\bar{i} + y(t)\bar{j}, \quad \bar{i} = (1, 0),\ \bar{j} = (0, 1), \\ \langle r'(t), r'(t)\rangle_L > 0, \quad t \in I. \end{cases}$$

The Frenet apparatus of the curve C is provided by the vector fields (T, N) along C and the curvature k_L is defined by:

$$\begin{cases} T(t) = \dfrac{r'(t)}{\|r'(t)\|_L}, \\ N(t) = j \cdot T(t) = \dfrac{1}{\|r'(t)\|_L}(y'(t), x'(t)), \\ k_L(t) = \dfrac{1}{\|r'(t)\|_L}\langle T'(t), N(t)\rangle_L. \end{cases}$$

Here T is a unit spacelike vector field along C while N is a unit timelike vector field along C. We denote by $\mathcal{X}_C$ the set of vector fields along the curve C. The curvature function can be expressed using a 2×2 determinant:

$$\begin{aligned} k_L(t) &= \frac{1}{\|r'(t)\|_L^3}\langle r''(t), jr'(t)\rangle_L \\ &= \frac{1}{\|r'(t)\|_L^3}[x'(t)y''(t) - y'(t)x''(t)] = \frac{1}{\|r'(t)\|_L^3}\begin{vmatrix} x'(t) & y'(t) \\ x''(t) & y''(t)\end{vmatrix}, \end{aligned}$$

and the difference to the Euclidean curvature consists in the ratio in front of this determinant. In the Euclidean case there stands the Euclidean norm $\|r'(t)\|^{-3}$. The Frenet equations can be unified by means of the column matrix $\mathcal{F}(t) = \left(\begin{smallmatrix} T \\ N \end{smallmatrix}\right)(t)$ in the following way:

$$\frac{d}{dt}\mathcal{F}(t) = -\|r'(t)\|_L\, k_L(t)\, R'_L(0)\, \mathcal{F}(t), \quad R'_L(0) \in \mathsf{so}(1,1) \tag{1.1}$$

with $R'_L(0)$ the derivative at $t = 0$ of the Lorentz rotation $R_L(t) \in SO(1,1)$, the Lie group of the Lie algebra $\mathsf{so}(1,1)$, given by the symmetric matrices:

$$R_L(t) := \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \quad R_L^{-1}(t) := \begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}, \ t \in \mathbb{R},$$

Explicitly, the formula (1.1) can be written as follows:

$$\frac{d}{dt}\begin{pmatrix} T \\ N \end{pmatrix}(t) = -\|r'(t)\|_L \begin{pmatrix} 0 & k_L(t) \\ k_L(t) & 0 \end{pmatrix}\begin{pmatrix} T \\ N \end{pmatrix}(t).$$

The Lorentz rotated curve $jC : r_j(t) := j \cdot r(t) = (y(t), x(t))$ is a timelike curve since

$$\langle r'_j(t), r'_j(t)\rangle_L = -\langle r'(t), r'(t)\rangle_L.$$

2. The flow-frame and the flow-curvature

This short note defines a new frame and correspondingly a new curvature function for C:

Definition 2.1. *The* **flow-frame** *of C consists of the pair of vector fields $E_1^f, E_2^f \in \mathcal{X}_C$ given by:*

$$\mathcal{E}(t) := \begin{pmatrix} E_1^f \\ E_2^f \end{pmatrix}(t) = R_L(t)\mathcal{F}(t) = \begin{pmatrix} \cosh t T(t) + \sinh t N(t) \\ \sinh t T(t) + \cosh t N(t) \end{pmatrix}$$

the letter f being the initial of the word "flow". The **flow-curvature** *of C is the smooth function $k_{Lf} : I \to \mathbb{R}$ given by* **the flow-equations** *as the flow-variant of* (1.1)*:*

$$\frac{d}{dt}\mathcal{E}(t) = -\|r'(t)\|_L\, k_{Lf}(t)\, R'_L(0)\, \mathcal{E}(t).$$

Before starting its study we point out that this work is dedicated to the memory of Academician Radu Miron (1927-2022). He was always interested in the geometry of curves and besides his theory of *Myller configurations* ([10]) he generalizes also a type of curvature for space curves in [9]. Returning to our subject we note as theoretical result:

Proposition 2.2. *E^f_1 is a unit spacelike vector field and E^f_2 is a unit timelike vector field. The expression of the flow-curvature is:*

$$k_{Lf}(t) = k_L(t) - \frac{1}{\|r'(t)\|_L} < k_L(t).$$

If the decomposition of the position vector $r(t)$ with respect to the Frenet frame is $r(t) = A(t)T(t)+B(t)N(t)$ with the smooth functions $A, B\colon I \to \mathbb{R}$, then its decomposition with respect to the flow-frame is:

$$\begin{aligned} r(t) = {} & [A(t)\cosh t - B(t)\sinh t]\, E^f_1(t) + \\ & + [B(t)\cosh t - A(t)\sinh t]\, E^f_2(t). \end{aligned} \tag{2.1}$$

Proof. We have directly in the flow-frame:

$$-\|r'(t)\|_L k_{Lf}(t) R'_L(0) = r'_L(t)\, R^{-1}_L(t) - \|r'(t)\|_L\, k_L(t)\, R_L(t)\, R'_L(0)\, R^{-1}_L(t).$$

Then the conclusion follows by comparing the elements in the 2×2 matrices of both sides. Also the decomposition (2.1) follows immediately from direct calculus. □

Examples 2.3. i) If C is the line $r_0 + tu, t \in \mathbb{R}$ with the spacelike vector $u \neq \bar{0} = (0,0)$ then both k_L and k_{Lf} are constant:

$$k_{Lf} = -\frac{1}{\|u\|_L} = \mathsf{const} < 0 = k_L.$$

In particular, if u is a unit spacelike vector then $k_{Lf} = -1$.

ii) The Euclidean plane geometry corresponding to the circles

$$\mathcal{C}(O, R > 0)$$

is provided by the equilateral hyperbola as integral curve of ξ_L:

$$\begin{cases} H_e(R) : x^2 - y^2 = R^2, \qquad k_L = \mathsf{const} = -\frac{1}{R} < 0, \\ T(t) = (\sinh t, \cosh t), \qquad N(t) = \frac{1}{R} r(t) = (\cosh t, \sinh t), \\ (R_L(u) \circ r)(t) = r(t+u) = (R_L(t) \circ r)(u). \end{cases}$$

The parametrization by arc-length of $H_e(R)$ is:

$$r_e(s) = R\left(\cosh \frac{s}{R}, \sinh \frac{s}{R}\right).$$

The new frame and curvature are:

$$\begin{cases} E_1^f(t) = T(2t), \qquad E_2^f(t) = N(2t), \qquad k_{Lf} = -\dfrac{2}{R}, \\ r(t) = (-R\sinh t)\, E_1^f(t) + (R\cosh t)\, E_2^f(t). \end{cases}$$

We note that $H_e(R)$ is called *pseudo-circle* in [11, p. 110] and is denoted $H^1(-R)$.

iii) The quarter-circle $C_{1/4} : r(t) = R(\cos t, \sin t)$, $t \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ has:

$$\begin{cases} T(t) = \frac{1}{\sqrt{\cos 2t}}(-\sin t, \cos t), \quad N(t) = \frac{1}{\sqrt{\cos 2t}}(\cos t, -\sin t), \\ k_L(t) = \frac{1}{R(\cos 2t)^{\frac{3}{2}}} > 0. \end{cases}$$

Its flow-curvature and total flow-curvature are:

$$k_{Lf}(t) = \frac{2\sin^2 t}{R(\cos 2t)^{\frac{3}{2}}} \geqslant 0, \qquad \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} k_{Lf}(t)dt = \frac{2}{R}\left(2.66435... \times 10^8\right)$$

and the minimum value of k_{Lf}, namely 0, is attained at $t = 0$.

iv) The previous example of the equilateral hyperbola suggests to consider the curve $t \in I \to E_2^f(t)$. Since

$$\langle (E_2^f)'(t), (E_2^f)'(t)\rangle_L = \left[1 - \|r'(t)\|_L k_L(t)\right]^2,$$

it follows that this new curve is spacelike if and only if the initial curve C is not Lorentzian flow-flat, which means that $k_{Lf} \equiv 0$.

v) Fix a smooth function $f : I \to \mathbb{R}$ and its graph $\Gamma_f : r(t) = (t, f(t))$. This curve is spacelike if and only if the range of the derivative of f is included in $(-\infty, -1) \cup (1, +\infty)$ and then its curvatures are:

$$k_L(t) = \frac{f''(t)}{[(f'(t))^2 - 1]^{\frac{3}{2}}},$$

$$k_{Lf}(t) = \frac{f''(t)}{[(f'(t))^2 - 1]^{\frac{3}{2}}} - \frac{1}{[(f'(t))^2 - 1]^{\frac{1}{2}}}.$$

Looking for a flow-flat graph we obtain the Lorentz version of the Grim-Reaper:

$$f(t) = -\ln(\sinh t),\ t \in (0, +\infty), \qquad f'(t) = -\frac{\cosh t}{\sinh t} < -1. \tag{2.2}$$

Without the minus sign this curve appears in [2, Corollary 5.1, p. 761] and in [1, Eq. (5), p. 4]. We point out that another curve is also called

Lorentzian grim-reaper in [3, Corollary 8.1, p. 769]. The usual curvature of the graph of (2.2) is:

$$k_L(t) = \sinh t = \frac{1}{\|r'(t)\|_L}.$$

Since $\sinh(\ln(1+\sqrt{2})) = 1$, we compute two integrals:

$$\int_0^{\ln(1+\sqrt{2})} f(t)dt = 0.955202\ldots, \qquad \int_0^{\ln(1+\sqrt{2})} k_L(t)dt = \sqrt{2}-1 = 0.41421\ldots$$

The frames of this flow-flat graph are:

$$\begin{cases} T(t) = (\sinh t, -\cosh t), \qquad N(t) = (-\cosh t, \sinh t), \\ E_1^f = (0,-1), \quad E_2^f = (-1,0). \end{cases}$$

Remark 2.4. i) An important tool in dynamics is the Fermi-Walker derivative. Then the Lorentz Fermi-Walker derivative is the hyperbolic variant of derivative from [6], namely $\nabla_L^{FW} : \mathcal{X}_C \to \mathcal{X}_C$:

$$\nabla_L^{FW}(X)(\cdot) := \frac{d}{dt}X - \|r'(\cdot)\|_L \, k_L(\cdot) \left[\langle X, N\rangle_L \cdot T - \langle X, T\rangle_L \cdot N\right]$$

where $(\cdot)$ marks the application on a fixed time $t \in I$. It is natural to make here a remark concerning *rotation-minimizing fields* $X \in \mathcal{X}_C$ i.e. fields satisfying:

$$\frac{d}{dt}X(t) = \lambda(t)T(t), \qquad \langle X(t), T(t)\rangle_L = 0,$$

for a smooth function $\lambda = \lambda(t)$. Then the Fermi-Walker derivative of such X is parallel with the tangent vector field T:

$$\left(\nabla_L^{FW}(X)\right)(t) = \left[\lambda(t) - \|r'(t)\|_L \, k_L(t) \langle X(t), N(t)\rangle_L\right] T(t).$$

Computing the Fermi-Walker derivative on our frames we get:

$$\nabla_L^{FW}(T) = \nabla_L^{FW}(N) = 0,$$
$$\nabla_L^{FW}(E_1^f) = E_2^f, \qquad \nabla_L^{FW}(E_2^f) = E_1^f.$$

With the matrix notation we can express these relations in the following way:

$$\nabla_L^{FW}(\mathcal{F}) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \qquad \nabla_L^{FW}(\mathcal{E}) = R'_L(0)\,\mathcal{E}$$

and the Fermi-Walker derivative can be expressed in terms of k_{Lf} as:

$$\nabla_L^{FW}(X) = \frac{d}{dt}X - (\|r'\|_L \, k_{Lf} + 1)\left[\langle X, N\rangle_L \cdot T - \langle X, T\rangle_L \cdot N\right].$$

ii) The nature and the relationship between our frames can be put in the framework of *moving frames* of [7, p. 32]. Recall that the set of all orientation-preserving Lorentzian isometries forms a Lie group

$$L(2) := \mathbb{R}^2 \times SO(1,1)$$

so that the standard projection $\pi_1 : L(2) \to \mathbb{R}^2$ onto the first factor is an $SO(1,1)$-principal bundle. A moving frame along C is a map $F : I \to L(2)$ such that $\pi_1 \circ F = r$. But C defines also a 1-parameter family of bijections of $SO(1,1)$:

$$\begin{cases} L^C : I \to \mathsf{Bijections}(SO(1,1)), \\ t \to \big[L^C(t) : SO(1,1) \to SO(1,1),\ A \to R_L(t)A\big]. \end{cases}$$

Then our frames are

$$\mathcal{F} : I \to L(2), \qquad \mathcal{F}(t) = (r(t), T(t), N(t)),$$
$$\mathcal{E} : I \to L(2), \qquad \mathcal{E}(t) = (r(t), (L^C(t) \circ \pi_2 \circ \mathcal{F})(t)).$$

3. New variants of the curve shortening flow

We finish this note with the problem raised in the beginning: find possible variants of the curve shortening flow. Note that the self-similar solutions in the hyperbolic variant of this problem are studied in [12]. Recall that the (Lorentzian) setting of this question consists in a 1-parameter family of plane curves $C_u : r = r(t,u)$ satisfying:

$$\frac{\partial r(t,u)}{\partial u} = k_L(t,u)\, N(t,u) \tag{3.1}$$

and due to the commutativity of partial derivatives with respect to t and u we have the u-variation of the Frenet frame:

$$\begin{cases} \dfrac{\partial T(t,u)}{\partial u} = -\|r'_t(t,u)\|_L\, k_L^2(t,u)\, T(t,u) + \dfrac{\partial k_L(t,u)}{\partial t} N(t,u), \\ \dfrac{\partial N(t,u)}{\partial u} = \dfrac{\partial k_L(t,u)}{\partial t}\, T(t,u) - \|r'_t(t,u)\|_L\, k_L^2(t,u)\, N(t,u). \end{cases}$$

This immediately implies the following expression in terms of flow-apparatus:

$$\frac{\partial r(t,u)}{\partial u} = \left(k_{Lf}(t,u) + \frac{1}{\|r'_t(t,u)\|_L}\right)\big[-\sinh t\, E_1^f(t,u) + \cosh t\, E_2^f(t,u)\big].$$

The first variant which we propose as an open problem is to study the flow-variant of (3.1):

$$\frac{\partial r(t,u)}{\partial u} = k_{Lf}(t,u)\, E_2^f(t,u) \tag{3.2}$$

while the second variant is to restrict only to the normal component of E_2^f:

$$\frac{\partial r(t,u)}{\partial u} = k_{Lf}(t,u)\,\cosh t\, N(t,u). \tag{3.3}$$

We point out that a solution to the Lorentzian curve flow (3.1) is provided by the 1-parameter family of equilateral hyperbolas with an initial equilateral hyperbola $H_e(R_0 > 0)$:

$$H_e(u) : x^2 - y^2 = R_0^2 - 2u,\ 0 \leqslant u < \frac{R_0^2}{2}, \qquad k_{Lf}(u) = \frac{-2}{\sqrt{R_0^2 - 2u}}$$

for which the expression defined in (3.2) is:

$$\frac{\partial r(t,u)}{\partial u} - k_{Lf}(t,u)\, E_2^f(t,u) = \frac{1}{\sqrt{R_0^2 - 2u}}(2\cosh 2t - \cosh t, 2\sinh 2t - \sinh t)$$

and the expression of (3.3) is:

$$\frac{\partial r(t,u)}{\partial u} - k_{Lf}(t,u)\,\cosh t N(t,u) = \frac{\cosh 2t}{\sqrt{R_0^2 - 2u}}(\cosh t, \sinh t) = \mathsf{timelike}.$$

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