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On quasi-geodesic mappings of special pseudo-Riemannian spaces

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Abstract. The present paper continues the study of quasi-geodesic mappings $f:(V_n, g_{ij}, F_i^h)\to(\overline{V}_n, \overline{g}_{ij}, F_i^h)$ of pseudo-Riemannian spaces V_n, $\overline{V}_n$ with a generalized-recurrent structure F_i^h of parabolic type. By a generalized recurrent structure of parabolic type on V_n we mean an almost Hermitian affinor structure of parabolic type for which the covariant derivative of the structural affinor F_i^h satisfies the condition $F^h_{(i,j)} = q_{(i}F^h_{j)}$.

In the previous paper by the authors [Proc. Intern. Geom. Center, 13:3 (2020) 18-32] it was proved that the class of pseudo-Riemannian spaces with generalized-recurrent structure of parabolic type is closed with respect to the considered mappings and the generalized recurrence vectors in (V_n, g_{ij}, F_i^h) and $(\overline{V}_n, \overline{g}_{ij}, F_i^h)$ may be distinct. In this article, it is assumed that the mapping f preserves the generalized recurrence vector q_i.

We construct geometric objects that are invariant under the quasi-geodesic mapping of generalized-recurrent spaces of parabolic type and recurrent-parabolic spaces. A number of conditions are given on these objects, which lead to the fact that a generalized-recurrent space of parabolic type admits a parabolic K-structure, and a recurrent-parabolic space admits a Kählerian structure of parabolic type.

We study special types of these mappings that preserve some tensors of an intrinsic nature.

Анотація. У статті продовжуються дослідження квазі-геодезичних відображень $f:(V_n, g_{ij}, F_i^h)\to(\overline{V}_n, \overline{g}_{ij}, F_i^h)$ псевдоріманових просторів V_n, $\overline{V}_n$ з узагальнено-рекурентною структурою F_i^h параболічного типу. Узагальнено-рекурентною структурою параболічного типу на V_n називається майже ермітова афінорна структура параболічного типу, для якої коваріантна похідна структурного афінора F_i^h задовольняє умові $F^h_{(i,j)} = q_{(i}F^h_{j)}$. В попередній роботі авторів [Proc. Intern. Geom. Center, 13:3 (2020) 18-32] доведено, що клас псевдо-ріманових просторів з узагальнено-рекурентною структурою параболічного типу замкнений відносно розглядуваних відображень, але при цьому вектори узагальненої

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рекурентності просторів (V_n, g_{ij}, F_i^h) і $(\overline{V}_n, \overline{g}_{ij}, F_i^h)$ можуть виявитись не тотожними. У цій статті припускається, що відображення f зберігає вектор узагальненої рекурентності q_i.

Ми будуємо геометричні об'єкти, інваріантні щодо квазі-геодезичного відображення узагальнено-рекурентних просторів параболічного типу, а також рекурентно-параболічних просторів. Наводиться ряд умов на ці об'єкти, які гарантують, що узагальнено-рекурентний простір параболічного типу допускає параболічну K-структуру, а рекурентно-параболічний простір допускає келерову структуру параболічного типу.

Вивчено спеціальні типи цих відображень, що зберігають деякі тензори внутрішнього характеру.

1. INTRODUCTION

In [8] we considered diffeomorphisms of pseudo-Riemannian spaces being quasi-geodesic mappings (**QGM**). Also in [5–7, 9, 11] such maps were studied under the reciprocity condition, while almost-geodesic mappings of the second type were considered in [2–4, 10, 12–14]. Say that a **QGM**

$$f : (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, F_i^h)$$

satisfies the *reciprocity condition* whenever its inverse f^{-1} is a **QGM** as well.

The basic equations of such a mapping $f : (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, F_i^h)$ in the common coordinate system (x^i) with respect to the mapping f have the form:

$$\overline{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_{(i}(x)\delta_{j)}^h + \phi_{(i}(x)F_{j)}^h(x), \tag{1.1}$$

$$F_i^h(x) = -\overline{F}_i^h(x), \tag{1.2}$$

$$\overline{g}_{i\alpha}F_j^\alpha = -\overline{g}_{j\alpha}F_i^\alpha, \qquad g_{i\alpha}F_j^\alpha = -g_{j\alpha}F_i^\alpha, \tag{1.3}$$

$$F_{(i,j)}^h = q_{(i}F_{j)}^h, \tag{1.4}$$

$$F_\alpha^h F_i^\alpha = e\delta_i^h, \qquad e = 0, \pm 1, \tag{1.5}$$

where $i, h, j, ... = 1, 2, ...n$, $\Gamma_{ij}^h, \overline{\Gamma}_{ij}^h$ are the Christoffel symbols of V_n and $\overline{V}_n$ respectively; $\psi_i(x)$, $\phi_i(x)$, $q_i(x)$ are certain covectors; $F_i^h(x)$ is affinor; brackets (i, j) denote the symmetrization with respect to the corresponding indices; and comma «,» is a sign of the covariant derivative in respect to the connection of V_n. If in (1.1) $\phi_i \neq 0$ and $\psi_i = 0$, the quasi-geodesic mapping is called *canonical*.

Condition (1.5) defines an e-structure, which is called

- *elliptic* if $e = -1$,
- *hyperbolic* if $e = +1$,
- *m-parabolic* when $e = 0$, $\mathrm{rank}(F) = m$, $(2m < n)$,

- *parabolic* when $e = 0$, $\mathrm{rank}(F) = m$, $(2m = n)$.

1.1. In [5] were studied **QGM** of pseudo-Riemannian spaces (V_n, g_{ij}, F_i^h) with a K-structure [1] of elliptic and hyperbolic types, i.e. when F satisfies (1.5) for $e = \pm 1$, and the following differential condition holds:

$$F^h_{(i,j)} = 0.$$

1.2. An affinor structure F_i^h satisfying conditions (1.4) will be called a *generalized-recurrent (***GR***-) structure* of elliptic, hyperbolic or parabolic type depending on the values of e.

Obviously, the K-structure is a special case of a generalized-recurrent structure.

Another special case of the **GR**-structure is the recurrent-parabolic (**RP**-) structure, which was determined in [9] by the conditions

$$F^h_\alpha F^\alpha_i = 0, \qquad g_{i\alpha} F^\alpha_j = -g_{j\alpha} F^\alpha_i, \qquad F^h_{i,j} = q_j F^h_i. \tag{1.6}$$

In [8], we described properties of parabolic **GR**-structure (**GRP**). We call q_i in (1.4) the *generalized recurrence vector* of the **GR**-structure F_i^h. Note that **GR**-structure is a K-structure, whenever $q_i = 0$. Moreover, under the condition $q_i = \frac{\partial q(x)}{\partial x^i}$ **GR**-space (V_n, g_{ij}, F_i^h) admits the K-structure

$$\tilde{F}^h_i = e^{-q} F^h_i.$$

1.3. To make the calculations easier, let us introduce an operation of conjugation in (V_n, g_{ij}, F_i^h) :

$$T^{\cdots}_{j_1 \cdots j_{k-1} \alpha j_{k+1} \cdots j_r} F^\alpha_i = T^{\cdots}_{j_1 \cdots j_{k-1} \bar{i} j_{k+1} \cdots j_r}$$
$$T^{j_1 \cdots j_{k-1} \alpha j_{k+1} \cdots j_r}_{\cdots} F^h_\alpha = T^{j_1 \cdots j^{k-1} \overline{h} j_{k+1} \cdots j_r}_{\cdots}.$$

In (V_n, g_{ij}, F_i^h) with integrable parabolic structure F_i^h, there exists a local coordinate system (adapted to the affinor), in which the structure tensor can be reduced to the form

$$(F^h_i) = \begin{pmatrix} 0 & 0 \\ I_m & 0 \end{pmatrix},$$

where I_m is the identity matrix of order $m = \frac{n}{2}$.

Further, we will use the auxiliary tensor A_i^h, which in the adapted system is defined by the matrix

$$(A^h_i) = \begin{pmatrix} P & I_m \\ -P^2 & -P \end{pmatrix},$$

where P is an arbitrary square matrix of order m.

It can be checked that components of tensor A satisfy the following identities:

$$F^\beta_\alpha A^\alpha_\beta = m, \qquad A^h_\alpha A^\alpha_i = 0, \qquad F^h_\alpha A^\alpha_i + A^h_\alpha F^\alpha_i = \delta^h_i. \tag{1.7}$$

1.4. In [8], the properties of a parabolic type generalized-recurrent structure were studied. It was proved that the Riemannian tensor of the GR-space (V_n, g_{ij}, F^h_i) satisfies the relations

$$3\big(R_{\bar{h}jki} + R_{h\bar{j}ki} + R_{hj\bar{k}i} + R_{hjk\bar{i}}\big) = 2Q_{jhki} + Q_{jkhi} - Q_{hkji}, \tag{1.8}$$

where

$$Q_{hjki} = q_{[h,j]}F_{ki} + q_{[k,i]}F_{hj}, \qquad\qquad F_{hj} = g_{h\alpha}F^{\alpha}_{j}.$$

For $q_i = 0$ the GR-structure F^h_i is a K-structure. Therefore, in the K-space, the Riemannian tensor has the following property:

$$R_{\bar{h}jki} + R_{h\bar{j}ki} + R_{hj\bar{k}i} + R_{hjk\bar{i}} = 0. \tag{1.9}$$

In [8] it was proved that in the GRP-space (V_n, g_{ij}, F^h_i) equality (1.9) holds if and only if the generalized recurrence vector q_i is a gradient, *i.e.* there exists a K-structure $\tilde{F}^h_i = e^{-q}F^h_i$ in V_n.

1.5. The affinor structure of a GRP-space (V_n, g_{ij}, F^h_i) provided that it is integrable has the following properties [8]:

$$\begin{gathered} F^{\alpha}_{i,\alpha} = 0, \\ F^{h}_{\bar{j},i} = F^{h}_{j,\bar{i}} = F^{\bar{h}}_{j,i} = 0, \quad q_{\bar{i}} = 0. \end{gathered} \tag{1.10}$$

Note that, in contrast to the hyperbolic and elliptic types, an integrable GR-structure of parabolic type (in particular, a parabolic K-structure) may be not Kähler, *i.e.* covariant constancy of the affinor F^h_i does not follow from relation (1.10). Further in this paper, we consider only an integrable affinor structure.

1.6. In [8] it was proved that the image of a GRP-space under QGM is also a GRP-space, that is,

$$F^{h}_{(i|j)} = \tilde{q}_{(j}F^{h}_{i)},$$

where

$$\tilde{q}_i = q_i - \psi_i + \phi_{\bar{i}},$$

«|» is a sign of a covariant derivative in respect to the connection of $\overline{V}_n$. In other words, the affinor F^h_i in the space $\overline{V}_n$ also defines a GRP-structure.

Under the condition $\tilde{q}_i = q_i$ we will say that QGM *preserves the generalized recurrence vector*. In this case, the vectors ψ_i and ϕ_i in the basic QGM equations (1.1) are related by the following identity:

$$\psi_i = \phi_{\bar{i}}. \tag{1.11}$$

Hence, contracting (1.1) with respect to h and j, we get:

$$\overline{\Gamma}^{\alpha}_{i\alpha} = \Gamma^{\alpha}_{i\alpha} + (n+2)\psi_i,$$

that is, ψ_i is locally a gradient:

$$(n+2)\psi_i = \tfrac{\partial\psi(x)}{\partial x^i}, \qquad\qquad \psi(x) = \tfrac{1}{2}\ln\left|\tfrac{\overline{g}}{g}\right|$$

In what follows, we construct geometric objects that are invariant with respect to the QGM of parabolic type generalized-recurrent spaces which preserve generalized recurrence vector. We will also study some special types of these mappings provided that the affinor structure is integrable.

The investigations are carried out in tensor form, locally, in the class of real sufficiently smooth functions.

2. Some geometric objects that are invariants under QGM of GRP-spaces

2.1. Assume that there exists a QGM

$$f : (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, F_i^h)$$

between GRP-spaces. Then, the following basic equations hold:

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_{(i}(x)\delta^h_{j)} + \phi_{(i}(x)F^h_{j)}(x), \tag{2.1}$$

$$\overline{g}_{i\alpha}F^\alpha_j = -\overline{g}_{j\alpha}F^\alpha_i, \qquad g_{i\alpha}F^\alpha_j = -g_{j\alpha}F^\alpha_i, \tag{2.2}$$

$$F^h_\alpha F^\alpha_i = 0, \qquad F^h_{(i,j)} = q_{(j}F^h_{i)}. \tag{2.3}$$

Contracting (2.1) with respect to h, j we find that

$$\psi_i = \tfrac{1}{n+2}\big(\overline{\Gamma}^\alpha_{i\alpha} - \Gamma^\alpha_{i\alpha}\big).$$

Let us multiply (2.1) by A^h_i and contract for indices h and j. Then, according to (2.3) and (1.7), we get

$$\phi_i = \tfrac{2}{n+2}\big(\overline{\Gamma}^\alpha_{i\beta} - \Gamma^\alpha_{i\beta}\big)A^\beta_\alpha.$$

Substituting ϕ_i and ψ_i into (2.1), we can rewrite them in the following form

$$T^h_{ij} = \overline{T}^h_{ij},$$

where

$$T^h_{ij} = \Gamma^h_{ij} - \tfrac{1}{n+2}\big(\Gamma^\alpha_{\alpha(i}\delta^h_{j)} + 2A^\beta_\alpha\Gamma^\alpha_{\beta(i}F^h_{j)}\big). \tag{2.4}$$

Analogously we define $\overline{T}^h_{ij}$ in $\overline{V}_n$. In this way, we have proved the invariance of geometric object (2.4) under QGM of GRP-spaces. This object is similar to the Thomas parameters of the theory of geodesic mappings. The equality of Thomas-like objects (2.4) is a necessary and sufficient condition for existence of QGM between V_n and $\overline{V}_n$.

2.2. Due to (2.1), we can written a relationship between components of the curvature tensors V_n and $\overline{V}_n$ in the following form:

$$\begin{aligned}\overline{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik} + F^h_i \phi_{[kj]} + \\ + F^h_k \phi_{ij} - F^h_j \phi_{ik} + \phi_i F^h_{[k,j]} + \phi_k F^h_{i,j} - \phi_j F^h_{i,k},\end{aligned} \tag{2.5}$$

where we have introduced

$$\psi_{ij} = \psi_{i,j} - \psi_i \psi_j, \qquad \phi_{ij} = \phi_{i,j} - \phi_i \psi_j - \phi_j \psi_i,$$

and $[i, j]$ denotes an alternation with respect to the corresponding indices.

Multiplying (2.5) by F^h_l and contracting it with respect to h, we get:

$$\overline{R}^{\overline{h}}_{ijk} = R^{\overline{h}}_{ijk} + F^h_k \psi_{ij} - F^h_j \psi_{ik}. \tag{2.6}$$

Multiplying (2.6) by A^k_h and contracting it with respect to h and k, we obtain

$$\psi_{ij} = \tfrac{2}{n-2}\big(\overline{R}^{\overline{\alpha}}_{ij\beta} - R^{\overline{\alpha}}_{ij\beta}\big) A^\beta_\alpha.$$

Substituting ψ_{ij} into (2.6) and after simple computation we find that

$$\overset{1}{T}{}^h_{ijk} = \overset{1}{\overline{T}}{}^h_{ijk},$$

where

$$\overset{1}{T}{}^h_{ijk} = R^{\overline{h}}_{ijk} - \tfrac{2}{n-2}\big(F^h_k R^{\overline{\alpha}}_{ij\beta} - F^h_j R^{\overline{\alpha}}_{ik\beta}\big) A^\beta_\alpha. \tag{2.7}$$

We also define $\overset{1}{\overline{T}}{}^h_{ijk}$ in $\overline{V}_n$ in a similar way.

The last equation tells that tensor $\overset{1}{T}{}^h_{ijk}$ is preserved under the **QGM** of **GRP**-spaces. Its preservation is only a necessary condition for existence of **QGM** between V_n and $\overline{V}_n$.

2.3. Multiplying (2.5) by F^i_l and contracting it with respect to i, we get:

$$\overline{R}^h_{\bar{i}jk} = R^h_{\bar{i}jk} + F^h_k \phi_{\bar{i}j} - F^h_j \phi_{\bar{i}k} + \psi_i F^h_{[k,j]}. \tag{2.8}$$

Due (2.1) and (1.11), the following relations hold:

$$\begin{aligned} q_{i|k} &= q_{i,k} - q_{(i}\psi_{k)}, \\ F^h_{i|j} &= F^h_{i,j}. \end{aligned} \tag{2.9}$$

Multiplying (2.8) by q_l, and symmetrizing the resulting equality with respect to l, i, we will get from (2.9) that:

$$\overset{2}{T}{}^h_{lijk} = \overset{2}{\overline{T}}{}^h_{lijk},$$

where

$$\overset{2}{T}{}^h_{lijk} = q_{(l}\tilde{T}^h_{i)jk} + q_{i,l}\tilde{\tilde{T}}{}^h_{kj},$$

$$\tilde{T}^h_{ijk} = R^h_{\bar{i}jk} - \tfrac{2}{n-2} A^\beta_\alpha R^\alpha_{\bar{i}\beta[j} F^h_{k]}, \tag{2.10}$$

$$\tilde{\tilde{T}}{}^h_{kj} = F^h_{[k,j]} + \tfrac{2}{n-2}\big(F^h_k F^\alpha_{[\beta,j]} - F^h_j F^\alpha_{[\beta,k]}\big) A^\beta_\alpha.$$

Analogously we define $\overset{2}{\overline{T}}{}^h_{lijk}$ in $\overline{V}_n$.

The last equation shows that the tensor $\overset{2}{T}{}^h_{lijk}$ is preserved under the QGM of GRP-spaces. Its preservation is only a necessary condition for the existence of QGM between V_n and $\overline{V}_n$.

Thus, we have established the following

Theorem 2.3.1. *Geometric objects* (2.4), (2.7), (2.10) *are invariant under QGM of GRP-spaces.*

2.4. Let V_n be a GRP-space in which $\overset{1}{T}{}^h_{ijk}$ vanishes. Then (2.7) can be written as follows:

$$R^{\bar{h}}_{ijk} - \tfrac{2}{n-2}\big(F^h_k a_{ij} - F^h_j a_{ik}\big) = 0,$$

where

$$a_{ij} = R^{\bar{\alpha}}_{ij\beta} A^\beta_\alpha.$$

Lowering the index h in V_n we get

$$R_{\bar{h}ijk} + \tfrac{2}{n-2}\big(F_{hk} a_{ij} - F_{hj} a_{ik}\big) = 0. \tag{2.11}$$

Alternating the latter by h, i, j, k, we get the equation:

$$F_{hi} a_{[jk]} + F_{jk} a_{[hi]} = 0. \tag{2.12}$$

Let us multiply (2.12) by $g^{h\alpha} A^i_\alpha$ and contract with respect to h, i. Multiply further the result by $g^{j\alpha} A^k_\alpha$ and contract with respect to j, k. This will gives us:

$$\tfrac{n}{2} a_{[jk]} + F_{jk} g^{\beta\alpha} A^\gamma_\alpha a_{[\beta\gamma]} = 0,$$
$$g^{\beta\alpha} A^\gamma_\alpha a_{[\beta\gamma]} = 0.$$

Hence

$$a_{[jk]} = 0.$$

Substituting this to (2.11), we get

$$R_{\bar{h}jki} + R_{h\bar{j}ki} + R_{hj\bar{k}i} + R_{hjk\bar{i}} = 0.$$

The latter means that the generalized recurrence vector q_i is gradient, *i.e.* there exists a K-structure $\tilde{F}_i^h = e^{-q} F_i^h$ in the GRP-space V_n.

Note that multiplying (1.9) by g^{jk} and contracting with respect to j, k we obtain that the Ricci tensor of K-space (as well as of GRP-space with gradient generalized recurrence vector) satisfies the following conditions:

$$R_{\overline{h}j} = -R_{h\overline{j}}. \tag{2.13}$$

Let us multiply (2.13) by $g^{h\alpha} A^j_\alpha$ and contract it with respect to h, k. Then we will obtain that

$$R^{\overline{\alpha}}_{\beta} A^{\beta}_{\alpha} = \tfrac{R}{2},$$

where R is the scalar curvature $R_{\alpha\beta} g^{\alpha\beta} = R$. Obviously, if $R_{\overline{h}j} = 0$, then $R = 0$.

Multiplying (2.11) by g^{ij} and contracting it with respect to i, j, we get

$$R_{\overline{h}k} + \tfrac{2}{n-2}\big(a_{\alpha\beta} g^{\alpha\beta} F_{hk} + a_{\overline{h}k}\big) = 0. \tag{2.14}$$

Hence, taking into account (2.13), we have

$$a_{\overline{h}k} = -a_{\overline{k}h}.$$

The latter identity together with (2.13) implies that

$$a_{\alpha\beta} g^{\alpha\beta} = \tfrac{R}{2}, \qquad g^{\overline{\alpha}\gamma} a_{\gamma\beta} A^{\beta}_{\alpha} = \tfrac{R}{4}. \tag{2.15}$$

Multiplying (2.11) by $g^{\overline{i}\alpha} A^j_\alpha$, contracting it with respect to i, j, and taking into account (2.14) and (2.15), we obtain

$$R_{\overline{h}k} = \tfrac{R}{n} F_{kh}. \tag{2.16}$$

Theorem 2.4.1. *If in a GRP-space V_n the equality*

$$\overset{1}{T}{}^h_{ijk} = 0$$

holds, then its Ricci tensor satisfies the following condition:

$$R_{\overline{h}k} = \tfrac{R}{n} F_{kh}$$

and the generalized recurrence vector q_i is gradient, i.e. there is a K-structure $\tilde{F}_i^h = e^{-q} F_i^h$ in V_n,

3. Geometric objects that are invariants of QGM of RP-spaces

3.1. QGM between the RP-spaces $f : (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, F_i^h)$ is characterized by basic equations

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_{(i}(x)\delta^h_{j)} + \phi_{(i}(x)F^h_{j)}(x),$$

$$\overline{g}_{i\alpha} F^{\alpha}_j = -\overline{g}_{j\alpha} F^{\alpha}_i,$$

$$g_{i\alpha}F^{\alpha}_{j} = -g_{j\alpha}F^{\alpha}_{i},$$
$$F^{h}_{\alpha}F^{\alpha}_{i} = 0$$

and

$$F^{h}_{i,j} = F^{h}_{i|j} = q_j F^{h}_{i}. \tag{3.1}$$

According to (3.1), the Riemannian tensor in the RP-space satisfies the following relations, see [6]:

$$R^{\overline{h}}_{ijk} - R^{h}_{\bar{i}jk} = F^{h}_{i} q_{[j,k]}$$

which is the same as

$$R_{\overline{h}ijk} + R_{h\bar{i}jk} = F_{ih} q_{[j,k]} \tag{3.2}$$

3.2. Due to (2.1), (2.2), (2.3) and (3.1) we can written a relationship between components of the curvature tensors R^{h}_{ijk} and $\overline{R}^{h}_{ijk}$ in the following form

$$\overline{R}^{h}_{ijk} = R^{h}_{ijk} + \delta^{h}_{k}\psi_{ij} - \delta^{h}_{j}\psi_{ik} + F^{h}_{i}\tilde{\phi}_{[kj]} + F^{h}_{k}\tilde{\phi}_{ij} - F^{h}_{j}\tilde{\phi}_{ik}, \tag{3.3}$$

where

$$\psi_{ij} = \psi_{i,j} + \psi_i\psi_j,$$

$$\tilde{\phi}_{ij} = \phi_{i,j} - \psi_i\phi_j - \psi_j\phi_i + \phi_i q_j.$$

Taking into account $\psi_i = \phi_{\bar{i}}$, we get

$$\tilde{\phi}_{\bar{i}j} = \psi_{ij}.$$

Let us multiply (3.3) by A^{k}_{h} and contract it with respect to h, k. Then contracting further the result with F^{i}_{l} with respect to i, we get:

$$\begin{aligned}\overline{R}^{\alpha}_{ij\beta}A^{\beta}_{\alpha} = R^{\alpha}_{ij\beta}A^{\beta}_{\alpha} + \big(\tilde{\phi}_{i\overline{\alpha}} - \psi_{i\alpha}\big)A^{\alpha}_{j} + \\ + \big(\tilde{\phi}_{j\overline{\alpha}} - \psi_{j\alpha}\big)A^{\alpha}_{i} + \tfrac{n}{2}\tilde{\phi}_{ij} - \tilde{\phi}_{ij},\end{aligned} \tag{3.4}$$

$$\overline{R}^{\alpha}_{\bar{i}j\beta}A^{\beta}_{\alpha} = R^{\alpha}_{\bar{i}j\beta}A^{\beta}_{\alpha} + \tfrac{n-2}{2}\psi_{ij}. \tag{3.5}$$

Alternation (3.4) with respect to the indices i, j, and subsequent contraction of the result with F^{j}_{l} in the index j gives us:

$$\overline{R}^{\alpha}_{\beta ji}A^{\beta}_{\alpha} = R^{\alpha}_{\beta ji}A^{\beta}_{\alpha} + \tfrac{n+2}{2}\tilde{\phi}_{[ij]}, \tag{3.6}$$

$$\overline{R}^{\alpha}_{\beta\bar{j}i}A^{\beta}_{\alpha} = R^{\alpha}_{\beta\bar{j}i}A^{\beta}_{\alpha} + \tfrac{n+2}{2}\big(\tilde{\phi}_{i\bar{j}} - \psi_{ji}\big). \tag{3.7}$$

Taking into account (3.4)-(3.7), we get:

$$\tfrac{n-2}{2}\psi_{ij} = \big(\overline{R}^{\alpha}_{\bar{i}j\beta} - R^{\alpha}_{\bar{i}j\beta}\big)A^{\beta}_{\alpha},$$

$$\tfrac{n^2-4}{4}\tilde{\phi}_{ij} = \tfrac{n+2}{2}\big(\overline{R}^{\alpha}_{ij\beta} - R^{\alpha}_{ij\beta}\big)A^{\beta}_{\alpha} - \big(\overline{R}^{\alpha}_{\beta ji} - R^{\alpha}_{\beta ji}\big)A^{\beta}_{\alpha} -$$

$$-\big(\overline{R}^{\alpha}_{\beta\overline{\gamma}(i}-R^{\alpha}_{\beta\overline{\gamma}(i)}\big)A^{\gamma}_{j)}A^{\beta}_{\alpha}.$$

Substituting ψ_{ij} and $\tilde{\phi}_{ij}$ into the equations (3.3), we represent them in the form

$$\overset{3}{T}{}^{h}_{ijk}=\overset{3}{\overline{T}}{}^{h}_{ijk},$$

where

$$\overset{3}{T}{}^{h}_{ijk}=\tfrac{n^2-4}{4}R^{h}_{ijk}-\tfrac{n+2}{2}\big(\delta^{h}_{k}R^{\alpha}_{\bar{i}j\beta}-\delta^{h}_{j}R^{\alpha}_{\bar{i}k\beta}\big)A^{\beta}_{\alpha}+ \qquad (3.8)$$

$$+F^{h}_{i}D_{[jk]}-F^{h}_{k}D_{ij}+F^{h}_{j}D_{ik},$$

$$D_{ij}=\big(\tfrac{n+2}{2}R^{\alpha}_{ij\beta}-R^{\alpha}_{\beta ji}-R^{\alpha}_{\beta\overline{\gamma}(i}A^{\gamma}_{j)}\big)A^{\beta}_{\alpha}. \qquad (3.9)$$

Analogously we define $\overset{3}{\overline{T}}{}^{h}_{ijk}$ in $\overline{V}_n$.

The last equation shows that the tensor $\overset{3}{T}{}^{h}_{ijk}$ is preserved under the QGM of RP-spaces. Again, its preservation is only a necessary condition for existence of QGM between V_n and $\overline{V}_n$. So, we proved the following

Theorem 3.2.1. *Geometric object* (3.8) *is an invariant of QGM of RP-spaces.*

3.3. Let V_n be a RP-space in which $\overset{3}{T}{}^{h}_{ijk}$ vanishes. Then taking to account (3.8) we obtain the following relation:

$$\tfrac{n^2-4}{4}R^{h}_{ijk}=\tfrac{n+2}{2}\big(\delta^{h}_{k}R^{\alpha}_{\bar{i}j\beta}-\delta^{h}_{j}R^{\alpha}_{\bar{i}k\beta}\big)A^{\beta}_{\alpha}+F^{h}_{i}D_{[kj]}+F^{h}_{k}D_{ij}-F^{h}_{j}D_{ik}.$$

Lowering further the index h in V_n we get

$$\begin{aligned}\tfrac{n^2-4}{4}R_{hijk}=\tfrac{n+2}{2}\big(g_{hk}R^{\alpha}_{\bar{i}j\beta}-g_{hj}R^{\alpha}_{\bar{i}k\beta}\big)A^{\beta}_{\alpha}+&\\ +F_{hi}D_{[kj]}+F_{hk}D_{ij}-F_{hj}D_{ik}.&\end{aligned} \qquad (3.10)$$

Cycling (3.10) by indices h, i, k, and contracting it with $g^{h\alpha}A^{j}_{\alpha}$ in the indices h and j, we get

$$D_{\bar{i}k}=0. \qquad (3.11)$$

Taking account of (3.10) and (3.11) we obtain

$$R^{h}_{\bar{i}jk}=0. \qquad (3.12)$$

We will call a GRP-space V_n *almost flat* if the Riemannian tensor R^{h}_{ijk} of V_n satisfies the condition (3.12). According to (3.2), it follows that the recurrence vector q_i of almost flat RP-space V_n is gradient. Therefore, an almost flat V_n admits the Kähler structure

$$\tilde{F}_i^h = e^{-q(x)} F_i^h, \qquad q_i = \tfrac{\partial q(x)}{\partial x^i}.$$

Symmetrizing (3.10) with respect to h, i and taking account (3.11) we get:

$$F_{hk}D_{ij} + F_{ik}D_{hj} - F_{hj}D_{ik} - F_{ij}D_{hk} = 0.$$

Contracting the latter with $g^{h\alpha}A_\alpha^k$ we obtain:

$$D_{ij} = \tilde{D}F_{ij},$$

where $\tilde{D} = \frac{2}{n}D_{\alpha\beta}g^{\alpha\gamma}A_\gamma^\beta$.

Substituting further D_{ij} into 3.10, one can write down it in the follows form:

$$R_{hijk} = K\left(F_{hk}F_{ij} - F_{hj}F_{ik} + 2F_{hi}F_{kj}\right), \tag{3.13}$$

where $K = \frac{4\tilde{D}}{n^2-4}$. The latter means that $R_{ij} = 0$, that is, the space is Ricci-flat.

Taking account to (3.13) and (3.8) we obtain $\overset{3}{T}{}^h_{ijk} = 0$. Thus, we have proved the following

Theorem 3.3.1. *An object* $\overset{3}{T}{}^h_{ijk}$ *in the* RP*-space* V_n *satisfies the equation* $\overset{3}{T}{}^h_{ijk} = 0$ *if and only if its Riemannian tensor has the form* (3.13)*.*

It was proved in [9] that the Riemannian tensor of a RP-space V_n admitting QGM on a flat space, has the form (3.13), and $K = Ce^{-2q(x)}$, $C = \mathrm{const}$, $q_i = \frac{\partial q(x)}{\partial x^i}$.

It was also shown in [9] that such a RP-space is symmetric, that is, $R^h_{ijk,l} = 0$, and the components of metric tensor of all such spaces are found.

3.4. Let V_n be a RP-space in which $\overset{3}{T}{}^h_{\bar{i}jk}$ vanishes. Then taking to account (3.8) we obtain the following identity:

$$\tfrac{n-2}{2}R_{h\bar{i}jk} = \big(F_{hk}R^\alpha_{\bar{i}j\beta} - F_{hj}R^\alpha_{\bar{i}k\beta}\big)A^\beta_\alpha. \tag{3.14}$$

Lowering the index h and cycling with respect to indices h, k, j, we get

$$\big(F_{hk}R^\alpha_{\bar{i}j\beta} + F_{kj}R^\alpha_{\bar{i}h\beta} + F_{jh}R^\alpha_{\bar{i}k\beta}\big)A^\beta_\alpha = 0. \tag{3.15}$$

Conjugating (3.15) in the index j, we obtain

$$R^\alpha_{\overline{ij}\beta}A^\beta_\alpha = 0.$$

Also, contraction (3.15) with $g^{h\alpha}A^k_\alpha$ in the indices h, k gives us

$$\tfrac{n-4}{2} R^{\alpha}_{\bar{i}j\beta} A^{\beta}_{\alpha} = 0.$$

The latter and (3.14) imply that

$$R^{h}_{\bar{i}jk} = 0$$

provided $n \neq 4$, *i.e.* if the object $\overset{3}{T}{}^{h}_{ijk}$ of a RP-space V_n satisfies the condition $\overset{3}{T}{}^{h}_{\bar{i}jk} = 0$, then V_n is almost flat.

3.5. Let V_n be a RP-space in which $\overset{3}{T}{}^{h}_{ij\bar{k}}$ vanishes. Then taking to account (3.8) we obtain the following:

$$\begin{aligned} \tfrac{n^2-4}{4} R_{hij\bar{k}} - \tfrac{n+2}{2}\big(F_{hk}R^{\alpha}_{\bar{i}j\beta} - F_{hj}R^{\alpha}_{\bar{k}i\beta} - g_{hk}R^{\alpha}_{\bar{i}\bar{j}\beta}\big)A^{\beta}_{\alpha} - \\ - \tfrac{n-2}{2}\big(F_{hi}R^{\alpha}_{\beta j\bar{k}} + F_{hj}R^{\alpha}_{\beta i\bar{k}}\big)A^{\beta}_{\alpha} = 0. \end{aligned} \tag{3.16}$$

Conjugating (3.16) in the index j, we obtain:

$$R^{\alpha}_{\bar{i}\bar{j}\beta} A^{\beta}_{\alpha} = 0. \tag{3.17}$$

Multiplying (3.16) by $g^{h\alpha}A^j_\alpha$ and contracting it in the indices h, j, and taking into account (3.17), we find

$$R^{\alpha}_{\bar{i}k\beta} A^{\beta}_{\alpha} = R^{\alpha}_{\bar{k}i\beta} A^{\beta}_{\alpha}. \tag{3.18}$$

Cycling further (3.16) in the indices h, i, j we get

$$\big(F_{hk}R^{\alpha}_{\bar{i}j\beta} + F_{ik}R^{\alpha}_{\bar{j}h\beta} + F_{jk}R^{\alpha}_{\bar{h}i\beta} + F_{hi}R^{\alpha}_{\bar{k}j\beta} + F_{ij}R^{\alpha}_{\bar{k}h\beta} + F_{jh}R^{\alpha}_{\bar{k}i\beta}\big)A^{\beta}_{\alpha} = 0.$$

Symmetrizing the latter in the indices i, j taking into account (3.18), and contracting with $g^{h\alpha}A^k_\alpha$ in the indices h, k, we obtain

$$R^{\alpha}_{\bar{i}j\beta} A^{\beta}_{\alpha} = 0.$$

Hence (3.16) can be rewritten as follows:

$$\tfrac{n+2}{2} R_{hij\bar{k}} - \big(F_{hi}R^{\alpha}_{\beta j\bar{k}} + F_{hj}R^{\alpha}_{\beta i\bar{k}}\big)A^{\beta}_{\alpha} = 0.$$

Alternating this equality in the indices i, j we will see that

$$R_{h\bar{k}ji} = 0,$$

i.e. V_n is almost flat.

3.6. Let V_n be a RP-space in which $\overset{3}{T}{}^{\bar{h}}_{ijk}$ vanishes. Then taking to account (3.8) we obtain the following identity:

$$R^{\bar{h}}_{ijk} - \tfrac{2}{n-2}\big(F^h_k a_{ij} - F^h_j a_{ik}\big) = 0, \tag{3.19}$$

where

$$a_{ij} = R^{\alpha}_{\bar{i}j\beta} A^{\beta}_{\alpha}.$$

In §2.4 it is proved that in this case $a_{ij} = a_{ji}$.

In view of (3.19), relation (3.2) can be written as follows:

$$R_{hk} a_{ij} - F_{hj} a_{ik} - F_{ik} a_{hj} + F_{ij} a_{hk} = F_{hi} q_{[j,k]}. \tag{3.20}$$

Contraction (3.20) with $g^{h\alpha} A^i_\alpha$ in the indices i, h gives us

$$q_{[j,k]} = 0.$$

Therefore, cycling (3.20) in the indices h, j and contracting it further with $g^{h\alpha} A^k_\alpha$ in the indices h, k, we obtain that $a_{ij} = 0$ for $n \neq 4$. It then follows from (3.19) that $R^{\bar{h}}_{ijk} = 0$, that is, the RP-space V_n is almost flat.

One easily checks that in an almost flat RP-space the following identity holds:

$$\overset{3}{T}{}^{h}_{\bar{i}jk} = \overset{3}{T}{}^{h}_{ij\bar{k}} = \overset{3}{T}{}^{\bar{h}}_{ijk} = 0$$

Thus, we have established the following

Theorem 3.6.1. *Each of the following conditions:*

- $\overset{3}{T}{}^{h}_{\bar{i}jk} = 0$ *provided* $n \neq 4$,
- $\overset{3}{T}{}^{h}_{ij\bar{k}} = 0$,
- $\overset{3}{T}{}^{\bar{h}}_{ijk} = 0$ *provided* $n \neq 4$,

in the ***RP****-space* V_n *holds if and only if* V_n *is almost flat, i.e.* $R^{\bar{h}}_{ijk} = 0$. *Therefore, in that cases the recurrence vector* q_i *of* V_n *is gradient, that is,* V_n *admits a Kähler structure*

$$\tilde{F}^h_i = e^{-q(x)} F^h_i, \qquad q_i = \tfrac{\partial q(x)}{\partial x^i}.$$

4. Special type QGM of GRP-spaces

In this section, we consider the QGM of GRP-spaces under certain conditions on their Riemannian tensors.

4.1. Suppose that a QGM of GRP-spaces $f : (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, F_i^h)$ satisfies the following condition

$$\overline{R}^{\overline{h}}_{ijk} - \overline{R}^{h}_{\bar{i}jk} = R^{\overline{h}}_{ijk} - R^{h}_{\bar{i}jk}. \tag{4.1}$$

The relation between the components of the Riemannian tensors R^h_{ijk} and $\overline{R}^h_{ijk}$ of GRP-spaces under QGM has the form (2.5).

Since in (2.5)

$$\psi_{ij} = \psi_{i,j} - \psi_i\psi_j,$$
$$\phi_{ij} = \phi_{i,j} - \phi_i\psi_j - \phi_j\psi_i,$$
$$\phi_{\bar{i}} = \psi_i$$

we get the following equality:

$$\phi_{\bar{i}j} = \psi_{ij} - \phi_\alpha F^\alpha_{i,j}. \tag{4.2}$$

Taking into account (4.1), (2.6), (2.8), (2.3) and (4.2) we get

$$F^h_k\phi_\alpha F^\alpha_{i,j} - F^h_j\phi_\alpha F^\alpha_{i,k} + 2\psi_i F^h_{j,k} - \psi_i q_{(k}F^h_{j)} = 0. \tag{4.3}$$

Symmetrizing (4.3) with respect to the indices i, j and contracting with ϕ_h with respect to the index h, we obtain that

$$\psi_i\phi_\alpha F^\alpha_{j,k} + \psi_j\phi_\alpha F^\alpha_{i,k} - 2q_k\psi_i\psi_j = 0.$$

As $\psi_i \neq 0$ under QGM, there exists a certain vector a^i such that $\psi_\alpha a^\alpha = 1$.

Contracting the last equality with a^i in the index i, and then with a^j in the index j, we find that

$$\phi_\alpha F^\alpha_{j,k} + \psi_j\phi_\alpha F^\alpha_{\beta,k}a^\beta - 2q_k\psi_j = 0,$$
$$2\big(\phi_\alpha F^\alpha_{\beta,k}a^\beta - q_k\big) = 0.$$

The latter together with (4.3) implies that

$$F^h_{i,j} = q_j F^h_i.$$

Thus, if the QGM mapping of GRP-spaces

$$f : (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, F_i^h)$$

satisfies the condition (4.1), then V_n and $\overline{V}_n$ are RP-spaces. It can be verified that (4.1) is satisfied identically for the QGM of RP-spaces.

The following statement is true

Theorem 4.1.1. *Suppose a GRP-space (V_n, g_{ij}, F_i^h) admits a QGM onto a GRP-space $(\overline{V}_n, \overline{g}_{ij}, F_i^h)$. Then the following condition*

$$\overline{R}^{\overline{h}}_{ijk} - \overline{R}^{h}_{\bar{i}jk} = R^{\overline{h}}_{ijk} - R^{h}_{\bar{i}jk}$$

is satisfied if and only if the GRP-structure of V_n and $\overline{V}_n$ is in fact a RP-structure.

4.2. Suppose that a QGM of the GRP-spaces

$$f : (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, F_i^h)$$

satisfies the following condition

$$\overline{R}^{\bar{h}}_{ijk} - \overline{R}^{h}_{\bar{i}jk} - \overline{R}^{h}_{i\bar{j}k} - \overline{R}^{h}_{ij\bar{k}} = R^{\bar{h}}_{ijk} - R^{h}_{\bar{i}jk} - R^{h}_{i\bar{j}k} - R^{h}_{ij\bar{k}}. \tag{4.4}$$

Taking into account (2.5), (4.4) and (4.2) we obtain

$$\begin{aligned} F_i^h(\phi_{\bar{j}k} - \phi_{k\bar{j}} - \phi_{\bar{k}j} + \phi_{j\bar{k}}) + F_j^h(\phi_{\bar{i}k} + \phi_{i\bar{k}}) - F_k^h(\phi_{\bar{i}j} + \phi_{i\bar{j}}) - \\ - \psi_k F^h_{i,j} + \psi_j F^h_{i,k} - \psi_i F^h_{[k,j]} = 0. \end{aligned} \tag{4.5}$$

Let us lower the index h in V_n, symmetrize it with respect to the indices h, i, and then cycle for the indices h, i, j. Then we will get that

$$F_{hk} b_{ij} + F_{ik} b_{jh} + F_{jk} b_{hi} = 0, \tag{4.6}$$

where

$$b_{ij} = \phi_{\bar{i}j} + \phi_{i\bar{j}} + \phi_{\bar{j}i} + \phi_{i\bar{j}} + \psi_{(i} q_{j)}.$$

Contracting (4.6) with $g^{k\alpha} A^h_\alpha$ in the indices h, k we get $b_{ij} = 0$. The latter together with (2.3) means that

$$\begin{aligned} d_{ij} &= -d_{ji}, \\ d_{ij} &= \phi_{i\bar{j}} + \psi_{ij}. \end{aligned} \tag{4.7}$$

Contracting (4.5) with ϕ_h and taking into account (2.3) and (4.7), we obtain

$$\psi_j d_{ik} + \psi_k d_{ji} + 2\psi_i d_{jk} = 0. \tag{4.8}$$

Since $\psi_i \neq 0$ (QGM is not canonical), there exists a certain vector a^i such that $a^\alpha \psi_\alpha = 1$. Contracting (4.8) with a^i in the index i and with $a^i a^j$ in the indices i, j, we find $d_{ij} = 0$, or

$$\phi_{i\bar{j}} = -\psi_{ij}. \tag{4.9}$$

Then (4.5) can be written as follows

$$F_k^h \phi_\alpha F^\alpha_{i,j} - F_j^h \phi_\alpha F^\alpha_{i,k} - F_i^h \phi_\alpha F^\alpha_{[j,k]} - \psi_k F^h_{i,j} + \psi_j F^h_{i,k} - \psi_i F^h_{[k,j]} = 0.$$

Contracting this equation with A^i_h in the indices i, h and with A^j_h in j, h we obtain a system of equations, from which taking into account (2.3) by certain algebraic transformations we get that:

$$\phi_\alpha F^\alpha_{j,k} = \psi_j q_k.$$

Hence, according to (4.9) and (4.2) we can rewrite (4.5) as follows:

$$\psi_i\big(F^h_{[j,k]} - q_k F^h_j + q_j F^h_k\big) + \psi_j\big(F^h_{i,k} - q_k F^h_i\big) - \psi_k\big(F^h_{i,j} - q_j F^h_i\big) = 0.$$

Lowering the index h in V_n and symmetrizing with respect to h, k, we get

$$\psi_k\big(F_{hi,j} - q_j F_{hi}\big) + \psi_h\big(F_{ki,j} - q_j F_{ki}\big) = 0.$$

If we compare this equation with the result of its sequential contraction with a^k, a^h with respect to the indices k, h respectively, we will see that

$$F^h_{i,j} = q_j F^h_i. \tag{4.10}$$

Thus, we obtain that a **GRP**-structure F^h_i of the space V_n turns out to be a **RP**-structure. It can be verified that under condition (4.10) the relations (4.4) are satisfied identically.

The following statement is true

Theorem 4.2.1. *Suppose a* ***GRP****-space* (V_n, g_{ij}, F^h_i) *admits a quasi-geodesic mapping onto* $(\overline{V}_n, \overline{g}_{ij}, F^h_i)$*. Then the condition*

$$\overline{R}^{\bar h}_{ijk} - \overline{R}^{h}_{\bar i jk} - \overline{R}^{h}_{i\bar j k} - \overline{R}^{h}_{ij\bar k} = R^{\bar h}_{ijk} - R^{h}_{\bar i jk} - R^{h}_{i\bar j k} - R^{h}_{ij\bar k}$$

is satisfied if and only if the ***GRP****-structure of* V_n *and* $\overline{V}_n$ *is in fact a* ***RP****-structure.*

Note that [8, Theorem 3.3] is a consequence of the Theorem 4.2.1.

4.3. Note that for a canonical **QGM** of the **GRP**-spaces Theorems 4.1.1 and 4.2.1 do not hold. Namely, if a **GRP**-space (V_n, g_{ij}, F^h_i) admits a canonical **QGM** onto $(\overline{V}_n, \overline{g}_{ij}, F^h_i)$, *i.e.* satisfies one of the conditions (4.1) or (4.4), then the condition

$$F^h_{(i,j)} = q_{(j} F^h_{i)}$$

does not necessarily degenerate into

$$F^h_{i,j} = q_j F^h_i.$$

In particular, a parabolic K-space V_n admitting a canonical **QGM** does not necessarily degenerate into parabolic Kähler.

At the same time, it is easy to show that the canonical **QGM** of **RP**-spaces satisfies the condition (4.4) for $\phi_{\bar i \bar j} = 0$, and the condition (4.1) is holds identically.

5. Conclusion

In the present article, the geometric objects T^h_{ij}, $\overset{1}{T}{}^h_{ijk}$, $\overset{2}{T}{}^h_{lijk}$, being invariant with respect to QGM of the GRP-spaces, and $\overset{3}{T}{}^h_{ijk}$, being invariant with respect to QGM of the RP-spaces, are constructed, see Theorems 2.3.1 and 3.2.1.

It is proved, see Theorem 2.4.1, that if $\overset{1}{T}{}^h_{ijk} = 0$ holds in a GRP-space V_n, then the Ricci tensor satisfies the condition

$$R_{\overline{h}k} = \frac{R}{n} F_{kh}$$

and the generalized recurrence vector of the GRP-structure q_i is gradient, *i.e.* there exists a K-structure $\tilde{F}^h_i = e^{-q} F^h_i$, $q_i = \frac{\partial q(x)}{\partial x^i}$ in V_n.

Moreover, the object $\overset{3}{T}{}^h_{ijk}$ satisfies the condition $\overset{3}{T}{}^h_{ijk} = 0$ in a RP-space V_n, if and only if its Riemannian tensor has the form

$$R_{hijk} = K\big(F_{hk}F_{ij} - F_{hj}F_{ik} + 2F_{hi}F_{kj}\big),$$

where K is a certain invariant, see Theorem 3.3.1. In that case V_n is Ricci-symmetric, and the recurrence vector q_i of the RP-structure is gradient, that is, V_n admits a Kähler structure $\tilde{F}^h_i = e^{-q(x)} F^h_i$, see Theorem 2.4.1.

Further, in Theorem 3.6.1 it is proved that each of the following conditions:

- $\overset{3}{T}{}^h_{\overline{i}ik} = 0$ provided $n \neq 4$,
- $\overset{3}{T}{}^h_{ij\overline{k}} = 0$,
- $\overset{3}{T}{}^{\overline{h}}_{ijk} = 0$ provided $n \neq 4$

in a RP-space V_n, is satisfied if and only if V_n is almost flat, that is, $R^{\overline{h}}_{ijk} = 0$ in V_n. Therefore, the recurrence vector q_i of the RP-space V_n is gradient, that is, V_n admits a Kähler structure $\tilde{F}^h_i = e^{-q(x)} F^h_i$.

Further, we investigated QGM between GRP-spaces V_n, $\overline{V}_n$ which preserve one of the tensors

$$R^{\overline{h}}_{ijk} - R^h_{\overline{i}jk}, \qquad R^{\overline{h}}_{ijk} - R^h_{\overline{i}jk} - R^h_{i\overline{j}k} - R^h_{ij\overline{k}}. \tag{5.1}$$

It turns out that if such a mapping exists, then the GRP-space V_n is recurrent-parabolic, that is the condition $F^h_{(i,j)} = q_{(j}F^h_{i)}$ degenerates into

$$F^h_{i,j} = q_j F^h_i,$$

see Theorems 4.1.1 and 4.2.1.

Note that the later property does not hold for canonical QGM. That is, if under the canonical QGM between GRP-spaces V_n and $\overline{V}_n$, either of the tensors (5.1) is preserved, then GRP-structure F_i^h does not necessarily degenerate into RP-structure.

From the above it follows that it makes sense to further study canonical QGM of GRP-spaces preserving (5.1), and also QGM of RP-spaces with non-gradient recurrence vector.

Naturally, the question arises about the existence of RP-spaces that are not reducible to Kähler spaces, that is, RP-spaces with a non-gradient recurrence vector. We give an example of such a space V_4. Let us put

$$ds^2 = 2x^3x^4dx^1dx^2 + 2e^{x^1+x^2}(dx^1dx^4 - dx^2dx^3),$$

$$(F_i^h) = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix},$$

$$(q_i) = \big(-\tfrac{1}{2}(1 + x^4e^{-x^1-x^2}),\ \tfrac{1}{2}(x^3e^{-x^1-x^2} - 1), 0, 0\big).$$

A direct calculation of the Christoffel symbols of V_4 and the covariant derivative of the affinor, shows that $F_{i,j}^h = q_jF_i^h$. Moreover, the vector q_i is non-gradient, since $\frac{\partial q_1}{\partial x^2} \neq \frac{\partial q_2}{\partial x^1}$.

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