On quasi-geodesic mappings of special pseudo-Riemannian spaces

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Abstract. The present paper continues the study of quasi-geodesic mappings $f : (V_n, g_{ij}, F^h_i) \rightarrow (\overline{V}_n, \overline{g}_{ij}, F^h_i)$ of pseudo-Riemannian spaces $V_n, \overline{V}_n$ with a generalized-recurrent structure $F^h_i$ of parabolic type. By a generalized recurrent structure of parabolic type on $V_n$ we mean an almost Hermitian affinor structure of parabolic type for which the covariant derivative of the structural affinor $F^h_i$ satisfies the condition $F^h_i (i,j) = q_i F^h_j$.

In the previous paper by the authors [Proc. Intern. Geom. Center, 13:3 (2020) 18-32] it was proved that the class of pseudo-Riemannian spaces with generalized-recurrent structure of parabolic type is closed with respect to the considered mappings and the generalized recurrence vectors in $(V_n, g_{ij}, F^h_i)$ and $(\overline{V}_n, \overline{g}_{ij}, F^h_i)$ may be distinct. In this article, it is assumed that the mapping $f$ preserves the generalized recurrence vector $q_i$

We construct geometric objects that are invariant under the quasi-geodesic mapping of generalized-recurrent spaces of parabolic type and recurrent-parabolic spaces. A number of conditions are given on these objects, which lead to the fact that a generalized-recurrent space of parabolic type admits a parabolic $K$-structure, and a recurrent-parabolic space admits a Kählerian structure of parabolic type.

We study special types of these mappings that preserve some tensors of an intrinsic nature.

Keywords: affinor structure, quasi-geodesic mapping

Anotečie. У статті продовжуються дослідження квазі-геодезичних відображень $f : (V_n, g_{ij}, F^h_i) \rightarrow (\overline{V}_n, \overline{g}_{ij}, F^h_i)$ псевдоріманових просторів $V_n, \overline{V}_n$ з узагальнено-рекурентною структурою $F^h_i$ параболічного типу. Узагальнено-рекурентною структурою параболічного типу на $V_n$ називається майже ермітов афінорна структура параболічного типу, для якої коваріантна похідна структурного афінора $F^h_i$ задовольняє умову $F^h_i (i,j) = q_i F^h_j$. В попередній роботі авторів [Proc. Intern. Geom. Center, 13:3 (2020) 18-32] доведено, що клас псевдо-ріманових просторів з узагальнено-рекурентною структурою параболічного типу замкнений відносно розглядуваних відображень, але при цьому вектора узагальненої

Keywords: affinor structure, quasi-geodesic mapping

Ключові слова: афінорна структура, квазі-геодезичне відображення

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On quasi-geodesic mappings (QGM) and their recurrence in spaces $(V_n, g_{ij}, F^h_i)$ and $(\overline{V}_n, \overline{g}_{ij}, F^h_i)$ may not be identical. It is assumed in this paper that the mapping $f$ preserves the vector of generalized recurrence $q_i$.

We construct geometric objects that are invariant under quasigeodesic mappings of para-hyperbolic type and also under recurrence-para-hyperbolic spaces. Various conditions on these objects guarantee that a generalized-para-hyperbolic space para-hyperbolically admits a $K$-structure, whereas a recurrence-para-hyperbolic space admits a Kähler structure of para-hyperbolic type.

We study special types of these mappings that preserve certain tensors of internal character.

1. Introduction

In [8] we considered diffeomorphisms of pseudo-Riemannian spaces being quasi-geodesic mappings (QGM). Also in [5–7, 9, 11] such maps were studied under the reciprocity condition, while almost-geodesic mappings of the second type were considered in [2–4, 10, 12–14]. Say that a QGM

$$f : (V_n, g_{ij}, F^h_i) \to (\overline{V}_n, \overline{g}_{ij}, F^h_i)$$

satisfies the reciprocity condition whenever its inverse $f^{-1}$ is also a QGM.

The basic equations of such a mapping $f : (V_n, g_{ij}, F^h_i) \to (\overline{V}_n, \overline{g}_{ij}, F^h_i)$ in the common coordinate system $(x^i)$ with respect to the mapping $f$ have the form:

$$\Gamma^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi(x)\delta^h_{ij} + \phi(x)F^h_{ij}(x),$$

$$F^h_i(x) = -\overline{F}^h_i(x),$$

$$\overline{g}_{\alpha\beta}F^\alpha_j = -\overline{g}_{\alpha\beta}F^\alpha_i,$$

$$g_{\alpha\beta}F^\alpha_i = -g_{\alpha\beta}F^\alpha_i,$$

$$F^h_{(i,j)} = q(iF^h_j),$$

$$F^h_\alpha F^\alpha_i = e\delta^h_i, \quad e = 0, \pm 1,$$

where $i, h, j, ..., n$, $\Gamma^h_{ij}, \overline{\Gamma}^h_{ij}$ are the Christoffel symbols of $V_n$ and $\overline{V}_n$ respectively; $\psi(x), \phi(x), q_i(x)$ are certain covectors; $F^h_i(x)$ is affinor; brackets $(i, j)$ denote the symmetrization with respect to the corresponding indices; and comma «,» is a sign of the covariant derivative in respect to the connection of $V_n$. If in (1.1) $\phi_i \neq 0$ and $\psi_i = 0$, the quasi-geodesic mapping is called canonical.

Condition (1.5) defines an $e$-structure, which is called

- elliptic if $e = -1$,
- hyperbolic if $e = +1$,
- $m$-parabolic when $e = 0$, rank($F$) = $m$, $(2m < n)$,
• parabolic when \( e = 0 \), \( \text{rank}(F) = m \), \( (2m = n) \).

1.1. In [5] were studied QGM of pseudo-Riemannian spaces \((V_n, g_{ij}, F^h_i)\) with a \( K \)-structure [1] of elliptic and hyperbolic types, i.e. when \( F \) satisfies (1.5) for \( e = \pm 1 \), and the following differential condition holds:

\[ F^h_{(i,j)} = 0. \]

1.2. An affinor structure \( F^h_i \) satisfying conditions (1.4) will be called a generalized-recurrent (GR-) structure of elliptic, hyperbolic or parabolic type depending on the values of \( e \).

Obviously, the \( K \)-structure is a special case of a generalized-recurrent structure.

Another special case of the GR-structure is the recurrent-parabolic (RP-) structure, which was determined in [9] by the conditions

\[ F^h_{\alpha} F^\alpha_i = 0, \quad g_{\alpha j} F^\alpha_j = - g_{j \alpha} F^\alpha_i, \quad F^h_{i,j} = q_j F^h_i. \] (1.6)

In [8], we described properties of parabolic GR-structure (GRP). We call \( q_i \) in (1.4) the generalized recurrence vector of the GR-structure \( F^h_i \). Note that GR-structure is a \( K \)-structure, whenever \( q_i = 0 \). Moreover, under the condition \( q_i = \frac{\partial q(x)}{\partial x} \) GR-space \((V_n, g_{ij}, F^h_i)\) admits the \( K \)-structure

\[ \tilde{F}^h_i = e^{-q} F^h_i. \]

1.3. To make the calculations easier, let us introduce an operation of conjugation in \((V_n, g_{ij}, F^h_i)\):

\[
T_{\ldots j_1 \ldots j_k}^{i_1 \ldots i_r} F^h_i = T_{\ldots j_1 \ldots j_k}^{i_1 \ldots i_r} \tilde{F}^h_i = T_{\ldots j_1 \ldots j_k}^{i_1 \ldots i_r} F^h_i + \tilde{T}_{\ldots j_1 \ldots j_k}^{i_1 \ldots i_r} F^h_i.
\]

In \((V_n, g_{ij}, F^h_i)\) with integrable parabolic structure \( F^h_i \), there exists a local coordinate system (adapted to the affinor), in which the structure tensor can be reduced to the form

\[
(F^h_i) = \begin{pmatrix} 0 & 0 \\ I_m & 0 \end{pmatrix},
\]

where \( I_m \) is the identity matrix of order \( m = \frac{n}{2} \).

Further, we will use the auxiliary tensor \( A^h_i \), which in the adapted system is defined by the matrix

\[
(A^h_i) = \begin{pmatrix} P & I_m \\ - P^2 & -P \end{pmatrix},
\]

where \( P \) is an arbitrary square matrix of order \( m \).

It can be checked that components of tensor \( A \) satisfy the following identities:

\[
F^\alpha_\beta A^\alpha_\beta = m, \quad A^h_\alpha A^\alpha_i = 0, \quad F^h_\alpha A^\alpha_i + A^h_\alpha F^\alpha_i = \delta^h_i. \] (1.7)
1.4. In [8], the properties of a parabolic type generalized-recurrent structure were studied. It was proved that the Riemannian tensor of the GR-space \((V_n, g_{ij}, F^h_i)\) satisfies the relations

\[
3(R_{hjki}^h + R_{hjki} + R_{hjki}) = 2Q_{jhki} + Q_{jhki} - Q_{hkji},
\]

(1.8)

where

\[
Q_{hjki} = q_{[h,j]}F_{ki} + q_{[k,i]}F_{hj}, \quad F_{hj} = g_{h\alpha}F^\alpha_j.
\]

For \(q_i = 0\) the GR-structure \(F^h_i\) is a K-structure. Therefore, in the K-space, the Riemannian tensor has the following property:

\[
R_{hjki} + R_{hjki} + R_{hjki} = 0.
\]

(1.9)

In [8] it was proved that in the GRP-space \((V_n, g_{ij}, F^h_i)\) equality (1.9) holds if and only if the generalized recurrence vector \(q_i\) is a gradient, i.e.

there exists a K-structure \(F^h_i = e^{-q}F^h_i\) in \(V_n\).

1.5. The affinor structure of a GRP-space \((V_n, g_{ij}, F^h_i)\) provided that it is integrable has the following properties [8]:

\[
F^\alpha_{i,\alpha} = 0, \quad F^h_{j,i} = F^h_{j,i}, \quad F^h_{j,i} = 0, \quad q_i = 0.
\]

(1.10)

Note that, in contrast to the hyperbolic and elliptic types, an integrable GR-structure of parabolic type (in particular, a parabolic K-structure) may be not Kähler, i.e. covariant constancy of the affinor \(F^h_i\) does not follow from relation (1.10). Further in this paper, we consider only an integrable affinor structure.

1.6. In [8] it was proved that the image of a GRP-space under QGM is also a GRP-space, that is,

\[
F^h_{(i|j)} = \tilde{q}(j \tilde{F}^h_{i}),
\]

where

\[
\tilde{q}_i = q_i - \psi_i + \phi_i,
\]

«|» is a sign of a covariant derivative in respect to the connection of \(\overline{V}_n\). In other words, the affinor \(F^h_i\) in the space \(\overline{V}_n\) also defines a GRP-structure.

Under the condition \(\tilde{q}_i = q_i\) we will say that QGM preserves the generalized recurrence vector. In this case, the vectors \(\psi_i\) and \(\phi_i\) in the basic QGM equations (1.1) are related by the following identity:

\[
\psi_i = \phi_i.
\]

(1.11)

Hence, contracting (1.1) with respect to \(h\) and \(j\), we get:

\[
\Gamma^\alpha_{i\alpha} = \Gamma^\alpha_{i\alpha} + (n + 2)\psi_i,
\]
that is, ψ is locally a gradient:

\[(n + 2)\psi = \frac{\partial \psi(x)}{\partial x^i}, \quad \psi(x) = \frac{1}{2} \ln \left| \frac{g}{g} \right| \]

In what follows, we construct geometric objects that are invariant with respect to the QGM of parabolic type generalized-recurrent spaces which preserve generalized recurrence vector. We will also study some special types of these mappings provided that the affinor structure is integrable.

The investigations are carried out in tensor form, locally, in the class of real sufficiently smooth functions.

2. SOME GEOMETRIC OBJECTS THAT ARE INVARIANTS UNDER QGM OF GRP-SPACES

2.1. Assume that there exists a QGM

\[ f : (V_n, g, F^h_i) \rightarrow (\overline{V}_n, \overline{g}_{ij}, F^h_i) \]

between GRP-spaces. Then, the following basic equations hold:

\[ \overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_i(x)\delta^h_{ij} + \phi_i(x)F^h_j(x), \]

\[ \overline{g}_{i\alpha}F^\alpha_j = -\overline{g}_{j\alpha}F^\alpha_i, \quad \overline{g}_{i\alpha}F^\alpha_j = -\overline{g}_{j\alpha}F^\alpha_i, \]

\[ F^h_\alpha F^\alpha_i = 0, \quad F^h_{(i,j)} = q_{(j}F^h_{i)}. \]

Contracting (2.1) with respect to \( h, j \) we find that

\[ \psi_i = \frac{1}{n+2} \left( \Gamma^\alpha_{i\alpha} - \Gamma^\alpha_{i\alpha} \right). \]

Let us multiply (2.1) by \( A^h_i \) and contract for indices \( h \) and \( j \). Then, according to (2.3) and (1.7), we get

\[ \phi_i = \frac{2}{n+2} \left( \Gamma^\alpha_{i\beta} - \Gamma^\alpha_{i\beta} \right)A^\beta_\alpha. \]

Substituting \( \phi_i \) and \( \psi_i \) into (2.1), we can rewrite them in the following form

\[ T^h_{ij} = \overline{T}^h_{ij}, \]

where

\[ T^h_{ij} = \Gamma^h_{ij} - \frac{1}{n+2} \left( \Gamma^\alpha_{i\alpha} \delta^h_{ij} + 2A^\beta_\alpha \Gamma^\alpha_{\beta(i}F^h_{j)} \right). \]

Analogously we define \( \overline{T}^h_{ij} \) in \( \overline{V}_n \). In this way, we have proved the invariance of geometric object (2.4) under QGM of GRP-spaces. This object is similar to the Thomas parameters of the theory of geodesic mappings. The equality of Thomas-like objects (2.4) is a necessary and sufficient condition for existence of QGM between \( V_n \) and \( \overline{V}_n \).
2.2. Due to (2.1), we can write a relationship between components of the curvature tensors $V_n$ and $\overline{V}_n$ in the following form:

$$\overline{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik} + F^h_i \phi_{[k,j]} + F^h_k \phi_{ij} - F^h_j \phi_{ik} + \phi_k F^h_{i,j} - \phi_j F^h_{i,k},$$

(2.5)

where we have introduced

$$\psi_{ij} = \psi_{i,j} - \psi_i \psi_j, \quad \phi_{ij} = \phi_{i,j} - \phi_i \psi_j - \phi_j \psi_i,$$

and $[i,j]$ denotes an alternation with respect to the corresponding indices.

Multiplying (2.5) by $F^h_l$ and contracting it with respect to $h$, we get:

$$\overline{R}^h_{ijk} = R^h_{ijk} + F^h_k \psi_{ij} - F^h_j \psi_{ik}.$$  

(2.6)

Multiplying (2.6) by $A^k_h$ and contracting it with respect to $h$ and $k$, we obtain

$$\psi_{ij} = \frac{2}{n-2} (\overline{R}^{\alpha}_{ij\beta} - \overline{R}^{\alpha}_{ij\beta}) A^\beta_\alpha.$$  

Substituting $\psi_{ij}$ into (2.6) and after simple computation we find that

$$\frac{1}{T^h_{ijk}} = \frac{1}{\overline{T}^h_{ijk}},$$

where

$$\frac{1}{T^h_{ijk}} = R^h_{ijk} - \frac{2}{n-2} (F^h_k \overline{R}_{ij\beta} - F^h_j \overline{R}_{ik\beta}) A^\beta_\alpha.$$  

(2.7)

We also define $\frac{1}{T^h_{ijk}}$ in $\overline{V}_n$ in a similar way.

The last equation tells that tensor $\frac{1}{T^h_{ijk}}$ is preserved under the QGM of GRP-spaces. Its preservation is only a necessary condition for existence of QGM between $V_n$ and $\overline{V}_n$.

2.3. Multiplying (2.5) by $F^h_l$ and contracting it with respect to $i$, we get:

$$\overline{R}^h_{ljk} = R^h_{ljk} + F^h_k \phi_{lj} - F^h_j \phi_{lk} + \psi_i F^h_{[k,j]},$$

(2.8)

Due (2.1) and (1.11), the following relations hold:

$$q_{i|k} = q_{i,k} - q(i \psi_k), \quad F^h_{i|j} = F^h_{i,j}.$$  

(2.9)

Multiplying (2.8) by $q_l$, and symmetrizing the resulting equality with respect to $l, i$, we will get from (2.9) that:

$$\frac{2}{T^h_{ijk}} = \frac{2}{\overline{T}^h_{ijk}}.$$
where
\[ 2T_{lijk}^h = q(l\tilde{T}_{lijk}^h) + q_i l\tilde{T}_{kj}^h, \]
\[ \tilde{T}_{lijk}^h = R_{lijk}^h - \frac{2}{n-2} A^\beta R_{\alpha[j}^\beta F_{k]}^h, \]  \hspace{1cm} (2.10)\]
\[ \tilde{T}_{kj}^h = F_{[kj]}^h + \frac{2}{n-2} \left( F_{h}^k F_{[\beta,j]}^\alpha - F_{j}^h F_{[\beta,k]}^\alpha \right) A^\alpha. \]

Analogously we define \( T_{lijk}^2 \) in \( V_n \).

The last equation shows that the tensor \( T_{lijk}^h \) is preserved under the QGM of GRP-spaces. Its preservation is only a necessary condition for the existence of QGM between \( V_n \) and \( \overline{V_n} \).

Thus, we have established the following

**Theorem 2.3.1.** Geometric objects (2.4), (2.7), (2.10) are invariant under QGM of GRP-spaces.

2.4. Let \( V_n \) be a GRP-space in which \( T_{lijk}^1 \) vanishes. Then (2.7) can be written as follows:
\[ R_{lijk}^h - \frac{2}{n-2} \left( F_{k}^h a_{ij} - F_{j}^h a_{ik} \right) = 0, \]
where
\[ a_{ij} = R_{ij\beta}^\alpha A^\beta_\alpha. \]

Lowering the index \( h \) in \( V_n \) we get
\[ R_{lijk}^h + \frac{2}{n-2} \left( F_{hk} a_{ij} - F_{hj} a_{ik} \right) = 0. \]  \hspace{1cm} (2.11)\]

Alternating the latter by \( h, i, j, k \), we get the equation:
\[ F_{hi} a_{[jk]} + F_{jk} a_{[hi]} = 0. \]  \hspace{1cm} (2.12)\]

Let us multiply (2.12) by \( g^{h\alpha} A^i_\alpha \) and contract with respect to \( h, i \). Multiply further the result by \( g^{j\alpha} A^k_\alpha \) and contract with respect to \( j, k \). This will gives us:
\[ \frac{n}{2} a_{[jk]} + F_{jk} g^{\beta\alpha} A^\gamma_\alpha a_{[\beta\gamma]} = 0, \]
\[ g^{\beta\alpha} A^\gamma_\alpha a_{[\beta\gamma]} = 0. \]
Hence
\[ a_{[jk]} = 0. \]

Substituting this to (2.11), we get
\[ R_{lijk}^h + R_{hjk}^i + R_{hj}^i + R_{hjk}^i = 0. \]
The latter means that the generalized recurrence vector \( q_i \) is gradient, i.e. there exists a \( K \)-structure \( \tilde{F}^h_i = e^{-q} F^h_i \) in the GRP-space \( V_n \).

Note that multiplying (1.9) by \( g^{jk} \) and contracting with respect to \( j, k \) we obtain that the Ricci tensor of \( K \)-space (as well as of GRP-space with gradient generalized recurrence vector) satisfies the following conditions:

\[
R_{\mu \nu} = -R_{\nu \mu}.
\] (2.13)

Let us multiply (2.13) by \( g^{h \alpha} A^j_\alpha \) and contract it with respect to \( h, k \). Then we will obtain that

\[
R^{\alpha \beta} A^\beta_\alpha = \frac{R}{2},
\]

where \( R \) is the scalar curvature \( R_{\alpha \beta} g^{\alpha \beta} = R \). Obviously, if \( R_{\mu \nu} = 0 \), then \( R = 0 \).

Multiplying (2.11) by \( g^{ij} \) and contracting it with respect to \( i, j \), we get

\[
R_{\mu \nu} + \frac{2}{n-2} (a_{\alpha \beta} g^{\alpha \beta} F_{\mu \nu} + a_{\mu \nu}) = 0.
\] (2.14)

Hence, taking into account (2.13), we have

\[
a_{\mu \nu} = -a_{\nu \mu}.
\]

The latter identity together with (2.13) implies that

\[
a_{\alpha \beta} g^{\alpha \beta} = \frac{R}{2}, \quad g^{\alpha \gamma} a_{\gamma \beta} A^\beta_\alpha = \frac{R}{4}.
\] (2.15)

Multiplying (2.11) by \( g^{i \alpha} A^j_\alpha \), contracting it with respect to \( i, j \), and taking into account (2.14) and (2.15), we obtain

\[
R_{\mu \nu} = \frac{R}{n} F_{\mu \nu}.
\] (2.16)

**Theorem 2.4.1.** If in GRP-space \( V_n \) equality \( T^h_{ij \alpha} = 0 \) holds, then the Ricci tensor satisfies the following condition:

\[
R_{\mu \nu} = \frac{R}{n} F_{\mu \nu}
\]

and the generalized recurrence vector \( q_i \) is gradient, i.e. there is a \( K \)-structure \( \tilde{F}^h_i = e^{-q} F^h_i \) in \( V_n \).

3. Geometric objects that are invariants of QGM of RP-spaces

3.1. QGM between the RP-spaces \( f : (V_n, g_{ij}, F^h_i) \to (V_n, \tilde{g}_{ij}, F^h_i) \) is characterized by basic equations

\[
\Gamma^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_{(i}(x) \delta^h_{j)} + \phi_{(i}(x) F^h_{j)}(x),
\]

\[
\tilde{g}_{i \alpha} F^\alpha_j = -\tilde{g}_{j \alpha} F^\alpha_i, \quad g_{i \alpha} F^\alpha_j = -g_{j \alpha} F^\alpha_i, \quad F^h_i F^\alpha_i = 0
\]

and

\[
F^h_{i, j} = F^h_{i|j} = q_j F^h_i.
\] (3.1)
According to (3.1), the Riemannian tensor in the $\mathbb{RP}$-space satisfies the following relations, see [6]:

$$ R^h_{ijk} - R^h_{ijk} = F^h_i q_{[j,k]} $$

which is the same as

$$ R^h_{ijk} + R^h_{iijk} = F^h_i q_{[j,k]} \quad (3.2) $$

3.2. Due to (2.1), (2.2), (2.3) and (3.1) we can written a relationship between components of the curvature tensors $R^h_{ijk}$ and $\overline{R}^h_{ijk}$ in the following form

$$ \overline{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik} + F^h_i \tilde{\phi}_{[kj]} + F^h_k \tilde{\phi}_{ij} - F^h_j \phi_{ik}, \quad (3.3) $$

where

$$ \psi_{ij} = \psi_{i,j} + \psi_i \psi_j, $$
$$ \tilde{\phi}_{ij} = \phi_{i,j} - \psi_i \phi_j - \psi_j \phi_i + \phi_i q_j. $$

Taking into account $\psi_i = \phi_i$, we get

$$ \tilde{\phi}_{ij} = \psi_{ij}. $$

Let us multiply (3.3) by $A^k_i$ and contract it with respect to $h, k$. Then contracting further the result with $F^l_j$ with respect to $i$, we get:

$$ \overline{R}^\alpha_{ij\beta} A^\beta = R^\alpha_{ij\beta} A^\beta + \left( \tilde{\phi}_{[i\alpha]} - \psi_{i\alpha} \right) A^\beta_j + \left( \tilde{\phi}_{j\alpha} - \psi_{j\alpha} \right) A^\alpha_i + \frac{n+2}{2} \tilde{\phi}_{ij} - \tilde{\phi}_{ij}, \quad (3.4) $$

$$ \overline{R}^\alpha_{ij\beta} A^\beta = R^\alpha_{ij\beta} A^\beta + \frac{n+2}{2} \tilde{\phi}_{ij}. \quad (3.5) $$

Alternation (3.4) with respect to the indices $i, j$, and subsequent contraction of the result with $F^l_j$ in the index $j$ gives us:

$$ \overline{R}^\alpha_{ij\beta} A^\alpha = R^\alpha_{ij\beta} A^\alpha + \frac{n+2}{2} \tilde{\phi}_{ij}, \quad (3.6) $$

$$ \overline{R}^\alpha_{ij\beta} A^\alpha = R^\alpha_{ij\beta} A^\alpha + \frac{n+2}{2} \left( \tilde{\phi}_{ij} - \psi_{ij} \right). \quad (3.7) $$

Taking into account (3.4)-(3.7), we get:

$$ \frac{n+2}{2} \psi_{ij} = \left( \overline{R}^\alpha_{ij\beta} - R^\alpha_{ij\beta} \right) A^\beta_\alpha, $$

$$ \frac{n^2+4}{4} \tilde{\phi}_{ij} = \frac{n^2+2}{2} \left( \overline{R}^\alpha_{ij\beta} - R^\alpha_{ij\beta} \right) A^\beta_\alpha - \left( \overline{R}^\alpha_{ij\beta} - R^\alpha_{ij\beta} \right) A^\alpha_\beta - \left( \overline{R}^\alpha_{ij\gamma} - R^\alpha_{ij\gamma} \right) A^\beta_\gamma A^\gamma_\alpha. $$

Substituting $\psi_{ij}$ and $\tilde{\phi}_{ij}$ into the equations (3.3), we represent them in the form

$$ T^h_{ijk} = \overline{T}^h_{ijk}. $$
where
\[
\frac{3}{4}T_{ijk}^h = \frac{n^2 - 4}{4} R_{ijk}^h - \frac{n + 2}{2} (\delta^h_k R_{ij\beta}^\alpha - \delta^h_j R_{ik\beta}^\alpha) A_\beta^\alpha + 
\]
\[+ F_i^h D_{[kj]} - F_k^h D_{ij} + F_j^h D_{ik}, \tag{3.8}\]

\[D_{ij} = \left( \frac{n + 2}{2} R_{ij\beta}^\alpha - R_{\beta ji}^\alpha - R_{\beta i\gamma}^\alpha A_\gamma^\alpha \right) A_\alpha^\beta. \tag{3.9}\]

Analogously we define \(T_{ijk}^h\) in \(V_n\).

The last equation shows that the tensor \(T_{ijk}^h\) is preserved under the QGM of RP-spaces. Again, its preservation is only a necessary condition for existence of QGM between \(V_n\) and \(\overline{V}_n\). So, we proved the following

**Theorem 3.2.1.** Geometric object (3.8) is an invariant of QGM of RP-spaces.

3.3. Let \(V_n\) be a RP-space in which \(T_{ijk}^h\) vanishes. Then taking to account (3.8) we obtain the following relation:

\[\frac{n^2 - 4}{4} R_{hijk}^h = \frac{n + 2}{2} (\delta^h_k R_{ij\beta}^\alpha - \delta^h_j R_{ik\beta}^\alpha) A_\beta^\alpha + F_i^h D_{[kj]} + F_k^h D_{ij} - F_j^h D_{ik}. \tag{3.10}\]

Lowering further the index \(h\) in \(V_n\) we get

\[\frac{n^2 - 4}{4} R_{hijk} = \frac{n + 2}{2} (\delta^h_k R_{ij\beta}^\alpha - \delta^h_j R_{ik\beta}^\alpha) A_\beta^\alpha + F_i^h D_{[kj]} + F_k^h D_{ij} - F_j^h D_{ik}. \tag{3.11}\]

Cycling (3.10) by indices \(h, i, k\), and contracting it with \(g^{h\alpha} A_j^\alpha\) in the indices \(h\) and \(j\), we get

\[D_{ik} = 0. \tag{3.11}\]

Taking account of (3.10) and (3.11) we obtain

\[R_{ijk}^h = 0. \tag{3.12}\]

We will call a GRP-space \(V_n\) almost flat if the Riemannian tensor \(R_{ijk}^h\) of \(V_n\) satisfies the condition (3.12). According to (3.2), it follows that the recurrence vector \(q_i\) of almost flat RP-space \(V_n\) is gradient. Therefore, an almost flat \(V_n\) admits the Kähler structure

\[\tilde{F}_i^h = e^{-q(x)} F_i^h, \quad q_i = \frac{\partial q(x)}{\partial x_i}. \]

Symmetrizing (3.10) with respect to \(h, i\) and taking account (3.11) we get:

\[F_{hk} D_{ij} + F_{ik} D_{hj} - F_{hj} D_{ik} - F_{ij} D_{hk} = 0.\]
Contracting the latter with $g^{h\alpha}A^k_\alpha$ we obtain:

$$D_{ij} = \tilde{D}F_{ij},$$

where $\tilde{D} = \frac{2}{n}D_{\alpha\beta}g^{\alpha\gamma}A^\gamma_\beta$.

Substituting further $D_{ij}$ into 3.10, one can write down it in the follows form:

$$R_{hijk} = K(F_{hk}F_{ij} - F_{hj}F_{ik} + 2F_{hi}F_{kj}), \quad (3.13)$$

where $K = \frac{4\tilde{D}}{n^2-4}$. The latter means that $R_{ij} = 0$, that is, the space is Ricci-flat.

Taking account to (3.13) and (3.8) we obtain $T^3_{ijk} = 0$. Thus, we have proved the following

**Theorem 3.3.1.** An object $T^3_{ijk}$ in the $RP$-space $V_n$ satisfies the equation

$$T^3_{ijk} = 0 \text{ if and only if its Riemannian tensor has the form (3.13).}$$

It was proved in [9] that the Riemannian tensor of a $RP$-space $V_n$ admitting QGM on a flat space, has the form (3.13), and $K = Ce^{-2q(x)}$, $C = \text{const}$, $q_i = \frac{\partial q(x)}{\partial x^i}$.

It was also shown in [9] that such a $RP$-space is symmetric, that is, $R^h_{ijk,l} = 0$, and the components of metric tensor of all such spaces are found.

3.4. Let $V_n$ be a $RP$-space in which $T^3_{ijk}$ vanishes. Then taking to account (3.8) we obtain the following identity:

$$\frac{n-2}{2}R_{hijk} = (F_{hk}R^\alpha_{ij\beta} - F_{hj}R^\alpha_{ik\beta})A^\beta_\alpha. \quad (3.14)$$

Lowering the index $h$ and cycling with respect to indices $h, k, j$, we get

$$\left(F_{hk}R^\alpha_{ij\beta} + F_{kj}R^\alpha_{ih\beta} + F_{jh}R^\alpha_{ik\beta}\right)A^\beta_\alpha = 0. \quad (3.15)$$

Conjugating (3.15) in the index $j$, we obtain

$$R^\alpha_{ij\beta}A^\beta_\alpha = 0.$$

Also, contraction (3.15) with $g^{h\alpha}A^k_\alpha$ in the indices $h, k$ gives us

$$\frac{n-4}{2}R^\alpha_{ij\beta}A^\beta_\alpha = 0.$$

The latter and (3.14) imply that

$$R^h_{ijk} = 0.$$
provided \( n \neq 4 \), \( i.e. \) if the object \( T_{ijk}^h \) of a \( \text{RP} \)-space \( V_n \) satisfies the condition \( T_{ijk}^h = 0 \), then \( V_n \) is almost flat.

3.5. Let \( V_n \) be a \( \text{RP} \)-space in which \( T_{ijk}^h \) vanishes. Then taking to account \((3.8)\) we obtain the following:

\[
\frac{n^2-4}{4} R_{hijk} - \frac{n+2}{2} \left( F_{hk} R_{ij\beta}^\alpha - F_{hj} R_{ki\beta}^\alpha - g_{hk} R_{ij\beta}^\alpha \right) A_\alpha^\beta - \frac{n-2}{2} \left( F_{hi} R_{\beta jk}^\alpha + F_{hj} R_{\beta ik}^\alpha \right) A_\alpha^\beta = 0. \tag{3.16}
\]

Conjugating \((3.16)\) in the index \( j \), we obtain:

\[
R_{ij\beta}^\alpha A_\alpha^\beta = 0. \tag{3.17}
\]

Multiplying \((3.16)\) by \( g^{ha} A^k_\alpha \) and contracting it in the indices \( h, j \), and taking into account \((3.17)\), we find

\[
R_{\alpha ki\beta} A^\beta_\alpha = R_{\alpha ki\beta} A^\beta_\alpha. \tag{3.18}
\]

Cycling further \((3.16)\) in the indices \( h, i, j \) we get

\[
\left( F_{hk} R_{ij\beta}^\alpha + F_{ik} R_{jh\beta}^\alpha + F_{jk} R_{hi\beta}^\alpha + F_{hi} R_{jk\beta}^\alpha + F_{ij} R_{kh\beta}^\alpha + F_{jh} R_{ki\beta}^\alpha \right) A_\alpha^\beta = 0.
\]

Symmetrizing the latter in the indices \( i, j \) taking into account \((3.18)\), and contracting with \( g^{ha} A^k_\alpha \) in the indices \( h, k \), we obtain

\[
R_{ij\beta}^\alpha A_\alpha^\beta = 0.
\]

Hence \((3.16)\) can be rewritten as follows:

\[
\frac{n+2}{2} R_{hijk} - \left( F_{hi} R_{\beta jk}^\alpha + F_{hj} R_{\beta ik}^\alpha \right) A_\alpha^\beta = 0.
\]

Alternating this equality in the indices \( i, j \) we will see that

\[
R_{hki} = 0,
\]

\( i.e. \) \( V_n \) is almost flat.

3.6. Let \( V_n \) be a \( \text{RP} \)-space in which \( T_{ijk}^h \) vanishes. Then taking to account \((3.8)\) we obtain the following identity:

\[
R_{ijk}^h - \frac{2}{n-2} \left( F_{ik}^h a_{ij} - F_{jk}^h a_{ik} \right) = 0, \tag{3.19}
\]

where

\[
a_{ij} = R_{ij\beta}^\alpha A_\alpha^\beta.
\]

In §2.4 it is proved that in this case \( a_{ij} = a_{ji} \).
In view of (3.19), relation (3.2) can be written as follows:

\[ R_{hk}a_{ij} - F_{hj}a_{ik} - F_{ik}a_{hj} + F_{ij}a_{hk} = F_{hi}q_{[j,k]} . \]  

(3.20)

Contraction (3.20) with \( g^{h}A^i_{\alpha} \) in the indices \( i,h \) gives us

\[ q_{[j,k]} = 0 . \]

Therefore, cycling (3.20) in the indices \( h,j \) and contracting it further with \( g^{h}A^k_{\alpha} \) in the indices \( h,k \), we obtain that \( a_{ij} = 0 \) for \( n \neq 4 \). It then follows from (3.19) that \( R_{ijk}^\overline{h} = 0 \), that is, the RP-space \( V_n \) is almost flat.

One easily checks that in an almost flat RP-space the following identity holds:

\[ T^3_{ijk} = 0 \text{ for } n \neq 4. \]

Thus, we have established the following

**Theorem 3.6.1.** Each of the following conditions:

- \( T^h_{ijk} = 0 \) provided \( n \neq 4 \),
- \( T^h_{ij\overline{k}} = 0 \),
- \( T^h_{ij\overline{k}} = 0 \) provided \( n \neq 4 \),

in the RP-space \( V_n \) holds if and only if \( V_n \) is almost flat, i.e. \( R_{ijk}^\overline{h} = 0 \). Therefore, in that cases the recurrence vector \( q_i \) of \( V_n \) is gradient, that is, \( V_n \) admits a Kähler structure

\[ \tilde{F}_i^h = e^{-q(x)} F_i^h, \quad q_i = \frac{\partial q(x)}{\partial x^i}. \]

4. SPECIAL TYPE QGM OF GRP-SPACES

In this section, we consider the QGM of GRP-spaces under certain conditions on their Riemannian tensors.

4.1. Suppose that a QGM of GRP-spaces \( f : (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, F_i^h) \) satisfies the following condition

\[ R_{ijk}^h - R_{ijk}^\overline{h} = R_{ijk}^\overline{h} - R_{ijk}^h. \]  

(4.1)

The relation between the components of the Riemannian tensors \( R_{ijk}^h \) and \( R_{ijk}^\overline{h} \) of GRP-spaces under QGM has the form (2.5).

Since \( \psi_{ij} = \psi_{i,j} - \psi_i \psi_j \), \( \phi_{ij} = \phi_{i,j} - \phi_i \psi_j - \phi_j \psi_i \) and \( \phi_i = \psi_i \) in (2.5), the following equality holds

\[ \phi_{ij} = \psi_{ij} - \phi_{i}F_{ij}^\alpha. \]  

(4.2)

Taking into account (4.1), (2.6), (2.8), (2.3) and (4.2) we get

\[ F_k^h \phi_{i}F_{ij}^\alpha - F_j^h \phi_{i}F_{ik}^\alpha + 2\psi_i F_{ji,k}^h - \psi_i q_{(k}F_{j)}^h = 0. \]  

(4.3)
Symmetrizing (4.3) with respect to the indices $i, j$ and contracting with $\phi_h$ with respect to the index $h$, we obtain that
\[
\psi_i \phi_\alpha F^\alpha_{j,k} + \psi_j \phi_\alpha F^\alpha_{i,k} - 2q_k \psi_i \psi_j = 0.
\]
As $\psi_i \neq 0$ under QGM, there exists a certain vector $a^i$ such that $\psi_i a^\alpha = 1$.

Contracting the last equality with $a^i$ in the index $i$, and then with $a^j$ in the index $j$, we find that
\[
\phi_\alpha F^\alpha_{j,k} + \psi_j \phi_\alpha F^\alpha_{i,k} a^\beta - 2q_k \psi_j = 0,
\]
\[
2(\phi_\alpha F^\alpha_{\beta,k} a^\beta - q_k) = 0.
\]
The latter together with (4.3) implies that
\[
F^h_{i,j} = q_j F^h_i.
\]

Thus, if the QGM mapping of GRP-spaces $f : (V_n, g_{ij}, F^h_i) \to (\overline{V}_n, \overline{g}_{ij}, F^h_i)$ satisfies the condition (4.1), then $V_n$ and $\overline{V}_n$ are RP-spaces. It can be verified that (4.1) is satisfied identically for the QGM of RP-spaces.

The following statement is true

**Theorem 4.1.1.** Suppose a GRP-space $(V_n, g_{ij}, F^h_i)$ admits a QGM onto a GRP-space $(\overline{V}_n, \overline{g}_{ij}, F^h_i)$. Then the following condition
\[
\overline{R}^h_{ijk} - R^h_{ijk} = R^h_{ijk} - R^h_{ijk}
\]
is satisfied if and only if the GRP-structure of $V_n$ and $\overline{V}_n$ is in fact a RP-structure.

4.2. Suppose that a QGM of the GRP-spaces $f : (V_n, g_{ij}, F^h_i) \to (\overline{V}_n, \overline{g}_{ij}, F^h_i)$ satisfies the following condition
\[
R^\overline{h}_{ijk} - R^h_{ijk} - R^h_{ij} - R^h_{ijk} = R^\overline{h}_{ijk} - R^h_{ijk} - R^h_{ijk} - R^h_{ijk}, \quad (4.4)
\]
Taking into account (2.5), (4.4) and (4.2) we obtain
\[
F^h_i (\phi_{i\overline{j}k} - \phi_{i\overline{k}j} + \phi_{j\overline{k}}) + F^h_j (\phi_{\overline{i}k} + \phi_{i\overline{k}}) - F^h_k (\phi_{i\overline{j}} + \phi_{i\overline{j}}) - \\
- \psi_k F^h_{i,j} + \psi_j F^h_{i,k} - \psi_i F^h_{k,j} = 0. \quad (4.5)
\]
Let us lower the index $h$ in $V_n$, symmetrize it with respect to the indices $h, i, j$, and then cycle for the indices $h, i, j$. Then we will get that
\[
F_{hk} b_{ij} + F_{ik} b_{jh} + F_{jk} b_{hi} = 0, \quad (4.6)
\]
where
\[
b_{ij} = \phi_{ij} + \phi_{ij} + \phi_{ij} + \phi_{ij} + \psi_i q_j.
\]
Contracting (4.6) with $g^{k\alpha}A^h_\alpha$ in the indices $h, k$ we get $b_{ij} = 0$. The latter together with (2.3) means that
\[ d_{ij} = -d_{ji}, \quad d_{ij} = \phi_{i\overline{j}} + \psi_{ij}. \] 
(4.7)

Contracting (4.5) with $\phi_h$ and taking into account (2.3) and (4.7), we obtain
\[ \psi_j d_{ik} + \psi_k d_{ji} + 2\psi_i d_{jk} = 0. \] 
(4.8)

Since $\psi_i \neq 0$ (QGM is not canonical), there exists a certain vector $a_i$ such that $a^\alpha\psi_\alpha = 1$. Contracting (4.8) with $a_i$ in the index $i$ and with $a_i a_j$ in the indices $i, j$, we find $d_{ij} = 0$, or
\[ \phi_{i\overline{j}} = -\psi_{ij}. \] 
(4.9)

Then (4.5) can be written as follows
\[ F^h_i F^\alpha_{j,k} - F^h_j F^\alpha_{i,k} - F^h_i F^\alpha_{j,k} - \psi_k F^h_{i,j} + \psi_j F^h_{i,k} - \psi_i F^h_{[k,j]} = 0. \]

Contracting this equation with $A^i_h$ in the indices $i, h$ and with $A^j_h$ in $j, h$ we obtain a system of equations, from which taking into account (2.3) by certain algebraic transformations we get that:
\[ \phi_{i\alpha} F^\alpha_{j,k} = \psi_j q_k. \]

Hence, according to (4.9) and (4.2) we can rewrite (4.5) as follows:
\[ \psi_i (F^h_{[i,k]} - q_k F^h_j + q_j F^h_k) + \psi_j (F^h_{i,k} - q_k F^h_i) - \psi_k (F^h_{i,j} - q_j F^h_i) = 0. \]

Lowering the index $h$ in $V_n$ and symmetrizing with respect to $h, k$, we get
\[ \psi_k (F_{hi,j} - q_j F_{hi}) + \psi_h (F_{ki,j} - q_j F_{ki}) = 0. \]

If we compare this equation with the result of its sequential contraction with $a^k, a^h$ with respect to the indices $k, h$ respectively, we will see that
\[ F^h_{i,j} = q_j F^h_i. \] 
(4.10)

Thus, we obtain that a GRP-structure $F^h_i$ of the space $V_n$ turns out to be a RP-structure. It can be verified that under condition (4.10) the relations (4.4) are satisfied identically.

The following statement is true

**Theorem 4.2.1.** Suppose a GRP-space $(V_n, g_{ij}, F^h_i)$ admits a quasi-geodesic mapping onto $(\overline{V}_n, \overline{g}_{ij}, F^h_i)$. Then the condition
\[ \overline{R}^h_{i\overline{j}k} - \overline{R}^h_{ij\overline{k}} - \overline{R}^h_{i\overline{j}k} - \overline{R}^h_{i\overline{j}k} = \overline{R}^h_{i\overline{j}k} - \overline{R}^h_{ij\overline{k}} - \overline{R}^h_{i\overline{j}k} - \overline{R}^h_{i\overline{j}k} \]
is satisfied if and only if the GRP-structure of $V_n$ and $\overline{V}_n$ is in fact a RP-structure.
Note that [8, Theorem 3.3] is a consequence of the Theorem 4.2.1.

4.3. Note that for a canonical QGM of the GRP-spaces Theorems 4.1.1 and 4.2.1 do not hold. Namely, if a GRP-space \((V_n, g_{ij}, F^h_i)\) admits a canonical QGM onto \((\overline{V}_n, \overline{g}_{ij}, F^h_i)\), i.e. satisfies one of the conditions (4.1) or (4.4), then the condition

\[ F^h_{(i,j)} = q(j)F^h_i \]

does not necessarily degenerate into

\[ F^h_{i,j} = q_j F^h_i. \]

In particular, a parabolic K-space \(V_n\) admitting a canonical QGM does not necessarily degenerate into parabolic Kähler.

At the same time, it is easy to show that the canonical QGM of RP-spaces satisfies the condition (4.4) for \(\phi_{ij} = 0\), and the condition (4.1) holds identically.

5. Conclusion

In the present article, the geometric objects \(T^h_{ij} \), \( T^h_{ijk} \), being invariant with respect to QGM of the GRP-spaces, and \( T^h_{ijk} \), being invariant with respect to QGM of the RP-spaces, are constructed, see Theorems 2.3.1 and 3.2.1.

It is proved, see Theorem 2.4.1, that if \( T^h_{ijk} = 0 \) holds in a GRP-space \(V_n\), then the Ricci tensor satisfies the condition \( R^h_{kk} = \frac{R}{n} F^h_{kh} \) and the generalized recurrence vector of the GRP-structure \( q_i \) is gradient, i.e. there exists a K-structure \( \tilde{F}^h_i = e^{-q} F^h_i \), \( q_i = \frac{\partial q(x)}{\partial x_i} \) in \(V_n\).

Moreover, the object \( T^h_{ijk} \) satisfies the condition \( T^h_{ijk} = 0 \) in a RP-space \(V_n\), if and only if its Riemannian tensor has the form

\[ R_{hijk} = K \left( F_{hk} F_{ij} - F_{hj} F_{ik} + 2 F_{hi} F_{kj} \right), \]

where \( K \) is a certain invariant, see Theorem 3.3.1. In that case \(V_n\) is Ricci-symmetric, and the recurrence vector \( q_i \) of the RP-structure is gradient, that is, \(V_n\) admits a Kähler structure \( \tilde{F}^h_i = e^{-q(x)} F^h_i \), see Theorem 2.4.1.

Further, in Theorem 3.6.1 it is proved that each of the following conditions:

- \( T^h_{iik} = 0 \) provided \( n \neq 4 \),
- \( T^h_{ijk} = 0 \),
- \( T^h_{ijk} = 0 \) provided \( n \neq 4 \)
in a RP-space $V_n$, is satisfied if and only if $V_n$ is almost flat, that is, $\check{R}_{ijk}^h = 0$ in $V_n$. Therefore, the recurrence vector $q_i$ of the RP-space $V_n$ is gradient, that is, $V_n$ admits a Kähler structure $\check{F}_i^h = e^{-q(x)} F_i^h$.

Further, we investigated QGM between GRP-spaces $V_n$, $\overline{V}_n$ which preserve one of the tensors $R_{ijk}^h$, $\check{R}_{ijk}^h$, $\check{R}_{ij}^h$, $\check{R}_{i}^h$. Therefore, the recurrence vector $q_i$ of the RP-space $V_n$ is gradient, that is, $V_n$ admits a Kähler structure $\check{F}_i^h = e^{-q(x)} F_i^h$.

It turns out that if such a mapping exists, then the GRP-space $V_n$ is recurrent-parabolic, that is the condition $F_i^h = q_j F_i^h$ degenerates into $F_i^h = q_j F_i^h$, see Theorems 4.1.1 and 4.2.1.

Note that the later property does not hold for canonical QGM. That is, if under the canonical QGM between GRP-spaces $V_n$ and $\overline{V}_n$, either of the tensors (5.1) is preserved, then GRP-structure $F_i^h$ does not necessarily degenerate into RP-structure.

From the above it follows that it makes sense to further study canonical QGM of GRP-spaces preserving (5.1), and also QGM of RP-spaces with non-gradient recurrence vector.

Naturally, the question arises about the existence of RP-spaces that are not reducible to Kähler spaces, that is, RP-spaces with a non-gradient recurrence vector. We give an example of such a space $V_4$. Let us put

$$ds^2 = 2x^3 x^4 dx^1 dx^2 + 2e^{x_1+x_2}(dx^1 dx^4 - dx^2 dx^3),$$

$$(F_i^h) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$(q_i) = (-\frac{1}{2}(1 + x^4 e^{-x_1-x_2}), \frac{1}{2}(x^3 e^{-x_1-x_2} - 1), 0, 0).$$

A direct calculation of the Christoffel symbols of $V_4$ and the covariant derivative of the affinor, shows that $F_i^{h, ij} = q_j F_i^h$. Moreover, the vector $q_i$ is non-gradient, since $\frac{\partial q_1}{\partial x^2} \neq \frac{\partial q_2}{\partial x^1}$.

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