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Geodesic Ricci-symmetric pseudo-Riemannian spaces

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Abstract. We introduced special pseudo-Riemannian spaces, called geodesic A-symmetric spaces, into consideration. It is proven that there are no geodesic symmetric spaces and no geodesic Ricci symmetric spaces, which differ from spaces of constant curvature and Einstein spaces respectively. The research is carried out locally, by tensor methods, without any limitations imposed on a metric and a sign.

Анотація. Введено в розгляд спеціальні псевдоріманові простори, які називаються геодезично A-симетричними. Доведено, що не існує геодезично симетричних просторів та геодезично Річчі симетричних просторів відмінних від просторів сталої кривини та просторів Ейнштейна відповідно. Дослідження ведуться локально, тензорними методами без обмежень на метрику та сигнатуру.

1. Introduction

The work of T. Levi-Civita [16] holds a particular place among the studies on the geodesic mappings of pseudo-Riemannian spaces. There, starting from equations of dynamics, he formulated the problem and obtained the main equations. He constantly applied tensor methods.

Since tensor methods took over the field of differential geometry, H. Weyl, L. P. Eisenhart, V. F. Kagan, G. I. Kruchkovich, A. S. Solodovnikov and others built an elegant theory of geodesic mappings of pseudo-Riemannian spaces, invariant in relation to a choice of coordinate system [2,4,15,25–27].

A new impulse to this theory was given by M. S. Sinyukov, who reduced the problem to the study of a linear system of differential equations [20].

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Ключові слова: псевдо-ріманові простори, геодезично Річі-симетричні простори, геодезичні відображення

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Recall that a bijection between points of pseudo-Riemannian spaces V_n with a metric tensor g_{ij} and $\bar{V}_n$ with a metric tensor $\bar{g}_{ij}$ is called *a geodesic mapping*, when it results in a situation, when every geodesic line of V_n corresponding to a geodesic line of $\bar{V}_n$.

Pseudo-Riemannian spaces V_n and $\bar{V}_n$, which permit a geodesic mapping of one on the other, are called spaces, which are in geodesic correspondence, or, in other words, spaces, which do belong to a single geodesic class [8, 9].

The necessary and sufficient condition of a geodesic mapping one onto the other for a pair of given pseudo-Riemannian spaces V_n and $\bar{V}_n$ can be formulated as follows:

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \varphi_i \delta^h_j + \varphi_j \delta^h_i, \tag{1.1}$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are connections of V_n and $\bar{V}_n$ respectively, δ^h_i are Kronecker symbols, and $\varphi_i = \partial_i \varphi$ is a gradient vector.

A geodesic mapping, which is not homothetic is called *non-trivial.*

In the course of research, it turned out that there are large classes of pseudo-Riemannian spaces, which do not permit non-trivial geodesic mappings. In other words, these spaces are unequivocally defined by their geodesic lines [6, 18].

Pseudo-Riemannian space V_n, which contains a tensor $A_{i_1 i_2 \dots i_k}$ satisfying

$$A_{i_1 i_2 \dots i_k, j} = 0, \tag{1.2}$$

is called *A-symmetric.* Here comma ",“ is a sign of the covariant derivative by the connection of V_n.

Namely, if the condition (1.2) is true for Riemann tensor of V_n, then spaces are called *symmetric.* Also, if the condition (1.2) is true for Ricci tensor, then the given spaces are called *Ricci-symmetric.*

Tensors of Riemann and Ricci, respectively, are defined as follows:

$$R^h_{.ijk} = \partial_j \Gamma^h_{ik} + \Gamma^\alpha_{ik} \Gamma^h_{j\alpha} - \partial_k \Gamma^h_{ij} - \Gamma^\alpha_{ij} \Gamma^h_{k\alpha},$$
$$R_{ij} = R^\alpha_{ij\alpha}.$$

It is already known that symmetric spaces, which are not spaces of constant curvature, as well as Ricci-symmetric spaces, which are not Einstein spaces, do not permit non-trivial geodesic mappings [10, 21–24].

A Geodesic A-symmetric space is a pseudo-Riemannian space, in which the condition (1.2) is true for a covariant derivative by the connection of a pseudo-Riemannian space $\bar{V}_n$, which is in a geodesical correspondence with the given space V_n. Namely, when the following condition is true for the Ricci tensor of a pseudo-Riemannian space

$$\nabla_k R_{ij} = 0, \tag{1.3}$$

(where ∇_k is a sign of the covariant derivative by the connection of $\bar{V}_n$), the space is called *Ricci-symmetric*.

This paper treats the spaces of this type.

2. PROPERTIES OF GEODESIC RICCI-SYMMETRIC SPACES

In the case, when a tensor field S belongs to type $\binom{p}{q}$, the covariant derivative by the connection of V_n in any system of coordinates $(x^1, x^2, \ldots, x^n)$ can be defined as follows:

$$\begin{aligned} S^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q, k}(x) &= \partial_k S^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}(x) + \\ &+ \Gamma^{i_1}_{k\alpha}(x) S^{\alpha i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}(x) + \ldots + \Gamma^{i_p}_{k\alpha}(x) S^{i_1 i_2 \ldots i_{p-1}\alpha}_{j_1 j_2 \ldots j_q}(x) - \\ &- \Gamma^{\beta}_{kj_1}(x) S^{i_1 i_2 \ldots i_p}_{\beta j_2 \ldots j_q}(x) - \ldots - \Gamma^{\beta}_{kj_q}(x) S^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_{q-1}\beta}(x), \end{aligned} \tag{2.1}$$

for $i_1, \ldots, i_p, j_1, \ldots j_q, k = 1, 2, \ldots, n$.

The following identity holds for a space $\bar{V}_n$ and covariant derivative ∇ in the latter space, in a common system of coordinates:

$$\begin{aligned} \nabla_k S^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}(x) &= \partial_k S^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}(x) + \\ &+ \bar{\Gamma}^{i_1}_{k\alpha}(x) S^{\alpha i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}(x) + \ldots + + \bar{\Gamma}^{i_p}_{k\alpha}(x) S^{i_1 i_2 \ldots i_{p-1}\alpha}_{j_1 j_2 \ldots j_q}(x) - \\ &- \bar{\Gamma}^{\beta}_{kj_1}(x) S^{i_1 i_2 \ldots i_p}_{\beta j_2 \ldots j_q}(x) - \ldots - \bar{\Gamma}^{\beta}_{kj_q}(x) S^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_{q-1}\beta}(x), \end{aligned} \tag{2.2}$$

for $i_1, \ldots, i_p, j_1, \ldots j_q, k = 1, 2, \ldots, n$. Subtracting (2.1) off (2.2) and taking into account (1.1) we get:

$$\begin{aligned} \nabla_k S^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} - S^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q, k} &= \left(\delta^{i_1}_k \varphi_\alpha + \delta^{i_1}_\alpha \varphi_k\right) S^{\alpha i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} + \ldots + \\ + \left(\delta^{i_p}_k \varphi_\alpha + \delta^{i_p}_\alpha \varphi_k\right) S^{i_1 i_2 \ldots i_{p-1}\alpha}_{j_1 j_2 \ldots j_q} &- \left(\delta^{\beta}_k \varphi_{j_1} + \delta^{\beta}_{j_1} \varphi_k\right) S^{i_1 i_2 \ldots i_p}_{\beta j_2 \ldots j_q} - \ldots - \\ &- \left(\delta^{\beta}_k \varphi_{j_q} + \delta^{\beta}_{j_q} \varphi_k\right) S^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_{q-1}\beta}, \end{aligned}$$

for $i_1, \ldots, i_p, j_1, \ldots j_q, k = 1, 2, \ldots, n$. Applying further the latter formula to the Ricci tensor and taking into account (1.1) and (1.3), we obtain

$$R_{ij,k} = 2\varphi_k R_{ij} + \varphi_i R_{jk} + \varphi_j R_{ik}. \tag{2.3}$$

Wrapping the latter by indices i, j:

$$R_{,k} = 2R\varphi_k + 2\varphi_\alpha R^\alpha_k,$$

and then by i, k:

$$R^\alpha_{j,\alpha} = 3\varphi_\alpha R^\alpha_j + R\varphi_j,$$

we get that

$$\varphi_\alpha R^\alpha{}_k = 0 \tag{2.4}$$

and

$$\frac{R_{,i}}{R} = 2\varphi_i,$$

or

$$\partial_i \log |R| = 2\varphi_i.$$

Here R is a scalar curvature, chosen so that $R = R_{\alpha\beta} g^{\alpha\beta}$, g^{ij} are elements of the matrix inverse to the matrix $\|g_{ij}\|$, and $R^i_j = R_{\alpha j} g^{\alpha i}$.

Wrapping the equation (1.1) by indices h, j we get the following expression:

$$\bar{\Gamma}^{\alpha}_{i\alpha}(x) = \Gamma^{\alpha}_{i\alpha}(x) + (n+1)\varphi_i(x). \tag{2.5}$$

The rules of differentiating the determinant imply that in every pseudo-Riemannian space V_n the following formula holds:

$$\Gamma^{\alpha}_{i\alpha}(x) = \frac{1}{2}\partial_i \log |g|,$$

where $g = \det \|g_{ij}\|$. Then, when constructing a geodesic mapping for Riemannian space, (2.5) implies that

$$2(n+1)\varphi_i = \partial_i \log \left|\frac{\bar{g}}{g}\right|. \tag{2.6}$$

Since the ratio $\dfrac{\bar{g}}{g}$ is invariant, then (2.6) implies that vector φ_i is gradient. This allows us to see that, in case of geodesic Ricci-symmetric spaces, determinants of matrices of metric tensors, which are in geodesic correspondence, are proportional.

3. GEODESIC MAPPINGS

Definition of geodesic Ricci-symmetric spaces implies that they permit non-trivial geodesic mappings. In other words, the following equation

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} \tag{3.1}$$

has a solution with respect to the tensor $a_{ij} = a_{ji} \neq c g_{ij}$ and the vector $\lambda_i = \lambda_{,i} = \partial_i \lambda \neq 0$, [20].

Integrability conditions for (3.1) can be written as follows:

$$a_{\alpha i} R^{\alpha}_{jkl} + a_{\alpha j} R^{\alpha}_{ikl} = \lambda_{li} g_{jk} + \lambda_{lj} g_{ik} - \lambda_{ki} g_{jl} - \lambda_{kj} g_{il}, \tag{3.2}$$

where $\lambda_{ij} = \lambda_{i,j}$. The latter implies

$$a_{\alpha j} R^{\alpha}_{k} - a_{\alpha k} R^{\alpha}_{j} = 0. \tag{3.3}$$

Differentiating (3.3) and taking into account (2.3), we get

$$\begin{aligned}\lambda_\alpha R^\alpha_k g_{ji} + R_{ik}\left(\lambda_j + \varphi_\alpha a^\alpha_j\right) - \lambda_\alpha R^\alpha_j g_{ki} - R_{ij}\left(\lambda_k + \varphi_\alpha a^\alpha_k\right) = \\ = \varphi_j a^\alpha_k R_{\alpha i} - \varphi_k a^\alpha_j R_{\alpha i},\end{aligned} \tag{3.4}$$

where $a^i_j = a_{\alpha j} g^{\alpha i}$. Differentiating (3.2) we obtain

$$\begin{aligned}\lambda_\alpha R^\alpha_{jkl} g_{im} + \lambda_i R_{mjkl} + a_{\alpha i} R^\alpha_{jkl,m} + \lambda_\alpha R^\alpha_{ikl} g_{jm} + \lambda_j R_{mikl} + \\ + a_{\alpha j} R^\alpha_{ikl,m} = \lambda_{li,m} g_{jk} + \lambda_{lj,m} g_{ik} - \lambda_{ki,m} g_{jl} - \lambda_{kj,m} g_{il}.\end{aligned}$$

Wrapping the latter identity by indices l and m we get

$$\begin{aligned}\lambda_\alpha R^\alpha_{jki} + \lambda_i R_{jk} + a^\alpha_i R^\beta_{kj\alpha,\beta} + \lambda_\alpha R^\alpha_{ikj} + \lambda_j R_{ik} + \\ + a^\alpha_j R^\beta_{ki\alpha,\beta} = \lambda_{\alpha i,\cdot}{}^{\alpha} g_{jk} + \lambda_{\alpha j,\cdot}{}^{\alpha} g_{ik} - \lambda_{ki,j} - \lambda_{kj,i},\end{aligned} \tag{3.5}$$

where $\lambda_{ij,}{}^{k} = \lambda_{ij,\alpha} g^{\alpha k}$.

Recall that the Riemann tensor satisfies the following identities:

$$R^h_{ijk,l} + R^h_{ikl,j} + R^h_{ilj,k} = 0. \tag{3.6}$$

Wrapping it by indices h and l:

$$R^\alpha_{ijk,\alpha} = R_{ij,k} - R_{ik,j}. \tag{3.7}$$

and substituting the latter into (3.5), we will arrive to

$$\begin{aligned}\lambda_\alpha R^\alpha_{jki} + \lambda_i R_{jk} + a^\alpha_i R_{kj,\alpha} - a^\alpha_i R_{k\alpha,j} + \lambda_\alpha R^\alpha_{ikj} + \lambda_j R_{ik} + \\ + a^\alpha_j R_{ki,\alpha} - a^\alpha_j R_{k\alpha,i} = \lambda_{\alpha i,\cdot}{}^{\alpha} g_{jk} + \lambda_{\alpha j,\cdot}{}^{\alpha} g_{ik} - \lambda_{ki,j} - \lambda_{kj,i}.\end{aligned}$$

Now, taking into account (2.3), we see that

$$\begin{aligned}\lambda_\alpha R^\alpha_{jki} + \lambda_\alpha R^\alpha_{ikj} + \lambda_i R_{jk} + \lambda_j R_{ik} + a^\alpha_i \varphi_\alpha R_{kj} + a^\alpha_j \varphi_\alpha R_{ki} - \\ - \varphi_j a^\alpha_i R_{\alpha k} - \varphi_i a^\alpha_j R_{\alpha k} = \lambda_{\alpha i,\cdot}{}^{\alpha} g_{jk} + \lambda_{\alpha j,\cdot}{}^{\alpha} g_{ik} - \lambda_{ki,j} - \lambda_{kj,i}.\end{aligned}$$

Alternating further by indices j and k, we also get

$$\begin{aligned}4\lambda_\alpha R^\alpha_{ikj} + R_{ik}\left(\lambda_j + \varphi_\alpha a^\alpha_j\right) - R_{ij}\left(\lambda_k + \varphi_\alpha a^\alpha_k\right) + \\ + \varphi_k a^\alpha_j R_{\alpha i} - \varphi_j a^\alpha_k R_{\alpha i} = \lambda_{\alpha j,\cdot}{}^{\alpha} g_{ik} - \lambda_{\alpha k,\cdot}{}^{\alpha} g_{ij}.\end{aligned}$$

Substituting (3.4) into the latter identity:

$$4\lambda_\alpha R^\alpha_{ikj} + \lambda_\alpha R^\alpha_j g_{ki} - \lambda_\alpha R^\alpha_k g_{ij} = \lambda_{\alpha j,\cdot}{}^{\alpha} g_{ik} - \lambda_{\alpha k,\cdot}{}^{\alpha} g_{ij}. \tag{3.8}$$

and then wrapping it by indices i and k, we can see that

$$\lambda_{\alpha j,\cdot}{}^{\alpha} = \frac{n+3}{n-1} \lambda_\alpha R^\alpha_j.$$

Now (3.8) implies that

$$\lambda_\alpha R^\alpha_{ikj} = \frac{1}{n-1}\left(\lambda_\alpha R^\alpha_j g_{ik} - \lambda_\alpha R^\alpha_k g_{ij}\right). \tag{3.9}$$

Multiplying (3.9) by λ^i and wrapping by index i we get

$$\frac{1}{n-1}\lambda_\alpha R^\alpha_j \lambda_k - \frac{1}{n-1}\lambda_\alpha R^\alpha_k \lambda_j = 0.$$

Since $\lambda_k \neq 0$, we can choose a vector ξ^k so that $\xi^k \lambda_k = 1$, which implies that

$$\frac{1}{n-1}\lambda_\alpha R^\alpha_j = B\lambda_j, \tag{3.10}$$

where $B = \frac{1}{n-1}\lambda_\alpha R^\alpha_\beta \xi^\beta$. Substitute (3.10) into (3.9):

$$\lambda_\alpha R^\alpha_{ijk} = B\left(\lambda_k g_{ij} - \lambda_j g_{ik}\right). \tag{3.11}$$

Multiplying (3.2) by vector λ^l and wrapping by index l we get from (3.11) that

$$\begin{aligned}\left(Ba^\alpha_i \lambda_\alpha - \lambda^\alpha \lambda_{\alpha i}\right) g_{jk} + \left(Ba^\alpha_j \lambda_\alpha - \lambda^\alpha \lambda_{\alpha j}\right) g_{ik} &= \\ = \lambda_j (Ba_{ki} - \lambda_{ki}) + \lambda_i (Ba_{kj} - \lambda_{kj}).&\end{aligned} \tag{3.12}$$

Wrapping (3.12) by indices j and k we obtain

$$\left(Ba^\alpha_i \lambda_\alpha - \lambda^\alpha \lambda_{\alpha i}\right) = -\mu \lambda_i, \tag{3.13}$$

where $-\mu = \frac{1}{n}(Ba_{\alpha\beta} - \lambda_{\alpha\beta})g^{\alpha\beta}$.

Substitute now (3.13) into (3.12):

$$\lambda_j\left(\mu g_{ik} + Ba_{ki} - \lambda_{ki}\right) + \lambda_i\left(\mu g_{jk} + Ba_{jk} - \lambda_{kj}\right) = 0. \tag{3.14}$$

Alternate further by indices j and k:

$$\lambda_j\left(\mu g_{ik} + Ba_{ki} - \lambda_{ki}\right) - \lambda_k\left(\mu g_{ij} + Ba_{ji} - \lambda_{ji}\right) = 0,$$

and also swap indices i and k:

$$\lambda_j\left(\mu g_{ki} + Ba_{ki} - \lambda_{ki}\right) - \lambda_i\left(\mu g_{kj} + Ba_{jk} - \lambda_{jk}\right) = 0.$$

Then adding the latter identity to (3.14) we obtain that

$$\lambda_j\left(\mu g_{ik} + Ba_{ki} - \lambda_{ki}\right) = 0,$$

which implies that

$$\lambda_{i,j} = \mu g_{ij} + Ba_{ij}. \tag{3.15}$$

Differentiating (3.11) and taking into account (3.15), we get that

$$\begin{aligned}\mu R_{hijk} + Ba_{\alpha h} R^h_{ijk} + \lambda_\alpha R^\alpha_{ijk,h} &= \\ = B_h(\lambda_k g_{ij} - \lambda_j g_{ik}) + B(\lambda_{k,h} g_{ij} - \lambda_{j,h} g_{ik}),&\end{aligned} \tag{3.16}$$

where $B_i = B_{,i} = \partial_i B$.

Cycling by indices h, j, k, we pass to the following expression

$$B_h(\lambda_k g_{ij} - \lambda_j g_{ik}) + B_j(\lambda_h g_{ik} - \lambda_k g_{ih}) + B_k(\lambda_j g_{ih} - \lambda_h g_{ij}) = 0.$$

Finally, wrapping it, we get the following identity:

$$(n-2)(B_h \lambda_k - B_k \lambda_h) = 0,$$

whence

$$B_h = \tau \lambda_h, \tag{3.17}$$

where $\tau = B_\alpha \xi^\alpha$.

Lemma 3.1. *For geodesic Ricci-symmetric pseudo-Riemannian spaces the system of equations* (3.1)*,* (3.15)*,* (3.17) *always has a solution.*

Multiplying (3.10) by vector φ^j, wrapping further by j, and taking into account (2.4), we will get the following identity:

$$B \lambda_\alpha \varphi^\alpha = 0.$$

Suppose $B \neq 0$. Then $\lambda_\alpha \varphi^\alpha = 0$. In other words, the vectors λ_i and φ_i are orthogonal. Differentiating (3.10) we obtain

$$\tfrac{1}{n-1}\left(\mu R_{ij} + B a_{\alpha i} R^\alpha_j\right) + 2B\varphi_i \lambda_j + B\varphi_j \lambda_i = \tau \lambda_i \lambda_j + B\mu g_{ij} + B^2 a_{ij}. \tag{3.18}$$

Alternating (3.18), and taking into account that $B \neq 0$, we arrive at

$$\varphi_i \lambda_j - \varphi_j \lambda_i = 0. \tag{3.19}$$

The equation (3.19) multiplied by vector ξ^j and wrapped by index j, can be re-written as follows:

$$\varphi_i = \overset{1}{\tau} \lambda_i, \tag{3.20}$$

where $\overset{1}{\tau} = \varphi_\alpha \xi^\alpha$.

Substituting (3.20) into (2.4) and comparing it with (3.10), we get a contradiction, whence $B = 0$. Therefore the equations (3.15) should be re-written in the following way

$$\lambda_{i,j} = \mu g_{ij}. \tag{3.21}$$

Differentiating (3.21) and alternating the latter while taking into account (3.11) and the fact that $B = 0$ we obtain the expression:

$$\mu_k g_{ij} - \mu_j g_{ik} = 0,$$

which is the same as

$$\mu_k = 0. \tag{3.22}$$

Lemma 3.2. *For geodesic Ricci-symmetric pseudo-Riemannian spaces contain the system of equations* (3.1)*,* (3.21)*,* (3.22) *always has a solution.*

Pseudo-Riemannian spaces V_n, admitting non-trivial geodesic mappings and satisfying (3.15), are called spaces $V_n(B)$ [19].

In the case when the system (3.1) has known solutions, the metric tensor $\bar{V}_n$ and vector φ_i, which defines non-trivial geodesic mappings on it, can be found from the following formulas

$$\bar{g}^{ij} = e^{-2\varphi} a^{ij}, \qquad \lambda^i = -a^{i\alpha}\varphi_\alpha,$$

where $a^{ij} = a_{\alpha\beta} g^{\alpha i} g^{\alpha j}$, $\lambda^i = \lambda_\alpha g^{\alpha i}$, and $\bar{g}^{ij}$ are elements of the matrix inverse to $\|\bar{g}_{ij}\|$.

Differentiating the latter, it is easy to see that equations (3.15) imply the following formula

$$\varphi_{ij} = \bar{B}\bar{g}_{ij} - Bg_{ij}, \tag{3.23}$$

where

$$\bar{B} = -e^{-2\varphi}\left(\lambda^\alpha\varphi_\alpha + \mu\right), \qquad \varphi_{ij} = \varphi_{i,j} - \varphi_i\varphi_j.$$

It is known that pseudo-Riemannian spaces $V_n(B)$ are closed in respect to non-trivial geodesic mappings. Namely, a pseudo-Riemannian space $\bar{V}_n$ which is in a geodesic correspondence with a space $V_n(B)$, is a space $\bar{V}_n(\bar{B})$ as well. Meanwhile, if $B = \mathsf{const}$, then $\bar{B}$ is also some constant [19]. Let us remind that is was proved above that $B = 0$, which implies $\bar{B} = \mathsf{const}$, [7, 12].

For geodesic Ricci-symmetric spaces equations (3.23) can be rewritten as follows:

$$\varphi_{ij} = \bar{B}\bar{g}_{ij}.$$

Existence of geodesic mappings implies the following relation between tensors of geodesic Ricci-symmetric spaces:

$$\bar{R}_{ij} = R_{ij} + (n-1)\varphi_{ij}.$$

If $B = 0$, then

$$\bar{R}_{ij} = R_{ij} + (n-1)\bar{B}\bar{g}_{ij}. \tag{3.24}$$

Differentiating (3.24) by the connection of $\bar{V}_n$ and taking into account (1.3), it is easy to see, that $\bar{V}_n$ is a Ricci-symmetric space. However, Ricci symmetric spaces, which are not Einstein spaces, do not admit non-trivial geodesic mappings. So the following theorem is true.

Theorem 3.3. *Every geodesic Ricci-symmetric space is an Einstein space.*

Let us turn our attention to pseudo-Riemannian spaces V_n, where geodesic symmetry is observed in relation to Riemann tensor, namely

$$\nabla_l R^h_{ijk} = 0. \tag{3.25}$$

We will call these spaces *geodesic symmetric.*

Wrapping (3.25) it can be seen that geodesic symmetric spaces are also geodesic Ricci-symmetric spaces. It implies that geodesic symmetric spaces exist only when they are Einstein spaces, namely

$$R_{ij} = \frac{R}{n} g_{ij}, \qquad\qquad R = \mathsf{const}.$$

Geodesic mappings of Einstein spaces and their properties were studied in [1, 3, 6, 17]. Einstein spaces admitting non-trivial geodesic mappings are spaces $V_n(B)$ with $B = \frac{R}{n(n-1)}$. The class of Einstein spaces can be extended via the notion of quasi-Einstein spaces. Their geodesic mappings were described in [5, 11, 13, 14].

Taking (1.1) into account, one can rewrite (3.25) as follows:

$$R^h_{ijk,m} + \delta^h_m \varphi_\alpha R^\alpha_{ijk} = 2\varphi_m R^h_{ijk} + \varphi_i R^h_{mjk} + \varphi_j R^h_{imk} + \varphi_k R^h_{ijm}.$$

Wrapping by indices h and m, we obtain the following expression for Einstein spaces

$$\varphi_\alpha R^\alpha_{ijk} = 0$$

implying

$$R^h_{ijk,m} = 2\varphi_m R^h_{ijk} + \varphi_i R^h_{mjk} + \varphi_j R^h_{imk} + \varphi_k R^h_{ijm}. \tag{3.26}$$

Let us put down index h in (3.26) index and symmetrize it by indices h and i. Then,

$$\varphi_i R_{hmjk} + \varphi_h R_{imjk} = 0.$$

It then follows from (3.26) that

$$R_{hijk,m} = -\varphi_m R_{hijk}. \tag{3.27}$$

Spaces in which (3.27) holds true are called *recurrent*. These spaces admit non-trivial geodesic mappings only when they are spaces of a constant curvature.

Theorem 3.4. *Every geodesic symmetric pseudo-Riemannian spaces is a space of constant curvature.*

Therefore, geodesic Ricci symmetric and geodesic symmetric spaces exist only when they are Einstein spaces and spaces of constant curvature respectively.

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