

Homeotopy groups of leaf spaces of one-dimensional foliations on non-compact surfaces with non-compact leaves

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Abstract. Let Z be a non-compact two-dimensional manifold obtained from a family of open strips $\mathbb{R} \times (0, 1)$ with boundary intervals by gluing those strips along some pairs of their boundary intervals. Every such strip has a natural foliation into parallel lines $\mathbb{R} \times t$, $t \in (0, 1)$, and boundary intervals which gives a foliation Δ on all of Z . Denote by $\mathcal{H}(Z, \Delta)$ the group of all homeomorphisms of Z that maps leaves of Δ onto leaves and by $\mathcal{H}(Z/\Delta)$ the group of homeomorphisms of the space of leaves endowed with the corresponding compact open topologies. Recently, the authors identified the *homeotopy* group $\pi_0\mathcal{H}(Z, \Delta)$ with a group of automorphisms of a certain graph G with additional structure which encodes the combinatorics of gluing Z from strips. That graph is in a certain sense dual to the space of leaves Z/Δ .

On the other hand, for every $h \in \mathcal{H}(Z, \Delta)$ the induced permutation k of leaves of Δ is a homeomorphism of Z/Δ and the correspondence $h \mapsto k$ is a homomorphism $\psi : \mathcal{H}(Z, \Delta) \rightarrow \mathcal{H}(Z/\Delta)$. The aim of the present paper is to show that ψ induces a homomorphism of the corresponding *homeotopy* groups $\psi_0 : \pi_0\mathcal{H}(Z, \Delta) \rightarrow \pi_0\mathcal{H}(Z/\Delta)$ which turns out to be either injective or having a kernel \mathbb{Z}_2 . This gives a dual description of $\pi_0\mathcal{H}(Z, \Delta)$ in terms of the space of leaves.

Анотація. Нехай некомпактний двовимірний многовид Z отримано з сім'ї відкритих смуг $\mathbb{R} \times (0, 1)$, що мають на межі відкриті інтервали, за допомогою склеювання цих смуг вздовж деяких пар їх межових інтервалів. Кожна така смуга має природне шарування, шарами якого є паралельні прями $\mathbb{R} \times t$, $t \in (0, 1)$, а також межові інтервали. Ці шарування породжують однозначно визначене шарування Δ на Z . Позначимо

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Ключові слова: шарування, простір шарів, смугаста поверхня

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через $\mathcal{H}(Z, \Delta)$ групу гомеоморфізмів Z , які відображають шари шарування Δ на шари, а через $\mathcal{H}(Z/\Delta)$ – групу гомеоморфізмів простору шарів. Наділимо ці групи компактно-відкритими топологіями. Нещодавно автори ідентифікували групу гомеотопій $\pi_0\mathcal{H}(Z, \Delta)$ з групою автоморфізмів певного графу G з додатковою структурою, яка кодує комбінаторику склеювання Z зі смуг. Вказаний граф є в деякому сенсі дуальним до простору шарів Z/Δ .

З іншого боку, для кожного $h \in \mathcal{H}(Z, \Delta)$ індукована перестановка k шарів Δ є гомеоморфізмом Z/Δ , а відповідність $h \mapsto k$ є гомоморфізмом $\psi : \mathcal{H}(\Delta) \rightarrow \mathcal{H}(Z/\Delta)$. У даній роботі показано, що ψ індукує гомоморфізм відповідних груп гомеотопій $\psi_0 : \pi_0\mathcal{H}(Z, \Delta) \rightarrow \pi_0\mathcal{H}(Z/\Delta)$, причому він або ін'єктивний, або його ядро ізоморфне \mathbb{Z}_2 . Це дає дуальний опис $\pi_0\mathcal{H}(Z, \Delta)$ у термінах простору шарів.

1. INTRODUCTION

In the present paper we study foliations on non-compact surfaces similar to foliations on the plane (considered by W. Kaplan [12, 13], see also [6, 7]) with “sufficiently regular behavior”. Those foliations are studied in a series of papers by S. Maksymenko, Ye. Polulyakh, and Yu. Soroka, see [16–19, 21, 23, 24]. Our main result (Theorem 1.4) relates the groups of isotopy classes of foliated (*i.e.* sending leaves to leaves) self-homeomorphisms of such surfaces with the groups of isotopy classes of homeomorphisms of the corresponding space of leaves (being in those cases non-Hausdorff one-dimensional manifolds). This extends results by Yu. Soroka [24].

Let Z be a surface endowed with a one-dimensional foliation Δ . It will be convenient to say that the pair (Z, Δ) is a *foliated* surface. For an open subset $U \subset Z$ denote by $\Delta|_U$ the induced foliation on U consisting of connected components of non-empty intersections $\omega \cap U$ for all $\omega \in \Delta$.

Let $Y = Z/\Delta$ be the set of all leaves, and $p : Z \rightarrow Y$ be the natural projection associating to each $x \in Z$ the leaf of Δ containing x . Endow Y with a quotient topology, so a subset $A \subset Y$ is open if and only if $p^{-1}(A)$ is open in Z . Notice that usually spaces of leaves of foliations are non-Hausdorff.

By the *saturation* $\text{Sat}(U)$ of a subset $U \subset Z$ we mean the union of all leaves of Δ intersecting U . Evidently $\text{Sat}(U) = p^{-1}(p(U))$. A subset $U \subset Z$ is called *saturated* (with respect to a foliation Δ) whenever $U = \text{Sat}(U)$.

Let K be a one-dimensional manifold. Then by a *trivial* foliation on $\mathbb{R} \times K$ we will mean a foliation by lines $\{\mathbb{R} \times y\}_{y \in K}$.

A homeomorphism $h : Z \rightarrow Z$ is said to be *leaf-preserving* if $h(\omega) = \omega$ for each leaf $\omega \in \Delta$. Also, a homeomorphism $h : Z_1 \rightarrow Z_2$ between foliated surfaces (Z_1, Δ_1) and (Z_2, Δ_2) is *foliated* if $h(\omega) \in \Delta_2$ for each $\omega \in \Delta_1$. In particular, every leaf-preserving homeomorphism is foliated.

Let $\mathcal{H}(Z, \Delta)$ be the group of all foliated self-homeomorphisms of Z and $\mathcal{H}(Y)$ be the group of homeomorphisms of Y endowed with the corresponding compact open topologies. Let also $\mathcal{H}_{\text{id}}(Z, \Delta)$ be the identity path component of $\mathcal{H}(Z, \Delta)$ consisting of all homeomorphisms of Z isotopic to id_Z in $\mathcal{H}(Z, \Delta)$, and $\mathcal{H}_{\text{id}}(Y)$ be the identity path component of $\mathcal{H}(Y)$. Evidently, $\mathcal{H}_{\text{id}}(Z, \Delta)$ and $\mathcal{H}_{\text{id}}(Y)$ are normal subgroups of the corresponding groups $\mathcal{H}(Z, \Delta)$ and $\mathcal{H}(Y)$ with respect to composition operation. Then the quotients

$$\pi_0\mathcal{H}(Z, \Delta) \cong \mathcal{H}(Z, \Delta)/\mathcal{H}_{\text{id}}(Z, \Delta), \quad \pi_0\mathcal{H}(Y) \cong \mathcal{H}(Y)/\mathcal{H}_{\text{id}}(Y),$$

i.e. the groups of path components of $\mathcal{H}(Z, \Delta)$ and $\mathcal{H}(Y)$, are called *homeotopy groups* of the foliation Δ and Y respectively.

Notice that by definition each $h \in \mathcal{H}(Z, \Delta)$ induces a permutation of leaves of Δ , and therefore it induces a bijection $k : Y \rightarrow Y$ making commutative the following diagram:

$$\begin{array}{ccc} Z & \xrightarrow{h} & Z \\ p \downarrow & & \downarrow p \\ Y & \xrightarrow{k} & Y \end{array} \tag{1.1}$$

One easily checks that k is a homeomorphism of Y and that the correspondence $h \mapsto k$ is a homomorphism of groups

$$\psi : \mathcal{H}(Z, \Delta) \rightarrow \mathcal{H}(Y), \quad \psi(h) = k. \tag{1.2}$$

Definition 1.1. *We will say that a foliated surface (Z, Δ) belongs to the class \mathcal{F} , whenever it satisfies the following conditions:*

- (A1) *Z has a countable base;*
- (A2) *every leaf ω of Δ is a non-compact closed subset of Z , so it is an image of \mathbb{R} under some topological embedding $\mathbb{R} \subset Z$;*
- (A3) *each boundary component of Z is a leaf of Δ .*

Our aim is to show (Theorem 1.4) that for foliations from class \mathcal{F} satisfying certain additional “local finiteness” condition we have that

$$\psi(\mathcal{H}_{\text{id}}(Z, \Delta)) = \mathcal{H}_{\text{id}}(Y).$$

This will imply that the image of ψ is a union of path components of $\mathcal{H}(Y)$ and that ψ induces a homomorphism of the corresponding homeotopy groups:

$$\psi_0 : \pi_0\mathcal{H}(Z, \Delta) = \frac{\mathcal{H}(Z, \Delta)}{\mathcal{H}_{\text{id}}(Z, \Delta)} \longrightarrow \frac{\mathcal{H}(Y)}{\mathcal{H}_{\text{id}}(Y)} = \pi_0\mathcal{H}(Y). \tag{1.3}$$

Moreover, we also show that in this case for connected Z the kernel of ψ_0 is either trivial or is isomorphic with \mathbb{Z}_2 .

Definition 1.2. Let (Z, Δ) be a foliated surface from class \mathcal{F} .

1) Say that a leaf $\omega \in \Delta$ is **regular** if there exists an open saturated neighborhood U of ω such that the pair (\overline{U}, U) is “trivially foliated” in the sense that (\overline{U}, U) is foliated homeomorphic

- either to the pair $(\mathbb{R} \times [-1, 1], \mathbb{R} \times (-1, 1))$ if $\omega \subset \text{Int}(Z)$,
- or to the pair $(\mathbb{R} \times [0, 1], \mathbb{R} \times [0, 1))$ if $\omega \subset \partial Z$,

with trivial foliation into lines $\mathbb{R} \times t$, via a homeomorphism sending ω to $\mathbb{R} \times 0$.

2) A leaf which is not regular will be called **singular**¹.

3) For a leaf ω define its **Hausdorff closure** by

$$\text{hcl}(\omega) := \bigcap_{U \text{ is a neighborhood of } \omega} \overline{\text{Sat}(U)},$$

so it is the intersection of closures of all saturated neighborhoods of ω . Evidently, $\text{hcl}(\omega)$ is saturated and $\omega \subset \text{hcl}(\omega)$. We will say that ω is **special** whenever $\text{hcl}(\omega) \setminus \omega \neq \emptyset$.

Denote by Δ_{reg} , Δ_{spec} , and Δ_{sing} respectively the families of all regular, special and singular leaves of Δ . Then we have the following relations:

$$\Delta_{\text{spec}} \subset \Delta_{\text{sing}}, \quad \Delta = \Delta_{\text{reg}} \sqcup \Delta_{\text{sing}}.$$

Remark 1.3. Notice that by definition each one-dimensional foliation Δ on a surface Z is “locally regular”, i.e. every point $x \in Z$ has a neighborhood U such that the leaf of $\Delta|_U$ containing x is regular in the sense of Definition 1.2. Thus Definition 1.2 puts restriction to a global structure of the foliation near a leaf.

Let us also mention that it is possible that a leaf might have a trivially foliated saturated neighborhood U , however for any such neighborhood the pair (\overline{U}, U) will never be trivially foliated. For example, let $\Delta = \{\mathbb{R} \times y\}_{y \in \mathbb{R}}$ be a trivial foliation on \mathbb{R}^2 into horizontal lines and $Z = \mathbb{R}^2 \setminus 0$. Then it is easy to see that $\Delta|_Z$ has two singular leaves $\omega_1 = (-\infty; 0) \times 0$ and $\omega_2 = (0; +\infty) \times 0$. Indeed, any open saturated neighborhood U of either of ω_1 or ω_2 contains $\mathbb{R} \times (-\varepsilon, \varepsilon) \setminus (0, 0)$ for some small $\varepsilon > 0$, and therefore (\overline{U}, U) can not be foliated homeomorphic with $(\mathbb{R} \times [-1, 1], \mathbb{R} \times (-1, 1))$. This implies $\text{hcl}(\omega_1) = \text{hcl}(\omega_2) = \omega_1 \cup \omega_2$, whence both ω_1 and ω_2 are special.

¹ There is some abuse of terminology used in [16, Property (c3) before Lemma 3.2]. Such leaves for striped surfaces were called *special*.

The main result of the present paper is the following theorem relating the groups $\pi_0\mathcal{H}(Z, \Delta)$ and $\pi_0\mathcal{H}(Y)$.

Theorem 1.4. *Let (Z, Δ) be a foliated surface from class \mathcal{F} satisfying the following additional condition:*

(A4) *the family Δ_{sing} of all singular leaves of Δ is locally finite.*

Then $\psi(\mathcal{H}_{id}(Z, \Delta)) = \mathcal{H}_{id}(Y)$, whence

- (1) *the image $\psi(\mathcal{H}_{id}(Z, \Delta))$ of the homomorphism $\psi : \mathcal{H}(Z, \Delta) \rightarrow \mathcal{H}(Y)$, see (1.2), is a union of path components of $\mathcal{H}(Y)$;*
- (2) *ψ induces a homomorphism $\psi_0 : \pi_0\mathcal{H}(Z, \Delta) \rightarrow \pi_0\mathcal{H}(Y)$.*

Moreover, if Z is **connected**, then the kernel $\ker \psi_0$ is either

- (a) *trivial, i.e. ψ_0 is injective, or*
- (b) *$\ker \psi_0 \cong \mathbb{Z}_2$, which happens if and only if $\mathcal{H}(Z, \Delta)$ contains a leaf-preserving homeomorphism which reverses orientation of all leaves.*

In fact in a series of papers by the authors it was obtained a characterization of foliated surfaces (Z, Δ) from class \mathcal{F} satisfying also the assumption (A4) of Theorem 1.4. It is shown in [20, Theorem 4.4] that every such surface is glued from certain “model strips” foliated by parallel lines. Such surfaces were called *striped*, and the proof of Theorem 1.4 essentially exploits an existence of that gluing. This description is motivated by results of W. Kaplan, see Theorem 3.1.5 below, describing the structure of foliations on \mathbb{R}^2 .

Furthermore, in the joint paper with Yu. Soroka [21] the homeotopy group $\pi_0\mathcal{H}(Z, \Delta)$ of canonical foliation Δ of a striped surface Z is identified with the group of automorphisms of a certain graph G with additional structure encoding the gluing of Z from model strips. In fact, the vertices of G are strips and the edges are “seams” (*i.e.* leaves along which we glue the strips), so this graph is *in a certain sense dual* to the space of leaves Y . It is also proved in S. Maksymenko and O. Nikitchenko [15] that G (as a one-dimensional CW-complex) is homotopy equivalent to the surface Z .

Thus Theorem 1.4 also gives a dual description of the homeotopy group $\pi_0\mathcal{H}(Z, \Delta)$ in terms of the space of leaves.

Structure of the paper. In Section 2 we consider several properties of *non-Hausdorff* T_1 -spaces. In particular, we recall the exponential law for them as well as describe the relations between isotopies and paths in the groups of homeomorphisms for such spaces, see Corollary 2.1.2. We also present a characterization of the identity path component of the group of homeomorphisms of second countable T_1 -manifolds that are not necessarily Hausdorff, see Lemma 2.4.1. In Section 3 we recall the results about striped surfaces, and prove Theorem 1.4 in Section 4.

2. T_1 -SPACES

We will recall here several elementary properties of T_1 -spaces which are not necessarily Hausdorff. They are well-known and rather trivial for T_2 -spaces, while for T_1 -ones they should also be known for specialists, though we did not find their explicit exposition in the literature. Therefore to make the paper self-contained and for future references we will collect them together and present their short proofs. These results will be applied to the spaces of leaves of foliations.

Recall that a topological space Y is *compact* if every open cover of Y contains a finite subcover. In this paper a (not necessarily Hausdorff) topological space Y will be called *locally compact*² whenever for each $y \in Y$ and an open neighborhood U of y there exists a compact subset $B \subset Y$ such that $y \in \text{Int}(B) \subset \overline{B} \subset U$.

2.1. Exponential law. For two sets A and B denote by $\text{Map}(A, B)$ the set of all maps $f : A \rightarrow B$. Then for any other set C we have a natural bijection

$$E = E_{A,B,C} : \text{Map}(A \times B, C) \rightarrow \text{Map}(A, \text{Map}(B, C))$$

called an *exponential map* and defined as follows: if $F : A \times B \rightarrow C$ is a map, then $E(F) : A \rightarrow \text{Map}(B, C)$ is given by $E(F)(a)(b) = F(a, b)$ for all $(a, b) \in A \times B$.

If A, B are topological spaces, it is natural to consider the space $\mathcal{C}^0(A, B)$ of continuous maps $A \rightarrow B$ instead of $\text{Map}(A, B)$. We will always endow $\mathcal{C}^0(A, B)$ with the compact open topology. Then the variant of exponential law for topological spaces holds under additional assumptions of A and B :

Theorem 2.1.1. *Let A, B, C be any topological spaces. Then E induces a continuous injective map*

$$E = E_{A,B,C} : \mathcal{C}^0(A \times B, C) \rightarrow \mathcal{C}^0(A, \mathcal{C}^0(B, C)). \quad (2.1)$$

If B is locally compact (not necessarily Hausdorff), then (2.1) is a continuous bijection. Moreover, if, in addition, A is a T_3 -space, then (2.1) is a homeomorphism.

This theorem is well known and usually formulated for Hausdorff spaces [2]. For non-Hausdorff spaces its proof is presented in [14, Lemma 3.3].

² Notice that there are several variations of this notion for non-Hausdorff spaces which coincide for Hausdorff ones.

Corollary 2.1.2. *If B is locally compact, and C is an arbitrary topological space, then the exponential map*

$$E = E_{[0,1],B,C} : \mathcal{C}^0([0, 1] \times B, C) \rightarrow \mathcal{C}^0([0, 1], \mathcal{C}^0(B, C)) \tag{2.2}$$

is a homeomorphism. In particular, it induces a bijection

$$E_0 : \pi_0 \mathcal{C}^0([0, 1] \times B, C) \rightarrow \pi_0 \mathcal{C}^0([0, 1], \mathcal{C}^0(B, C)) .$$

between the corresponding sets of path components.

Moreover, for groups of homeomorphisms one can say more. Let

- $\mathcal{H}(B)$ be the group of homeomorphisms of a topological space B ;
- $\mathcal{H}_{\text{id}}(B)$ be the path component of id_B in $\mathcal{H}(B)$, so it consists of $h \in \mathcal{H}(B)$ such that there exists a continuous path $\gamma : [0, 1] \rightarrow \mathcal{H}(B)$ such that $\gamma(0) = \text{id}_B$ and $\gamma(1) = h$;
- $\tilde{\mathcal{H}}_{\text{id}}(B)$ be the subgroup of $\mathcal{H}(B)$ consisting of homeomorphisms isotopic to id_B .

Corollary 2.1.3. *For any topological space B we have that*

- (1) $\tilde{\mathcal{H}}_{\text{id}}(B) \subset \mathcal{H}_{\text{id}}(B)$ and these subgroups are normal in $\mathcal{H}(B)$;
- (2) if B is locally compact, $\tilde{\mathcal{H}}_{\text{id}}(B) = \mathcal{H}_{\text{id}}(B)$;
- (3) the set $\pi_0 \mathcal{H}(B)$ of path components of $\mathcal{H}(B)$ has a group structure under which the natural map $p : \mathcal{H}(B) \rightarrow \pi_0 \mathcal{H}(B)$ is an epimorphism with kernel $\mathcal{H}_{\text{id}}(B)$, whence p induces an isomorphism of groups $\mathcal{H}(B)/\mathcal{H}_{\text{id}}(B) \cong \pi_0 \mathcal{H}(B)$;

Proof. (1) By Theorem 2.1.1 every isotopy $F : [0, 1] \times B \rightarrow B$ induces a continuous path $E(F) : [0, 1] \rightarrow \mathcal{H}(B)$ between F_0 and F_1 . This implies that $\tilde{\mathcal{H}}_{\text{id}}(B) \subset \mathcal{H}_{\text{id}}(B)$. Normality of these groups is easy.

(2) If B is locally compact, then by Corollary 2.1.2 the exponential map $E_{[0,1],B,B}$ is a homeomorphism, so every continuous path $\gamma : [0, 1] \rightarrow \mathcal{H}(B)$ is induced by some isotopy $E^{-1}(\gamma) : [0, 1] \times B \rightarrow B$. This gives the inverse inclusion $\tilde{\mathcal{H}}_{\text{id}}(B) \supset \mathcal{H}_{\text{id}}(B)$.

Statement (3) is straightforward. □

Thus for locally compact spaces isotopies can equally be thought of as paths in the group of homeomorphisms endowed with compact open topologies.

2.2. Special points of T_1 spaces. Let Y be a topological space.

Definition 2.2.1. For a point $z \in Y$ define its *Hausdorff closure*, $\text{hcl}(z)$, to be the intersection of closures of all neighborhoods of z , that is

$$\text{hcl}(y) = \bigcap_{V \text{ is a neighborhood of } y} \bar{V} . \tag{2.3}$$

Evidently, $z \in \text{hcl}(z)$. We say that z is *special*³ whenever $\text{hcl}(z) \setminus z \neq \emptyset$. Denote by \mathcal{V} the set of all special points of Y .

One easily check that Y is Hausdorff iff $\text{hcl}(z) = \{z\}$ for all $z \in Y$, that is $\mathcal{V} = \emptyset$.

Further, let $L \subset Y$ be a subset, $y \in L$, and $\text{hcl}_L(y)$ be Hausdorff closure of z in L with respect to the induced topology from Y . It is straightforward that

$$\text{hcl}_L(y) \subset \text{hcl}(y) \cap L, \tag{2.4}$$

however the opposite inclusion can fail.

Lemma 2.2.2. *For a subset $A \subset Y$ the following statements hold true.*

- (1) *If $A \cap \text{hcl}(y) = \{y\}$ for all $y \in A$, then A is Hausdorff.*
- (2) *In particular, for every $z \in \mathcal{V}$ the following subspace*

$$A_z := (A \setminus \mathcal{V}) \cup \{z\} = A \setminus (\mathcal{V} \setminus \{z\})$$

of Y is Hausdorff.

- (3) *If A is compact, then $\overline{A} \subset A \cup \mathcal{V}$.*

Proof. Statement (1) is an immediate consequence of (2.4).

(2) To prove that $B := (L \setminus \mathcal{V}) \cup \{z\}$ is Hausdorff it suffices to verify that $\text{hcl}_B(y) = \{y\}$ for all $y \in B$.

If $y \in L \setminus \mathcal{V}$, then $\text{hcl}(y) = \{y\}$, whence $B \cap \text{hcl}(y) = \{y\}$. Since $z \in \text{hcl}(w)$ if and only if $w \in \text{hcl}(z)$, it follows that $\text{hcl}(z) \subset \mathcal{V}$. Therefore

$$\begin{aligned} B \cap \text{hcl}(z) &= ((L \setminus \mathcal{V}) \cup \{z\}) \cap \text{hcl}(z) = \\ &= ((L \setminus \mathcal{V}) \cap \text{hcl}(z)) \cup (\{z\} \cap \text{hcl}(z)) = \emptyset \cup \{z\} = \{z\}. \end{aligned}$$

(3) Let $y \notin L \cup \mathcal{V}$. We will show that then there exists an open neighborhood W of y such that $L \cap W = \emptyset$. This will imply that $y \notin \text{Cl}(L)$, whence $\text{Cl}(L) \subset L \cup \mathcal{V}$.

Since y is not a special point, *i.e.* $\text{hcl}(y) = \{y\}$, we get from (2.3) that

$$L \subset L \cup \mathcal{V} \subset Y \setminus \{y\} = \bigcup_{V \ni y} (Y \setminus \text{Cl}(V)),$$

where V runs over all open neighborhoods of y . Thus

$$\{Y \setminus \overline{V} \mid V \text{ is a neighborhood of } y\}$$

is an open cover of a compact set L , and so it contains a finite subcover, *i.e.* one can find open neighborhoods V_1, \dots, V_m of y such that $L \subset \bigcup_{i=1}^m (Y \setminus \text{Cl}(V_i))$. Hence $W = V_1 \cap \dots \cap V_m$ is an open neighborhood of y with $L \cap W = \emptyset$. □

³In [6, 8] such points were called *branch*.

Evidently, statement (3) of Lemma 2.2.2 extends the well known fact that a compact subset A of a Hausdorff space Y is closed.

Corollary 2.2.3. *Let A be a compact subset of the subspace $Y \setminus \mathcal{V}$. Then it is closed in Y .*

Proof. Let $x \in \overline{A} \setminus A \subset \mathcal{V}$. Consider the Hausdorff subspace A_x of Y . Then A is closed in A_x as a compact subset of the Hausdorff space A_x , whence $A = \overline{A} \cap A_x$. On the other hand $x \in \overline{A} \cap A_x$ by definition which is impossible. \square

2.3. Locally finite subsets. Say that a subset $A \subset Y$ is *locally finite* if for every point $z \in Y$ there exists an open neighborhood U such that the intersection $U \cap A$ is finite. In other words, the family $\{\{a\} \mid a \in A\}$ of one-point subsets of A is locally finite in Y .

Lemma 2.3.1. *Consider the following conditions on a subset A of a topological space Y :*

- (1) A is closed and discrete;
- (2) for each $y \in Y$ there exists a neighborhood V intersecting A in at most one point;
- (3) A is locally finite;

Then we have the following implications: (1) \Rightarrow (2) \Rightarrow (3).

If Y is T_1 then we also have that (3) \Rightarrow (1), i.e. all the above conditions are equivalent.

Proof. (1) \Rightarrow (2). Suppose A is closed and discrete and let $y \in Y$. If $y \notin A$, then $V = Y \setminus A$ is an open neighborhood of y that does not intersect A . If $y \in A$, then discreteness of A implies that there exists an open neighborhood V of y such that $V \cap A = \{y\}$.

The implication (2) \Rightarrow (3) is evident.

It remains to prove the implication (3) \Rightarrow (1) under the assumption that Y is T_1 . Suppose A is locally finite. Then each subset $B \subset A$ is locally finite as well. Moreover, as every point $y \in Y$ is a closed subset, it follows that B is closed as a union of a locally finite family of its closed one-point subsets. In other words every subset of A is closed in Y . Hence A is closed and discrete. \square

2.4. One-dimensional T_1 manifolds. Let Y be a T_1 topological space locally homeomorphic with $[0, 1)$. In other words, Y is a one-dimensional manifold which is T_1 but not necessarily Hausdorff. Then Y is also locally compact, and by Corollary 2.1.3 one can equally regard $\pi_0 \mathcal{H}(Y)$ as the

group of path components of $\mathcal{H}(Y)$ and also as the group of isotopy classes of homeomorphisms of Y .

As usual, a point $y \in Y$ is called *internal*, if it has an open neighborhood homeomorphic with $(0, 1)$. Otherwise, y has an open neighborhood homeomorphic with $[0, 1)$ and is called a *boundary point*. As usual, the sets of all internal and boundary points will be denoted by $\text{Int}(Y)$ and ∂Y respectively.

The following Lemma 2.4.1 characterizes the identity path component of Y under assumption that the set of its special points is locally finite.

Lemma 2.4.1. *Let Y be a T_1 space locally homeomorphic with $[0, 1)$ such that the set \mathcal{V} of its special points is locally finite and every connected component of the complement $A := Y \setminus \mathcal{V}$ is second-countable. Let also $z \in \mathcal{V}$ be any special point and $A_z := A \cup \{z\}$. Then the following statements hold.*

- (1) A and A_z are open and Hausdorff.
- (2) Every connected component C of A or of A_z is open in Y and is homeomorphic to one of the following spaces: the circle S^1 , $[0, 1]$, $[0, 1)$, $(0, 1)$. In particular, C is orientable.
- (3) If a connected component C of A is compact then it is a connected component of Y .
- (4) Let $k \in \mathcal{H}(Y)$. Then $k \in \mathcal{H}_{\text{id}}(Y)$ if and only if
 - (YA) k is fixed on $T := \partial Y \cup \mathcal{V}$;
 - (YB) $k(C) = C$ for every connected component C of $Y \setminus T$, and the restriction map $k|_C : C \rightarrow C$ is isotopic to id_C (which in the case of 1-dimensional manifolds is equivalent to the assumption that $k|_C$ preserves orientation of C).

If these conditions hold, then k is also fixed on ∂Y .

Proof. (1) By Lemma 2.2.2, A and A_z are Hausdorff. Moreover, as Y is T_1 and \mathcal{V} is locally finite, it follows that \mathcal{V} and $\mathcal{V} \setminus \{z\}$ are closed in Y , whence their complements A and A_z are open.

(2) This statement is proved in [17, Lemma 2.3]. Let us recall the arguments. Since A_z is open in Y and Y is locally path connected, it follows that every connected component C of A and A_z is open in Y as well. Moreover, by assumption every connected component of A is second countable. As z has a countable base of open neighborhoods each being a union of z and at most two open subsets of A , it follows that the connected components of A_z are also *second countable*. Hence C is a connected Hausdorff 1-manifold, and the results follows from the classification of such manifolds, e.g. [10, Exercises 1.2.6 and 1.4.9].

(3) The connected set C is both open, see (1) and closed, see Corollary 2.2.3.

(4) *Necessity.* Let $k \in \mathcal{H}_{\text{id}}(Y)$, so there is an isotopy $F : [0, 1] \times Y \rightarrow Y$ such that $F_0 = \text{id}_Y$ and $F_1 = k$. We need to check conditions (YA) and (YB) for k .

First notice that \mathcal{V} and ∂Y are defined in topological terms, *i.e.* they are invariant under self-homeomorphisms of Y . Moreover, each $z \in \partial Y \setminus \mathcal{V}$ has a neighborhood U_z such that $U_z \cap T = \{z\}$. Since \mathcal{V} is locally finite, and therefore discrete due to Lemma 2.3.1, we obtain that T is discrete as well. This implies that $F([0, 1] \times z) \subset T$ for any $z \in T$, and therefore $F([0, 1] \times z)$ is contained in some connected component of T , being a one-point set since T is discrete. Hence $F_t(z) = F_0(z) = z$ for all $t \in [0, 1]$. In particular, k is fixed on T , which proves (YA).

As $Y \setminus T$ is also invariant under self-homeomorphisms of Y , we obtain by similar arguments that $F([0, 1] \times C) \subset C$ for every connected component C of $Y \setminus T$. Thus the restriction

$$F|_{[0,1] \times C} : [0, 1] \times C \rightarrow C$$

is an isotopy between id_C and $k|_C$. Therefore, $k|_C : C \rightarrow C$ preserves orientation of C which proves (YB).

Sufficiency. Let $k \in \mathcal{H}(Y)$ be a homeomorphism satisfying (YA) and (YB). For every connected component C of $A = Y \setminus T$ fix an arbitrary isotopy $F^C : [0, 1] \times C \rightarrow C$ such that $F_0^C = \text{id}_C$ and $F_1^C = k|_C$ and define the map $G : [0, 1] \times Y \rightarrow Y$ by

$$G(t, x) = \begin{cases} x, & x \in T, \\ F^C(t, x), & x \in C, \text{ where } C \text{ is a connected component of } Y \setminus T. \end{cases}$$

Then $G_0 = \text{id}_Y$, $G_1 = k$, and each G_t is a bijection of Y . Notice that $G|_{A \times [0,1]}$ is an isotopy of A , and G is its extension to $Y \times [0, 1]$ fixed at special points. We need to show that G is an isotopy of Y .

For each $z \in T$ let C_z be a connected component of $A_z := A \cup \{z\}$ containing z . Then $A \cup \{C_z\}_{z \in T}$ is an open cover of Y . Since the restriction $G|_{A \times [0,1]} : A \times [0, 1] \rightarrow A$ is an isotopy, it suffices to show that for each $z \in T$, the restriction $G : C_z \times [0, 1] \rightarrow C_z$ is an isotopy as well.

If C_z is compact, *i.e.* it is homeomorphic either to S^1 or to $[0, 1]$, then C_z is a one-point compactification of C , and continuity of $G|_{C_z \times [0,1]}$ follows from [1, Theorem 2.1].

Otherwise, (C_z, z) is homeomorphic to one of the following pointed spaces:

$$((-1, 1), 0), \quad ([-1, 1), 0), \quad ([0, 1), 0).$$

In this case the proof can proceed similarly to the arguments of [1, Theorem 3.2] extending the previous case one-point compactification of C to the

case of the so-called ideal Freudental compactification. In fact, the proof of [1, Theorem 3.2] consists of proving continuity of G for each added point, which in our situation is z only. \square

Remark 2.4.2. The problem of one-point extensions of isotopies (as in the proof of continuity of G) were studied by R. H. Crowell [1, Theorem 2.1], H. Gluck [5] and many others. Also, certain variants of such results for concordancies of homeomorphisms considered in L. S. Husch and T. B. Rushing [11], applications to ambient isotopies of arcs in manifolds [4, 22], and applications to Nielsen numbers are given in P. Heath and X. Zhao [9, §4].

3. STRIPED SURFACES

In this section we will briefly recall the results by W. Kaplan, S. Maksymenko, E. Polulyakh, and Yu. Soroka.

3.1. Quasi-striped surfaces. The following notion of a quasi-strip is a “basic block” generating foliations on the plane.

Definition 3.1.1. A foliated surface (S, Δ_S) will be called a *quasi-strip* whenever

- a) every boundary component of S is an open arc being also a leaf of Δ_S ;
- b) the restriction of Δ_S to the interior $\text{Int}(S)$ is foliated homeomorphic with \mathbb{R}^2 endowed by trivial foliation into horizontal lines $\{\mathbb{R} \times y\}_{y \in \mathbb{R}}$.

Definition 3.1.2. Let $\Delta = \{\mathbb{R} \times y\}_{y \in [0,1]}$ be the trivial foliation on $\mathbb{R} \times [0, 1]$ into horizontal lines. Then for every *open* (in the induced topology) subset $S \subset \mathbb{R} \times [0, 1]$ such that $\mathbb{R} \times (0, 1) \subset S$ the foliated surface $(S, \Delta|_S)$ will be called a *model strip*. In this case we will use the following notation:

$$\begin{aligned} \partial_0 S &:= S \cap (\mathbb{R} \times \{0\}), & \partial_1 S &:= S \cap (\mathbb{R} \times \{1\}), \\ \partial S &:= \partial_0 S \cup \partial_1 S, & \text{Int}(S) &:= \mathbb{R} \times (0, 1), \end{aligned}$$

and call ∂S the *boundary* of S , and $\partial_0 S$ and $\partial_1 S$ the *boundary sides* of S .

Thus a model strip is a union of $\mathbb{R} \times (0, 1)$ with at most countably many (possibly zero) open mutually disjoint intervals contained in $\mathbb{R} \times \{0, 1\}$. Evidently, each model strip is a quasi-strip as well. The difference between these two notions is that in the model strip all boundary components are distributed between two boundary sides only, while quasi-strips may have boundary components between internal leaves.

Definition 3.1.3. Let Z be a two-dimensional topological manifold (a surface) and $Z_0 = \bigsqcup_{\lambda \in \Lambda} S_\lambda$ be a family of mutually disjoint *quasi-strips* $\{S_\lambda\}_{\lambda \in \Lambda}$.

A *quasi-striped atlas* on Z is a map $q : Z_0 \rightarrow Z$ such that

- 1) q is a *quotient* map, *i.e.* it is continuous, surjective, and a subset $U \subset Z$ is open if and only if $q^{-1}(U) \cap S_\lambda$ is open in S_λ for each $\lambda \in \Lambda$;
- 2) there exist two disjoint families $\mathcal{X} = \{X_j\}_{j \in \Gamma}$ and $\mathcal{Y} = \{Y_j\}_{j \in \Gamma}$ of mutually distinct boundary intervals of Z_0 enumerated by the same set of indexes Γ such that
 - (a) q is injective on $Z_0 \setminus (\mathcal{X} \cup \mathcal{Y})$;
 - (b) $q(X_j) = q(Y_j)$ for each $j \in \Gamma$, and the restrictions

$$X_j \xrightarrow{q|_{X_j}} q(X_j) = q(Y_j) \xleftarrow{q|_{Y_j}} Y_j$$

are bijections.

The images $q(X_j) = q(Y_j)$, $j \in \Gamma$, will be called *seams* of Z (as well as of the striped atlas q).

If all strips $\{S_\lambda\}_{\lambda \in \Lambda}$ are model, then the quasi-striped atlas q will be called *striped*.

Condition (b) implies that for each $j \in \Gamma$ we have a well-defined “gluing” homeomorphism $\phi_j = (q|_{X_j})^{-1} \circ q|_{Y_j} : Y_j \rightarrow X_j$. Thus a quasi-striped surface is obtained from a family of quasi-strips by gluing them along certain pairs of boundary intervals by homeomorphisms ϕ_j . It is allowed to glue two quasi-strips along more than one pair of boundary components, and one may also glue boundary components belonging to the same quasi-strip.

Notice that the foliations Δ_{S_λ} on strips S_λ give a unique foliation Δ_0 on Z_0 . In this case if γ is a leaf of Δ_0 contained in a strip S_λ , then we will say that S_λ is a *strip of γ* .

Also, the images of leaves of Δ_0 under q constitute a one-dimensional foliation Δ on Z which we will call the *canonical* foliation associated with the quasi-striped atlas q .

Definition 3.1.4. A foliated surface (Z, Δ) will be called a *(quasi-)striped* surface whenever it admits a (quasi-)striped atlas whose canonical foliation is Δ .

Now the results by W. Kaplan can be formulated as follows.

Theorem 3.1.5 (W. Kaplan [12, 13]). *Every one-dimensional foliation Δ on the plane \mathbb{R}^2 belongs to the class \mathcal{F} and admits a quasi-striped atlas, *i.e.* Δ is a canonical foliation associated with some quasi-striped atlas q on \mathbb{R}^2 .*

Remark 3.1.6. Let (Z, Δ) be any (oriented or non-oriented, compact or non-compact) connected surface distinct from S^2 and $\mathbb{R}P^2$. Then the universal covering of its interior $\text{Int}(Z)$ is homeomorphic with \mathbb{R}^2 , *e.g.* [3, Corollary 1.8]. Moreover, if $p : \mathbb{R}^2 \rightarrow \text{Int}(Z)$ is the corresponding universal

covering map, then one has a foliation F on \mathbb{R}^2 induced from Δ : the leaves of F are connected components of the inverse images $p^{-1}(\omega)$, $\omega \in \Delta$. There is also a natural free action of the fundamental group $\pi_1 Z$ on \mathbb{R}^2 by foliated homeomorphisms so that $(\text{Int}(Z), \Delta|_{\text{Int}(Z)})$ is the quotient $(\mathbb{R}^2, F)/\pi_1 Z$.

This observation shows that foliations on \mathbb{R}^2 determine all foliations on all connected boundaryless surfaces distinct from S^2 and $\mathbb{R}P^2$. However, this does not help so much, since usually it is hard to explicitly describe the action of $\pi_1 Z$ of \mathbb{R}^2 by covering transformations.

Also, due to Theorem 3.1.5 the foliated surface (\mathbb{R}^2, F) admits a quasi-striped atlas. However, again such a splitting into quasi-strips in general is not related with the action of $\pi_1 Z$. Therefore it is still better to work with the surface Z instead of \mathbb{R}^2 .

In fact, the splitting of a quasi-striped surface into quasi-strips proposed by Kaplan in Theorem 3.1.5 is not unique. This led the present authors to consider quasi-striped surfaces glued from model strips only, and it turned out that splitting into such strips (if it exists) is unique, see Theorem 3.1.10.

Remark 3.1.7. Let us discuss distinct types of leaves of Δ .

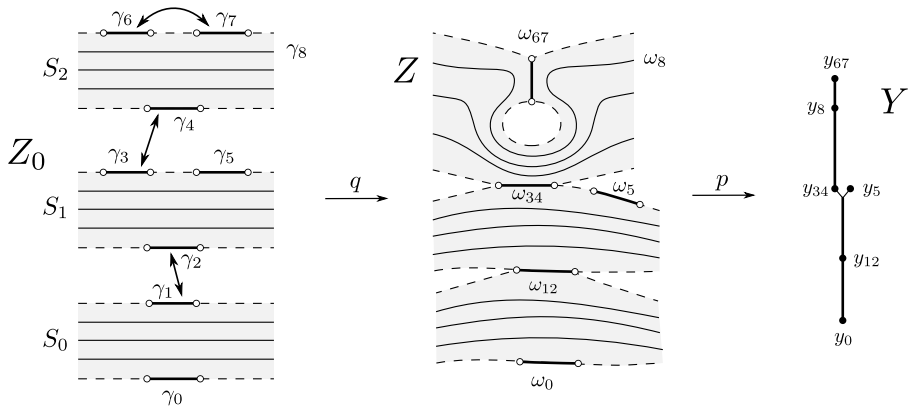


FIGURE 3.1.

- 1) For each leaf $\gamma = \mathbb{R} \times \{t\} \subset \text{Int}(S_\lambda)$, $t \in (0, 1)$,

$$\omega := q(\gamma) \in \Delta_{reg}, \quad y := p(\omega) \in \text{Int}(Y) \setminus \mathcal{V},$$

see the leaf ω_8 in Figure 3.1.

- 2) Moreover, suppose $\omega \in \Delta$ is a seam, so $q^{-1}(\omega) = X_j \cup Y_j$, where $X_j \subset \partial_\varepsilon S_\lambda$, and $Y_j \subset \partial_{\varepsilon'} S_{\lambda'}$ for some $\varepsilon, \varepsilon' \in \{0, 1\}$ and $\lambda, \lambda' \in \Lambda$. Then the following characterizations hold.

- a) $\omega \in \Delta_{reg}$ if and only if $X_j = \partial_\varepsilon S_\lambda$ and $Y_j = \partial_{\varepsilon'} S_{\lambda'}$, see leaf ω_{12} in Figure 3.1.

- b) $\omega \in \Delta_{sing} \setminus \Delta_{spec}$ if and only if $\lambda = \lambda'$, $\varepsilon = \varepsilon'$, and $\partial_\varepsilon S_\lambda = X_j \cup Y_j$. In other words, $\partial_\varepsilon S_\lambda$ consists of two boundary intervals, and ω is obtained by gluing those intervals. In this case $y := p(\omega) \in \partial Y \setminus \mathcal{V}$, so Y is Hausdorff at y , see the leaf ω_{67} in Figure 3.1.
- c) In all other cases, $\omega \in \Delta_{spec}$, and $\text{hcl}(\omega)$ contains all leaves from

$$q(\partial_\varepsilon S_\lambda \cup \partial_{\varepsilon'} S_{\lambda'}),$$

see leaves ω_{34} and ω_5 in Figure 3.1.

Notice that in the case a) one can replace $S_\lambda \cup S_{\lambda'}$ with a single strip, and get another striped atlas for (Z, Δ) having one less strip. This leads to the following notion.

A striped atlas will be called *reduced* whenever

$$q(\partial Z_0) = \partial Z \cup \Delta_{sing}. \tag{3.1}$$

The following statement is easy and we leave it for the reader.

Lemma 3.1.8 (cf. [17, Lemma 2.1]). *Let $q : Z_0 \rightarrow Z$ be a striped atlas, with Δ the corresponding canonical foliation on Z , $p : Z \rightarrow Y$ the projection to the space of leaves, and \mathcal{V} the set of special points of Y . Then*

$$p(\text{hcl}(\omega)) = \text{hcl}(p(\omega))$$

for every leaf $\omega \in \Delta$. Moreover,

$$p(\Delta_{spec}) = \mathcal{V}, \quad p(\Delta_{sing} \setminus \Delta_{spec}) \subset \partial Y \setminus \mathcal{V}, \quad p(\partial Z \cup \Delta_{sing}) = \partial Y \cup \mathcal{V}.$$

If q is **reduced**, then for every connected component C of $Y \setminus (\partial Y \cup \mathcal{V})$, $q^{-1}(p^{-1}(C))$ is an interior of some model strip $S_\lambda \subset Z_0$. Define the following path $\gamma_C : (0, 1) \rightarrow \text{Int}(S_\lambda) = \mathbb{R} \times (0, 1)$ by $\gamma_C(t) = (0, t)$, $t \in (0, 1)$. Then the composition $p \circ q \circ \gamma_C : (0, 1) \rightarrow C$ is a homeomorphism. \square

Theorem 3.1.9 ([20, Theorem 4.4]). *Let (Z, Δ) be a foliated surface from class \mathcal{F} . Then the following conditions are equivalent:*

- (A4) *the family Δ_{sing} of all singular leaves of Δ is locally finite;*
- (A5) *(Z, Δ) is a striped surface.*

Theorem 3.1.10 ([16, Theorem 3.7]). *Every connected striped surface Z with countable base is foliated homeomorphic either to*

- (1) *an open cylinder C or an open Möbius band M foliated by fibers of one-dimensional vector bundles $C \rightarrow S^1$ and $M \rightarrow S^1$ over the circle, or to*
- (2) *a striped surface with reduced atlas.*

Notice that in the case (1) all leaves of Δ are regular. Also in the case (2) the seams are precisely the singular leaves contained in $\text{Int}(Z)$ and thus we have a canonical decomposition of Z into strips.

3.2. Characterization of the homeotopy group $\mathcal{H}_{\text{id}}(Z, \Delta)$ for striped surfaces. Let $q : Z_0 \rightarrow Z$ be a striped atlas, and $h \in \mathcal{H}(Z, \Delta)$ a homeomorphism such that $h(q(S_\lambda)) = q(S_\lambda)$ for some $\lambda \in \Lambda$. Since q is injective on the interior of S_λ , as well as on each boundary component of S_λ , it follows that there exists a unique foliated homeomorphism $h_0 : S_\lambda \rightarrow S_\lambda$ making commutative the following diagram:

$$\begin{array}{ccc}
 S_\lambda & \xrightarrow{q} & q(S_\lambda) \\
 h_0 \downarrow & & \downarrow h \\
 S_\lambda & \xrightarrow{q} & q(S_\lambda)
 \end{array} \tag{3.2}$$

Moreover, see [16, Lemma 4.1], $h_0(x, y) = (\alpha(x, y), \beta(y))$, where

- $\beta : [0, 1] \rightarrow [0, 1]$ is a homeomorphism;
- for each $y \in (0, 1)$ the map $\alpha_y : \mathbb{R} \rightarrow \mathbb{R}$, $\alpha_y(x) = \alpha(x, y)$, $x \in \mathbb{R}$, is a homeomorphism;
- for each $y \in \{0, 1\}$ the map $\alpha_y : \partial_y S_\lambda \rightarrow \partial_{\beta(y)} S_\lambda$, $\alpha_y(x) = \alpha(x, y)$, is a *monotone* homeomorphism between linearly ordered subsets (being unions of mutually disjoint open intervals) $\partial_y S_\lambda$ and $\partial_{\beta(y)} S_\lambda$.

Theorem 3.2.1 ([16, Theorem 4.4]). *Let $q : Z_0 \rightarrow Z$ be a reduced striped atlas such that the family Δ_{spec} of all special leaves of the canonical foliation Δ is locally finite. Let also $h \in \mathcal{H}(Z, \Delta)$. Then $h \in \mathcal{H}_{\text{id}}(Z, \Delta)$ iff the following conditions hold:*

- (ZA) *h preserves each leaf $\omega \in \partial Z \cup \Delta_{\text{sing}}$ with its orientation;*
- (ZB) *$h(q(S_\lambda)) = q(S_\lambda)$ for all $\lambda \in \Lambda$, and if $h_0 = (\alpha, \beta) : S_\lambda \rightarrow S_\lambda$ is a unique lifting of $h|_{S_\lambda}$, see (3.2), then β and each α_y , $y \in [0, 1]$, is strictly increasing.*

4. PROOF OF THEOREM 1.4

Let (Z, Δ) be a connected foliated surface from class \mathcal{F} satisfying condition (A4). By Theorem 3.1.9, the latter condition is equivalent to the assumption (A5) that (Z, Δ) is a striped surface, *i.e.* there exists a striped atlas $q : Z_0 \rightarrow Z$ such that Δ is its canonical foliation. We need to prove that $\psi(\mathcal{H}_{\text{id}}(Z, \Delta)) = \mathcal{H}_{\text{id}}(Y)$ and that the kernel of the induced homomorphism $\psi_0 : \pi_0 \mathcal{H}_{\text{id}}(Z, \Delta) \rightarrow \mathcal{H}_{\text{id}}(Y)$ is either trivial or isomorphic to \mathbb{Z}_2 .

First suppose that Z is connected. Then due to Theorem 3.1.10 we have the following three cases.

1) $Z = \mathbb{R} \times S^1$ is an open cylinder, $Y = S^1$ is the circle, $p : \mathbb{R} \times S^1 \rightarrow S^1$ is given by $p(x, y) = y$, and Δ consists of lines $\mathbb{R} \times \{y\}$, $y \in S^1$. Then every $h \in \mathcal{H}(Z, \Delta)$ is given by $h(x, y) = (\alpha_h(x, y), \beta_h(y))$, where $\beta_h : S^1 \rightarrow S^1$ is some homeomorphism, and $\alpha_h : \mathbb{R} \times S^1 \rightarrow \mathbb{R}$ is a continuous map such that the correspondence $x \mapsto \alpha_h(x, y)$ is a homeomorphism $\mathbb{R} \rightarrow \mathbb{R}$ which preserves or reverses orientation mutually for all $y \in S^1$. In particular, we have a homomorphism $\mathbf{or} : \mathcal{H}(Z, \Delta) \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbf{or}(h) = (\mathbf{or}(\alpha_h), \mathbf{or}(\beta_h))$, where $\mathbf{or}(\beta_h) = 0$ if β_h preserves orientation, $\mathbf{or}(\beta_h) = 1$ otherwise, and similarly for α_h .

Evidently $\psi(h) = \beta_h$. This also implies that if $\beta \in \mathcal{H}(Y)$ and $h \in \mathcal{H}(Z, \Delta)$ is given by $h(x, y) = (x, \beta(y))$, then $\psi(h) = \beta$. Hence ψ is surjective.

It is well-known that $\mathcal{H}_{\text{id}}(Y)$ consists of homeomorphisms preserving orientation of Y . Also, see *e.g.* [21, Examples 6.5], one easily checks that $h \in \mathcal{H}_{\text{id}}(Z, \Delta)$ iff β_h and the correspondences $x \mapsto \alpha_h(x, y)$ preserve orientations. In particular, we get the following commutative diagram:

$$\begin{array}{ccccccc}
 \mathcal{H}_{\text{id}}(Z, \Delta) & \hookrightarrow & \mathcal{H}(Z, \Delta) & \xrightarrow{\mathbf{or}} & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \xrightarrow{\cong} & \pi_0 \mathcal{H}(Z, \Delta) \\
 \psi \downarrow & & \psi: h \mapsto \beta_h \downarrow & & (a, b) \mapsto b \downarrow & & \downarrow \psi_0 \\
 \mathcal{H}_{\text{id}}(Y) & \hookrightarrow & \mathcal{H}(Y) & \xrightarrow{\mathbf{or}} & \mathbb{Z}_2 & \xrightarrow{\cong} & \pi_0 \mathcal{H}(Y)
 \end{array}$$

This implies Theorem 1.4 for this case.

2) The proof of Theorem 1.4 for the case when Z is an open Möbius band and $p : Z \rightarrow S^1$ is a locally trivial fibration with fiber \mathbb{R} is similar, and we leave it to the reader.

3) In all other cases due to Theorem 3.1.10 one can assume that q is a reduced striped atlas.

a) First we prove that $\psi(\mathcal{H}_{\text{id}}(Z, \Delta)) \subset \mathcal{H}_{\text{id}}(Y)$ which will imply that ψ induces a homomorphism $\psi_0 : \pi_0 \mathcal{H}_{\text{id}}(Z, \Delta) \rightarrow \pi_0 \mathcal{H}_{\text{id}}(Y)$.

Let $h \in \mathcal{H}_{\text{id}}(Z, \Delta)$ and $k = \psi(h)$. Thus h satisfies conditions (ZA) and (ZB) of Theorem 3.2.1, and we should check that $k \in \mathcal{H}_{\text{id}}(Y)$, *i.e.* it satisfies conditions (YA) and (YB) of Lemma 2.4.1.

Due to (ZA), h preserves each leaf $\omega \subset \partial Z \cup \Delta_{\text{sing}}$, whence k fixes each point $y \in \partial Y \cup \mathcal{V} = p(\partial Z \cup \Delta_{\text{sing}})$, see Lemma 3.1.8, *i.e.* k satisfies condition (YA).

Furthermore, let C be a connected component of $Y \setminus (\partial Y \cup \mathcal{V})$. Then by Lemma 3.1.8, $q^{-1}(p^{-1}(C))$ is the interior of some model strip S_λ . Moreover, by property (ZB), $h(q(S_\lambda)) = q(S_\lambda)$, and h lifts to a homeomorphism $h_0 = (\beta, \alpha) : S_\lambda \rightarrow S_\lambda$ in which β is strictly increasing. But the latter is equivalent to the assumption that k preserves orientation of C , *i.e.* satisfies condition (YB).

b) Let us show that $\psi(\mathcal{H}_{\text{id}}(Z, \Delta)) = \mathcal{H}_{\text{id}}(Y)$.

Let $k \in \mathcal{H}_{\text{id}}(Y)$. We should find $h \in \mathcal{H}_{\text{id}}(Z, \Delta)$ such that $\psi(h) = k$. In fact we will define a homeomorphism $h_0 : Z_0 \rightarrow Z_0$ which will induce the desired h .

Let C be a connected component of $Y \setminus T$, so $r^{-1}(C)$ is the interior of some model strip S_λ . Then by Lemma 3.1.8, the map $\gamma : (0, 1) \rightarrow C$, $\gamma(y) = r(0, y)$, $y \in (0, 1)$, is a homeomorphism. Since by (YB), $k(C) = C$ and k preserves its orientation, the map

$$\beta = \gamma^{-1} \circ k \circ \gamma : (0, 1) \rightarrow (0, 1)$$

is an orientation preserving homeomorphism of $(0, 1)$. Therefore β extends to a self-homeomorphism $\beta : [0, 1] \rightarrow [0, 1]$ by setting $\beta(0) = 0$ and $\beta(1) = 1$, and we define $h_0 : S_\lambda \rightarrow S_\lambda$ by $h_0(x, y) = (x, \beta(y))$, so it is also fixed on S_λ .

It now follows that h_0 induces a homeomorphism $h \in \mathcal{H}_{\text{id}}(Z, \Delta)$ fixed on $\partial Z \cup \Delta_{\text{sing}}$ and such that $h_0 \circ q = q \circ h$. Then h in turn induces a homeomorphism $k' = \psi(h) : Y \rightarrow Y$ fixed on $p(\partial Z \cup \Delta_{\text{sing}}) = \partial Y \cup \mathcal{V}$ and coinciding with k on the complement $Y \setminus (\partial Y \cup \mathcal{V})$. But due to (YA), k is also fixed on $\partial Y \cup \mathcal{V}$, whence $k = k' = \psi(h)$.

c) Denote $\mathcal{Q} := \psi^{-1}(\mathcal{H}_{\text{id}}(Y))$. Then

$$\begin{aligned} \ker \left[\psi_0 : \pi_0 \mathcal{H}(Z, \Delta) \rightarrow \pi_0 \mathcal{H}(Y) \right] &= \ker \left[\psi_0 : \frac{\mathcal{H}(Z, \Delta)}{\mathcal{H}_{\text{id}}(Z, \Delta)} \rightarrow \frac{\mathcal{H}(Y)}{\mathcal{H}_{\text{id}}(Y)} \right] \cong \\ &\cong \frac{\psi^{-1}(\mathcal{H}_{\text{id}}(Y))}{\mathcal{H}_{\text{id}}(Z, \Delta)} = \mathcal{Q} / \mathcal{H}_{\text{id}}(Z, \Delta). \end{aligned}$$

Hence it remains show that $\mathcal{Q} / \mathcal{H}_{\text{id}}(Z, \Delta)$ is either trivial or isomorphic with \mathbb{Z}_2 .

Let $h \in \mathcal{Q}$, $k = \psi(h) \in \mathcal{H}_{\text{id}}(Y)$, and $h_0 : Z_0 \rightarrow Z_0$ be its unique lifting. Then h_0 preserves every boundary component of Z_0 and $h_0(S_\lambda) = S_\lambda$ for all $\lambda \in \Lambda$. Moreover, $h_0 : \text{Int}(S_\lambda) = \mathbb{R} \times (0, 1) \rightarrow \mathbb{R} \times (0, 1) = \text{Int}(S_\lambda)$ is given by $h_0(x, y) = (\alpha^\lambda(x, y), \beta^\lambda(y))$, where β^λ is increasing, while α_y^λ is either increasing or decreasing mutually for all $y \in (0, 1)$.

Therefore one can associate to each λ an element $\mathbf{or}_h(\lambda) \in \mathbb{Z}_2$ so that $\mathbf{or}_h(\lambda) = 0$ (resp. $\mathbf{or}_h(\lambda) = 1$) if all α_y^λ are increasing (resp. decreasing).

Note that if two strips S_λ and $S_{\lambda'}$ are glued along some of their boundary components, then $\mathbf{or}_h(\lambda) = \mathbf{or}_h(\lambda')$, whence \mathbf{or}_h is constant on the connected components of Z .

But Z is connected, whence \mathbf{or}_h is constant. Therefore, one can associate to each $h \in \mathcal{Q}$ a well-defined element $\mathbf{or}_h \in \mathbb{Z}_2$. It is easy to checks that the correspondence $h \mapsto \mathbf{or}_h$ is a homomorphism $\mathbf{or} : \mathcal{Q} \rightarrow \mathbb{Z}_2$. Finally, $h \in \ker(\mathbf{or})$ iff h satisfies conditions (ZA) and (ZB), i.e. $h \in \mathcal{H}_{\text{id}}(Z, \Delta)$.

Therefore $\ker(\psi_0) = \mathcal{Q}/\mathcal{H}_{\text{id}}(Z, \Delta)$ is either trivial or \mathbb{Z}_2 . This completes Theorem 1.4.

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