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Relative Gottlieb groups of mapping spaces and their rational cohomology

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Abstract. Let $f : X \to Y$ be a map of simply connected CW-complexes of finite type. Put $\max \pi_*(Y) \otimes \mathbb{Q} = \max\{i \mid \pi_i(Y) \otimes \mathbb{Q} \neq 0\}$. In this paper we compute the relative Gottlieb groups of f when X is an F_0-space and Y is a product of odd spheres. Also, under reasonable hypothesis, we determine these groups when X is a product of odd spheres and Y is an F_0-space. As a consequence, we show that the rationalized G-sequence associated to f splits into a short exact sequence. Finally, we prove that the rational cohomology of $\mathrm{map}(X, Y; f)$ is infinite dimensional whenever $\max \pi_*(Y) \otimes \mathbb{Q}$ is even.

Анотація. Нехай $f : X \to Y$ – відображення однозв'язних CW-комплексів скінченного типу. Покладемо $\max \pi_*(Y) \otimes \mathbb{Q} = \max\{i \mid \pi_i(Y) \otimes \mathbb{Q} \neq 0\}$. В роботі обчислено відносні групи Готліба відображення f для випадку, коли X є F_0-простом, а Y – добутком сфер непарних розмірностей. За досить природних припущень, ці групи також обчислено для протилежної ситуації, коли X – це добуток сфер непарних розмірностей, а Y – F_0-простір. Як наслідок, показано, що раціоналізована G-послідовність, пов'язана з f, розщеплюється в коротку точну послідовність. Також доведено, що раціональні когомології $\mathrm{map}(X, Y; f)$ мають нескінченну розмірність, за умови, що $\max \pi_*(Y) \otimes \mathbb{Q}$ є парним.

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1. Introduction

In this paper all spaces are simply connected CW-complex and are of finite type over $\mathbb{Q}$, i.e., have finite dimensional rational cohomology in each degree.

For two topological spaces X and Y, let $\mathrm{map}(X, Y)$ be the mapping space of all free continuous maps of X into Y. This space is generally disconnected with path-components corresponding to the set of free homotopy classes of maps from X to Y. We write $\mathrm{map}(X, Y; f)$ for the path-component containing a given map $f : X \to Y$.

The n-th generalized Gottlieb group $G_n(Y, X; f)$ of a map $f : X \to Y$ is the subgroup of $\pi_n(Y)$ consisting of homotopy classes of map $\alpha : \mathbb{S}^n \to Y$ such that the wedge

$$(\alpha \vee f) : \mathbb{S}^n \vee X \to Y$$

extends to a map $H : \mathbb{S}^n \times X \to Y$. Alternately, $G_n(Y, X; f)$ is the image of the map induced on homotopy groups by the evaluation map $\omega : \mathrm{map}(X, Y; f) \to Y$, [15]. The n-th Gottlieb group of X, denoted $G_n(X)$, is a special case $X = Y$ and $f = \mathrm{id}_X$, see [4, 5]. In particular,

$$G_n(X) = \mathrm{im}\big(\omega_\sharp : \pi_n(\mathrm{map}(X, X; \mathrm{id}_X)) \to \pi_n(X)\big).$$

Gottlieb groups have led to many interesting results in topology, especially, in fibration, in fixed point theory and in the theory of identification of spaces. The existence of cross-section can also be studied using the Gottlieb groups. For instance, the triviality of the n-th Gottlieb group of a space ensures that every fibration over an $(n+1)$-dimensional sphere with the space as a fiber has a cross-section, [5]. But, unfortunately there are not many explicit computations of these groups in literature. One reason is the fact that a map of spaces does not necessarily induce a corresponding homomorphism of Gottlieb groups. In [16], Lee and Woo attempted to circumvent this problem by introducing the n-th relative Gottlieb group $G_n^{\mathrm{rel}}(Y, X; f)$, and showed that they fit in a sequence

$$\cdots \to G_{n+1}^{\mathrm{rel}}(Y, X; f) \to G_n(X) \to \\ \to G_n(Y, X; f) \to G_n^{\mathrm{rel}}(Y, X; f) \to \cdots \quad (1.1)$$

called the G-sequence of f, which is not necessarily exact. The computation of the rational relative Gottlieb groups was limited to sporadic cases, see [2, 7, 9, 11]. Our goal in this paper is to compute these groups in some new cases. Our main results are the following:

Theorem 1.1. *Let $f : X \to Y \simeq \prod_j \mathbb{S}^{2n_j+1}$ be a map with X being an F_0-space. Then*

$$G_*^{\mathrm{rel}}(Y, X; f) \otimes \mathbb{Q} \;\cong\; \pi_*(Y) \otimes \mathbb{Q} \oplus \pi_{\mathrm{odd}}(X) \otimes \mathbb{Q}.$$

Theorem 1.2. *Let $f : X \simeq \prod_j \mathbb{S}^{2n_j+1} \to Y$ be a map with Y being an F_0-space. Suppose that*

(i) $\pi_{\mathrm{odd}}(Y) \otimes \mathbb{Q} \cong \pi_*(X) \otimes \mathbb{Q}$;

(ii) $\max \pi_{\mathrm{even}}(Y) \otimes \mathbb{Q} \leqslant \min \pi_*(X) \otimes \mathbb{Q}$.

Then, we have that $G_*^{\mathrm{rel}}(Y, X; f) \otimes \mathbb{Q} \cong \pi_{\mathrm{even}}(Y) \otimes \mathbb{Q}$.

Our last result focused on rational cohomology of $\mathrm{map}(X, Y; f)$.

Theorem 1.3. *Let $f : X \to Y$ be a map with finite X. If $\max \pi_*(Y) \otimes \mathbb{Q}$ is even, then the rational cohomology of $\mathrm{map}(X, Y; f)$ are infinite dimensional.*

The paper is organized as follows. In section 2 and 3, we introduce our notation and recall some background of rational homotopy theory, namely Sullivan minimal models, derivations and mapping cone. In section 4, we use this background to prove Theorem 1.1. Section 5 is devoted to establish Theorem 1.2. In section 6 we prove that the rational cohomology of mapping spaces is infinite dimensional, in particular we prove Theorem 1.3. The paper ends with section 7 in which we propose an open problem.

2. Sullivan minimal models

We will work with $\mathbb{Q}$ as ground field and our principal tools are Sullivan minimal models. A detailed description of these and the standard tools of rational homotopy theory can be found in [1]. For our purposes, we recall the following.

Definition 2.1. *A commutative differential graded algebra (cdga) is a graded algebra $A = \bigoplus_{i \geqslant 0} A^i$ with a differential $d : A^i \to A^{i+1}$ such that*

$$d^2 = 0, \qquad xy = (-1)^{ij} yx, \qquad d(xy) = d(x)y + (-1)^i x d(y),$$

for all $x \in A^i$ and $y \in A^j$.

In [14], D. Sullivan defined a contravariant functor A_{PL} which associates to each space X a cgda $A_{\mathrm{PL}}(X)$ which represents the rational homotopy type of X. He also constructed, for each simply connected cgda (A, d) (i.e., satisfying $H^0(A, d) = H^1(A, d) = 0$), another cgda $(\Lambda V, d)$ and a map

$$(\Lambda V, d) \xrightarrow{\cong} (A, d)$$

which induces an isomorphism in cohomology, where ΛV denotes the free commutative graded algebra on the graded vector space $V = \oplus_n V^n$, which has a well ordered, homogeneous basis $\{x_\alpha\}$ such that, if $V_{<\alpha}$ denotes span $\{x_\beta \mid \beta < \alpha\}$, we have $dx_\alpha \in \Lambda^{\geqslant 2}(V_{<\alpha})$. The cgda $(\Lambda V, d)$ is called a Sullivan minimal model of (A, d) or a Sullivan minimal model of X if $A = A_{\mathrm{PL}}(X)$. In particular, if $(\Lambda V, d)$ is a Sullivan minimal model of X, then there are isomorphism's:

$$H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q}) \text{ as commutative graded algebras,}$$
$$V \cong \pi_*(X) \otimes \mathbb{Q} \text{ as graded vector spaces.}$$

A map of spaces $f : X \to Y$ has a Sullivan minimal model which is a map of cdga $\varphi : (\Lambda W, d_W) \to (\Lambda V, d_V)$, where $(\Lambda W, d_W)$ and $(\Lambda V, d_V)$ are the Sullivan minimal models of Y and X respectively.

Next, we focus on elliptic spaces. A simply connected CW-complex X is *elliptic* if and only if $\pi_*(X) \otimes \mathbb{Q}$ and $H^*(X; \mathbb{Q})$ are both finite dimensional. Algebraically, this means that V and $H^*(\Lambda V, d)$ are both finite dimensional, where $(\Lambda V, d)$ is a Sullivan minimal model of X. There is a remarkable subclass of elliptic spaces called F_0-space.

Definition 2.2. *An elliptic space X is said to be an F_0-space if*

$$H^*(X; \mathbb{Q}) = H^{\mathrm{even}}(X; \mathbb{Q}).$$

Therefore, we remark that in terms of Sullivan minimal model, an F_0-space X has a model of the form $\big(\Lambda(x_1, \ldots, x_m, y_1, \ldots, y_m), d\big)$ where $|x_i|$ is even, $|y_j|$ is odd, $d(x_i) = 0$ for $1 \leqslant i \leqslant m$ and $d(y_j) \in \Lambda(x_1, \ldots, x_m)$ for $1 \leqslant j \leqslant m$.

Example 2.3. There are many examples of such spaces:

- Finite products of even dimensional spheres,
- Finite products of complex projective spaces,
- Homogeneous spaces G/H, where H is a closed sub-group of maximal rank of a compact connected Lie group G.

3. Derivation of a Sullivan model and the rationalized G-sequence

Our purpose in this section is to give a description in rational homotopy theory of all the terms involved in the G-sequence (1.1).

Definition 3.1. *Let $\varphi : (\Lambda W, d_W) \to (\Lambda V, d_V)$ be a morphisms of cdga. Define a φ-derivation θ of degree n to be a linear map $\theta : \Lambda W \to \Lambda V$ that* *reduces degree by n such that*

$$\theta(xy) = \theta(x)\varphi(y) + (-1)^{n|x|}\varphi(x)\theta(y).$$

Let $\mathrm{Der}_n(\Lambda W, \Lambda V; \varphi)$ *denote the vector space of* φ*-derivations of degree* n *for* $n > 0$*. Define a linear map*

$$\partial : \mathrm{Der}_n(\Lambda W, \Lambda V; \varphi) \to \mathrm{Der}_{n-1}(\Lambda W, \Lambda V; \varphi)$$

by

$$\partial(\theta) = d_V \circ \theta - (-1)^{|\theta|}\theta \circ d_W.$$

Here, we note that $(\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi), \partial)$ is a chain complex, where

$$\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi) = \bigoplus_n \mathrm{Der}_n(\Lambda W, \Lambda V; \varphi).$$

In the case $\Lambda W \cong \Lambda V$ and $\varphi = \mathrm{id}_{\Lambda V}$, the chain complex of derivations $\mathrm{Der}_*(\Lambda V, \Lambda V; \mathrm{id}_{\Lambda V})$ is just the usual complex of derivations on the cdga ΛV which we denoted by $\mathrm{Der}_*(\Lambda V)$. There is an isomorphism of graded vector spaces

$$\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi) \cong \mathrm{Hom}_*(W, \Lambda V).$$

Hence, we can denote by (y, x) the unique φ-derivation sending an element $y \in W$ to $x \in \Lambda V$ and the other generators to zero.

The detailed discussion of the following are in [7]. The post-composition with the augmentation $\varepsilon : \Lambda V \to \mathbb{Q}$ gives a chain complex map

$$\varepsilon_* : \mathrm{Der}_*(\Lambda W, \Lambda V; \varphi) \to \mathrm{Der}_*(\Lambda W, \mathbb{Q}; \varepsilon).$$

Definition 3.2. *For $n \geqslant 2$, the n-th generalized Gottlieb group of φ is defined as follows:*

$$G_n(\Lambda W, \Lambda V; \varphi) = \mathrm{im}\{H_n(\varepsilon_*) : H_n(\mathrm{Der}(\Lambda W, \Lambda V; \varphi)) \to \mathrm{Hom}_n(W, \mathbb{Q})\}.$$

Note that $w^* \in \mathrm{Hom}_n(W, \mathbb{Q})$, ($w^*$ is the dual of the basis element w of W^n) is in $G_n(\Lambda W, \Lambda V; \varphi)$ if and only if w^* extends to a derivation θ of $\mathrm{Der}_n(\Lambda W, \Lambda V; \varphi)$ such that $\partial(\theta) = 0$. In particular, we have

Definition 3.3. *The n-th Gottlieb group of $(\Lambda V, d_V)$ is defined as follows:*

$$G_n(\Lambda V) = \mathrm{im}\{H_n(\varepsilon_*) : H_n(\mathrm{Der}(\Lambda V)) \to \mathrm{Hom}_n(V, \mathbb{Q})\} \text{ for } n \geqslant 2.$$

Theorem 3.4 (cf. [7])**.** *If X is a finite CW-complex, then*

$$G_n(Y, X; f) \otimes \mathbb{Q} \cong G_n(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; f_{\mathbb{Q}}) \cong G_n(\Lambda W, \Lambda V; \varphi) \text{ for } n \geqslant 2.$$

We recall the definition of the mapping cone of a chain map $\varphi : A \to B$.

Definition 3.5. *Let $\varphi : (A, d_A) \to (B, d_B)$ be a map of differential graded vector spaces. The mapping cone of φ denoted by $\mathrm{Rel}_*(\varphi)$ is defined as follows: $\mathrm{Rel}_n(\varphi) = A_{n-1} \oplus B_n$ with the differential*

$$D(a,b) = \big(-d_A(a), \varphi(a) + d_B(b)\big).$$

Further, define the inclusion and projection maps $J : B_n \to \mathrm{Rel}_n(\varphi)$ by $J(b) = (0, b)$ and $P : \mathrm{Rel}_n(\varphi) \to A_{n-1}$ by $P(a, b) = a$. These yields a short exact sequence of chain complexes

$$0 \;\to\; B_* \xrightarrow{J} \mathrm{Rel}_*(\varphi) \xrightarrow{P} A_{*-1} \;\to\; 0.$$

This definition can be applied to the Sullivan minimal model

$$\varphi : (\Lambda W, d_W) \to (\Lambda V, d_V)$$

of the map $f : X \to Y$.

Note that the pre-composition with φ gives a map of chain complexes:

$$\varphi^* : \mathrm{Der}_*(\Lambda V) \to \mathrm{Der}_*(\Lambda W, \Lambda V; \varphi).$$

Following G. Lupton and S. B. Smith [7], we consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Der}_*(\Lambda V) & \xrightarrow{\varphi^*} & \mathrm{Der}_*(\Lambda W, \Lambda V; \varphi) & \xrightarrow{J} & \mathrm{Rel}_*(\varphi^*) \\
\downarrow{\scriptstyle \varepsilon_*} & & \downarrow{\scriptstyle \varepsilon_*} & & \downarrow{\scriptstyle (\varepsilon_*, \varepsilon_*)} \\
\mathrm{Der}_*(\Lambda V, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{\varphi^*}} & \mathrm{Der}_*(\Lambda W, \mathbb{Q}; \varepsilon) & \xrightarrow{\hat{J}} & \mathrm{Rel}_*(\widehat{\varphi^*})
\end{array}$$

Here, ε denotes the augmentation of either ΛV or ΛW. On passing to homology and using the naturality of the mapping cone construction, we obtain the following homology ladder for $n \geqslant 2$:

$$\begin{array}{ccccccc}
\cdots \to H_{n+1}(\mathrm{Rel}(\varphi^*)) & \xrightarrow{H(P)} & H_n(\mathrm{Der}(\Lambda V)) & \xrightarrow{H(\varphi^*)} & H_n(\mathrm{Der}(\Lambda W, \Lambda V; \varphi)) \to \cdots \\
\downarrow{\scriptstyle H(\varepsilon_*, \varepsilon_*)} & & \downarrow{\scriptstyle H(\varepsilon_*)} & & \downarrow{\scriptstyle H(\varepsilon_*)} \\
\cdots \to H_{n+1}(\mathrm{Rel}(\widehat{\varphi^*})) & \xrightarrow{H(\hat{P})} & H_n(\mathrm{Der}(\Lambda V, \mathbb{Q}; \varepsilon)) & \xrightarrow{H(\widehat{\varphi^*})} & H_n(\mathrm{Der}(\Lambda W, \mathbb{Q}; \varepsilon)) \to \cdots
\end{array}$$

The following definition is very useful to determine the rational relative Gottlieb groups of a map.

Definition 3.6. *Suppose $\varphi : (\Lambda W, d_W) \to (\Lambda V, d_V)$ is a map of cdga. We define the n-th rational relative Gottlieb group of φ, for $n \geqslant 2$ by:*

$$G_n^{\mathrm{rel}}(\Lambda W, \Lambda V; \varphi) = \mathrm{im}\big\{H(\varepsilon_*, \varepsilon_*) : H_n(\mathrm{Rel}(\varphi^*)) \to H_n(\mathrm{Rel}(\widehat{\varphi^*}))\big\}.$$

Then, the rationalized G-sequence of the map $\varphi : (\Lambda W, d_W) \to (\Lambda V, d_V)$ is given by

$$\cdots \xrightarrow{H(\hat{J})} G^{\mathrm{rel}}_{n+1}(\Lambda W, \Lambda V; \varphi) \xrightarrow{H(\hat{P})} \\ \to G_n(\Lambda V) \xrightarrow{H(\widehat{\varphi^*})} G_n(\Lambda W, \Lambda V; \varphi) \xrightarrow{H(\hat{J})} \cdots \quad (3.1)$$

which ends in $G_2(\Lambda W, \Lambda V; \varphi)$.

We finish this section by an overriding hypothesis. In general, we assume that all spaces appearing in the sequel are *rational simply connected CW-complex and are of finite type.*

4. Relative Gottlieb groups of mapping spaces between F_0-space and products of odd spheres

In this section, let $f : X \to Y$ be a map where X is an F_0-space and Y is products of odd dimensional spheres. We give a partial generalization of a result of G. Lupton and S. B. Smith (cf. [7, Theorem 4.3]). We begin our motivation by the following:

Example 4.1. Consider the map

$$f : \mathbb{S}^2 \times \mathbb{S}^4 \to \mathbb{S}^{2i+1} \times \mathbb{S}^{2j+1}$$

which rationally is given by

$$\varphi : (\Lambda W, 0) = \big(\Lambda(y_{2i+1}, y_{2j+1}), 0\big) \to (\Lambda V, d) = \big(\Lambda(x_2, y_3, x_4, y_7), d\big)$$

where subscripts denote degrees. The differential is given by

$$d(x_2) = d(x_4) = 0, \qquad d(y_3) = x_2^2, \qquad d(y_7) = x_4^2.$$

Hence, from degree reasons and $d \circ \varphi = 0$, we write $\varphi(y_{2i+1}) = \varphi(y_{2j+1}) = 0$. An easy argument shows that

$$G^{\mathrm{rel}}_*(\Lambda W, \Lambda V; \varphi) \cong G_*(\Lambda W, \Lambda V; \varphi) \oplus G_*(\Lambda V).$$

It follows that the G-sequence of f is exact. Though, if $i = 1$ and $j = 3$ we note that the homomorphism induces on rational homotopy groups $f_\sharp$ is non null.

Next, we will use the following result for our computations.

Lemma 4.2. *Let $f : X \to Y \simeq \prod_j \mathbb{S}^{2n_j+1}$ and $\varphi : (\Lambda W, 0) \to (\Lambda V, d)$ its Sullivan minimal model. Then φ is null homotopic.*

Proof. Denote the Sullivan minimal model of $\Pi_j \mathbb{S}^{2n_j+1}$ by

$$(\Lambda W, 0) = (\Lambda(z_1, \ldots, z_m), 0),$$

where $|z_i|$ are odd for $1 \leqslant i \leqslant m$. Further, the Sullivan minimal model of X is of the form

$$(\Lambda V, d) = (\Lambda(x_1, \ldots, x_p, y_1, \ldots, y_p), d),$$

where $|x_k|$ are even and $|y_k|$ are odd for $1 \leqslant k \leqslant p$. The differential is defined as follows, $d(x_k) = 0$ and $d(y_k) \in \Lambda(x_1, \ldots, x_p)$. Hence, for degree reasons, we may assume that the Sullivan minimal model of f

$$\varphi : (\Lambda(z_1, \ldots, z_m), 0) \to (\Lambda(x_1, \ldots, x_p, y_1, \ldots, y_p), d),$$

is given by $\varphi(z_i) = a\alpha\beta + b\gamma$ for coefficients $a, b \in \mathbb{Q}$, $\alpha \in \Lambda(x_1, \ldots, x_p)$, and $\beta, \gamma \in \Lambda(y_1, \ldots, y_p)$. Here, we note that $|a\alpha\beta + b\gamma|$ is odd. Next, we will use the fact $d \circ \varphi(z_i) = 0$. It implies that $a\alpha\beta + b\gamma$ is a d-cycle. However, since X is an F_0-space, it easy to see that $[a\alpha\beta + b\gamma] \neq 0$ in $H^*(\Lambda V, d) = H^{\text{even}}(\Lambda V, d)$. Consequently, we have necessarily $a = b = 0$, and the result follows. □

Theorem 4.3. *Let $f : X \to Y \simeq \Pi_j \mathbb{S}^{2n_j+1}$ and $\varphi : (\Lambda W, 0) \to (\Lambda V, d)$ its Sullivan minimal model. Then*

$$G_*^{\text{rel}}(\Lambda W, \Lambda V; \varphi) \cong W \oplus V^{\text{odd}}.$$

Proof. First, we recall that

$$\varphi : (\Lambda W, 0) = (\Lambda(z_1, \ldots, z_m), 0) \to (\Lambda V, d) = (\Lambda(x_1, \ldots, x_p, y_1, \ldots, y_p), d)$$

is defined by $\varphi(z_i) = 0$ for all $1 \leqslant i \leqslant m$. It induces the map

$$\varphi^* : \text{Der}_*(\Lambda V) \to \text{Der}_*(\Lambda W, \Lambda V; \varphi) \tag{4.1}$$

is also null. Now, let us calculate $G_*^{\text{rel}}(\Lambda W, \Lambda V; \varphi)$ as follows. Since X is finite CW-complex, it is well known that

$$G_*(\Lambda V) \cong \langle y_1^*, \ldots, y_p^* \rangle.$$

Furthermore, since Y is an H-space, it follows that

$$G_*(\Lambda W, \Lambda V; \varphi) \cong \langle z_1^*, \ldots, z_m^* \rangle.$$

Hence, a simple computation shows that

$$D(y_k, 1), 0) = 0 \text{ for } 1 \leqslant k \leqslant p,$$
$$D(0, (z_i, 1)) = 0 \text{ for } 1 \leqslant i \leqslant m.$$

Moreover, it is evident to verify that $((y_k, 1), 0)$ and $(0, (z_i, 1))$ represent non zero homology classes in $H_*(\text{Rel}(\varphi^*))$. Otherwise we consider the vector space,

$$\text{Rel}_*(\widehat{\varphi^*}) = \text{Der}_{*-1}(\Lambda V, \mathbb{Q}; \varepsilon) \oplus \text{Der}_*(\Lambda W, \mathbb{Q}; \varepsilon).$$

Thus, we define

$$\varepsilon_* : \text{Der}_*(\Lambda V) \to \text{Der}_*(\Lambda V, \mathbb{Q}; \varepsilon), \qquad \varepsilon_*(y_k, 1) = \widehat{(y_k, 1)},$$

$$\varepsilon_* : \mathrm{Der}_*(\Lambda W) \to \mathrm{Der}_*(\Lambda W, \mathbb{Q}; \varepsilon), \qquad \varepsilon_*(z_i, 1) = \widehat{(z_i, 1)}.$$

Now, consider the map

$$\widehat{\varphi^*} : \mathrm{Der}_*(\Lambda V, \mathbb{Q}; \varepsilon) \to \mathrm{Der}_*(\Lambda W, \mathbb{Q}; \varepsilon)$$

which is null from (4.1). So, in a similar fashion as above, we get for $1 \leqslant k \leqslant p$ and $1 \leqslant i \leqslant m$ that

$$\widehat{D}(\widehat{(y_k, 1)}, 0) = \widehat{D}(0, \widehat{(z_i, 1)}) = 0.$$

Therefore, it is easy to see that $(\widehat{(y_k, 1)}, 0)$ and $(0, \widehat{(z_i, 1)})$ are not boundaries in $\mathrm{Rel}_*(\widehat{\varphi^*})$. Combining all the above, we obtain that

$$H(\varepsilon_*, \varepsilon_*)\big([((y_k, 1), 0)]\big) = [(\widehat{(y_k, 1)}, 0)] \text{ for } 1 \leqslant k \leqslant p,$$
$$H(\varepsilon_*, \varepsilon_*)([(0, (z_i, 1))]) = [(0, \widehat{(z_i, 1)})] \text{ for } 1 \leqslant i \leqslant m.$$

In summary, we have computed that

$$G_*^{\mathrm{rel}}(\Lambda W, \Lambda V; \varphi) =$$
$$= \big\langle [(\widehat{(y_k, 1)}, 0)] \,\forall\, 1 \leqslant k \leqslant p \big\rangle \oplus \big\langle [(0, \widehat{(z_i, 1)})] \,\forall\, 1 \leqslant i \leqslant m \big\rangle. \quad \square$$

Corollary 4.4. *The G-sequence associated to φ splits into short exact sequence*

$$0 \to G_*(\Lambda W, \Lambda V; \varphi) \to G_*^{\mathrm{rel}}(\Lambda W, \Lambda V; \varphi) \to G_*(\Lambda V) \to 0.$$

Proof. Wee see directly that $G_*(\Lambda V) \cong V^{\mathrm{odd}}$ and $G_*(\Lambda W, \Lambda V; \varphi) \cong W$. Therefore, to finish this proof we use Theorem 4.3 and the rationalized G-sequence (3.1). □

5. Relative Gottlieb groups of mapping spaces between products of odd spheres and F_0-space

Given a simply connected space X which is finite dimensional rational homotopy groups, i.e., $\dim \pi_*(X) < \infty$, write

$$\max \pi_*(X) := \max\big\{i \mid \pi_i(X) \neq 0\big\} = \max\big\{i \mid V^i \neq 0\big\}$$

and also denote by,

$$\min \pi_*(X) := \min\big\{i \mid \pi_i(X) \neq 0\big\} = \min\big\{i \mid V^i \neq 0\big\}.$$

Here $(\Lambda V, d)$ denotes the Sullivan minimal model of X.

Now, let $f : X \to Y$ be a map such that $X \simeq \Pi_j \mathbb{S}^{2n_j+1}$ and Y is an F_0-space. In this section, we use Sullivan minimal models to compute the rational relative Gottlieb groups of f. As a consequence, we show that the G-sequence of the map f splits into a short exact sequence.

Theorem 5.1. *Given a map* $f : \Pi_j \mathbb{S}^{2n_j+1} \to Y$ *and* $\varphi : (\Lambda W, d) \to (\Lambda V, 0)$ *its Sullivan minimal model. If* $W^{\mathrm{odd}} \cong V$ *and* $\max W^{\mathrm{even}} \leqslant \min W^{\mathrm{odd}}$*, then:*

(1) $G_*(\Lambda W, \Lambda V; \varphi) \cong W$*;*
(2) $G_*^{\mathrm{rel}}(\Lambda W, \Lambda V; \varphi) \cong W^{\mathrm{even}}$.

Proof. We give the proof by dividing it into two cases.

(1) First, we denote the Sullivan minimal model of $\Pi_j \mathbb{S}^{2n_j+1}$ by

$$(\Lambda(y_1, \ldots, y_m), 0),$$

where $|y_j|$ are odd for $1 \leqslant j \leqslant m$. Otherwise, since Y is an F_0-space and $W^{\mathrm{odd}} \cong V$, then its Sullivan minimal model is given by

$$(\Lambda W, d) = (\Lambda(x_1, \ldots, x_m, y_1, \ldots, y_m), d)$$

where $|x_i|$ is even. The differential is defined as follows:

$$d(x_i) = 0 \text{ for } 1 \leqslant i \leqslant m,$$
$$d(y_j) \in \Lambda(x_1, \ldots, x_m) \text{ for } 1 \leqslant j \leqslant m.$$

Under the condition $\max W^{\mathrm{even}} \leqslant \min W^{\mathrm{odd}}$, the Sullivan minimal model of f,

$$\varphi : (\Lambda(x_1, \ldots, x_m, y_1, \ldots, y_m), d) \to (\Lambda(y_1, \ldots, y_m), 0)$$

is given on generators by

$$\varphi(x_i) = 0, \qquad \varphi(y_j) = y_j + \alpha,$$

where $\alpha \in \Lambda(y_1, \ldots, y_m)$; α may be null. Now, an easy computation reveals that each of the derivations $(y_j, 1)$ is a cycle and cannot be boundary for $1 \leqslant j \leqslant m$. We deduce that $[(y_j, 1)]$ is non null in $H_*(\mathrm{Der}(\Lambda W, \Lambda V; \varphi))$. Next, consider the derivations $(x_i, 1)$ for $1 \leqslant i \leqslant m$. It is clear that

$$\partial((x_i, 1)(y)) = -(x_i, 1)(d(y)) \text{ for } y \in \Lambda(y_1, \ldots, y_m).$$

So, by the minimality of $(\Lambda W, d)$ and $\varphi(x_i) = 0$, we get $(x_i, 1)(d(y)) = 0$ and further, $\partial(x_i, 1) = 0$. Moreover, $(x_i, 1)$ cannot be boundaries for degree reasons. Furthermore, due to the fact that

$$\varepsilon_*(x_i, 1) = x_i^* \text{ for } 1 \leqslant i \leqslant m,$$
$$\varepsilon_*(y_j, 1) = y_j^* \text{ for } 1 \leqslant j \leqslant m,$$

we deduce that

$$G_*(\Lambda W, \Lambda V; \varphi) \cong \mathrm{Hom}(W, \mathbb{Q}).$$

(2) Let us calculate $G_*^{\mathrm{rel}}(\Lambda W, \Lambda V; \varphi)$. First, we recall that

$$\varphi^* : \mathrm{Der}_*(\Lambda V) \to \mathrm{Der}_*(\Lambda W, \Lambda V; \varphi)$$

is defined by $\varphi^*(y_j, 1) = (y_j, 1) + \theta_j$, where $\theta_j \in \mathrm{Der}_{|y_j|}(\Lambda W, \Lambda V; \varphi)$. Now, it is easy to check that

$$D(0, (x_i, 1)) = 0 \text{ for } 1 \leqslant i \leqslant m.$$

On the other hand, since $(x_i, 1)$ is not in $\mathrm{im}(\varphi^*)$, we deduce that $(0, (x_i, 1))$ cannot be boundaries in $\mathrm{Rel}_*(\varphi^*)$. Hence, we obtain

$$[(0, (x_i, 1))] \text{ is non null in } H_*(\mathrm{Rel}(\varphi^*)) \text{ for } 1 \leqslant i \leqslant m.$$

Now, for $1 \leqslant i \leqslant m$ let $\widehat{(x_i, 1)} = \varepsilon_*(x_i, 1)$ and $\widehat{(y_j, 1)} = \varepsilon_*(y_j, 1)$ for $1 \leqslant j \leqslant m$. Define the map

$$\widehat{\varphi^*} : \mathrm{Der}_*(\Lambda V, \mathbb{Q}; \varepsilon) \to \mathrm{Der}_*(\Lambda W, \mathbb{Q}; \varepsilon)$$

by setting $\widehat{\varphi^*}\widehat{(y_j, 1)} = \widehat{(y_j, 1)}$. Further, a direct computation shows

$$\widehat{D}(\widehat{(y_j, 1)}, 0) = (0, \widehat{(y_j, 1)}) \text{ and } \widehat{D}(0, \widehat{(x_i, 1)}) = 0.$$

However, a similar argument as above shows that

$$[(0, \widehat{(x_i, 1)})] \text{ is non null in } H_*(\mathrm{Rel}(\widehat{\varphi^*})) \text{ for } 1 \leqslant i \leqslant m.$$

Thus summarizing the analysis above, the image of

$$H(\varepsilon_*, \varepsilon_*) : H_*(\mathrm{Rel}(\varphi^*)) \to H_*(\mathrm{Rel}(\widehat{\varphi^*}))$$

is spanned by $[(0, \widehat{(x_i, 1)})]$ for $1 \leqslant i \leqslant m$. As a consequence, we get

$$G_*^{\mathrm{rel}}(\Lambda W, \Lambda V; \varphi) \cong \big\langle [(0, \widehat{(x_i, 1)})] \text{ for } 1 \leqslant i \leqslant m \big\rangle.$$

This completes the proof. □

We finish this section with the following consequence.

Corollary 5.2. *The G-sequence associated to φ splits into short exact sequence*

$$0 \to G_*(\Lambda V) \to G_*(\Lambda W, \Lambda V; \varphi) \to G_*^{\mathrm{rel}}(\Lambda W, \Lambda V; \varphi) \to 0.$$

Proof. It follows immediately from Theorem 5.1 and the rationalized G-sequence (3.1). □

6. Rational cohomology of mapping spaces

Let $f : X \to Y$ be a map between simply connected CW-complexes. Recall that

$$\mathrm{map}(X, Y; f)$$

is the path components of the space of maps $X \to Y$ consisting of those maps that are homotopic to f. The first description of a Sullivan model, in particular of mapping spaces is due to A. Haefliger [6]. In [7], G. Lupton and S. B. Smith gave an elegant formula for the rational homotopy type of $\operatorname{map}(X, Y; f)$ in terms of Sullivan minimal models of X and Y. In particular, they proved that when X is finite

$$\pi_i(\operatorname{map}(X, Y; f)) \cong H_i(\operatorname{Der}(\Lambda W, \Lambda V; \varphi)) \text{ for } i \geqslant 2. \tag{6.1}$$

Here $\varphi : (\Lambda W, d_W) \to (\Lambda V, d_V)$ denotes the Sullivan minimal model of $f : X \to Y$. Recently, by using the theory of Sullivan minimal models, P. A. Otieno, J. B. Gatsinzi and V. O. Otieno calculated the rational cohomology of the component of the inclusion

$$i_{n,r} : G(k, n) \hookrightarrow G(k, n + r),$$

where $G(k, n)$ is the complex Grassmannians, [10]. However, there is no explicit and complete description of the rational cohomology of $\operatorname{map}(X, Y; f)$. We begin by the following:

Example 6.1. For $m > n \geqslant 1$, let

$$f : X \simeq K(\mathbb{Q}, 2n + 1) \to Y \simeq K(\mathbb{Q}, 2m)$$

which is rationally defined by

$$\varphi : (\Lambda W, 0) = (\Lambda(x), 0) \to (\Lambda V, 0) = (\Lambda(y), 0),$$

where $|x| = 2m$ and $|y| = 2n + 1$. Hence, $\operatorname{Der}_*(\Lambda W, \Lambda V; \varphi)$ is spanned by the derivations $(x, 1)$ and (x, y) of degree $2m$ and $2m - 2n - 1$ respectively. An easy computation shows that

$$D(x, 1) = D(x, y) = 0.$$

From (6.1), it follows that

$$\pi_i(\operatorname{map}(X, Y; f)) \cong \mathbb{Q} \text{ for } i = 2m, 2m - 2n - 1 \text{ and zero otherwise.}$$

Then for degree reasons, we deduce that

$$\operatorname{map}(X, Y; f) \cong K(\mathbb{Q}, 2m - 2n - 1) \times K(\mathbb{Q}, 2m).$$

Up to this point, our observation is well-known and also easily obtained by a number of standard methods. We can hence generalize this example as follows:

Theorem 6.2. *Let $f : X \to Y$ be a map with X is finite and* $\max \pi_*(Y)$ *is even. Then* $\operatorname{map}(X, Y; f)$ *has infinite dimensional rational cohomology.*

Proof. Without loss of generality, let

$$\varphi : (\Lambda W, d_W) \to (\Lambda V, d_V)$$

be the Sullivan minimal model of f. Our assumption $\max \pi_*(Y)$ is even means that the top degree of W is even. Further, define a derivation $(x, 1)$

in $\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi)$ with $|x| = \max W$. So, it is evident to see that

$$D(x, 1) = 0.$$

Now, since $\mathrm{Der}_*(\Lambda W, \Lambda V; \varphi)$ is zero in degree $|x|+1$ and higher, the derivation $(x, 1)$ cannot be boundary. Consequently, we obtain

$$[(x,1)] \text{ is non null in } H_{|x|}\big(\mathrm{Der}(\Lambda W, \Lambda V; \varphi)\big) \cong \pi_{|x|}(\mathrm{map}(X, Y; f)).$$

Further, denote by τ the class $[(x, 1)]$. So, $\mathrm{map}(X, Y; f)$ has a Sullivan model of the form

$$M(X, Y; f) \cong \big(\Lambda(U \oplus \mathbb{Q}(\tau)), d\big)$$

such that U is finite dimensional. Finally, we consider the pure model $(\Lambda(U \oplus \mathbb{Q}(\tau)), d_\sigma)$ associated to $(\Lambda(U \oplus \mathbb{Q}(\tau)), d)$ (cf. [1, 32(b)]), we obtain for $n \geqslant 1$,

$$[\tau^n] \neq 0 \text{ in } H^*(\Lambda(U \oplus \mathbb{Q}(\tau)), d_\sigma).$$

As a consequence, by [1, Proposition 32.4], we get

$$\dim H^*(\Lambda(U \oplus \mathbb{Q}(\tau)), d) = \infty. \qquad \square$$

We next recall an interesting invariant of a simply connected space X. The LS category of X, $\mathrm{cat}(X)$, is the least integer n such that X can be covered by $(n+1)$ open subsets contractible in X and is ∞ if no such n exists. The rational category of X, $\mathrm{cat}_0(X)$, is defined by $\mathrm{cat}_0(X) = \mathrm{cat}(X_{\mathbb{Q}})$. For example $\mathrm{cat}_0(\mathbb{S}^n) = 1$ and $\mathrm{cat}_0(\mathbb{C}P^n) = n$ for $n \geqslant 2$.

A much stronger consequence of Theorem 6.2 and [1, Proposition 32.4] is the following:

Corollary 6.3. *Let $f : X \to Y$ be a map with X is finite and $\max \pi_*(Y)$ is even. Then $\mathrm{cat}_0(\mathrm{map}(X, Y; f)) = \infty$.*

7. Open Problem

In rational homotopy theorey, the fundamental paper of G. Lupton and S. B. Smith [7] has played a central role in the development of results concerning rational Gottlieb groups, rational generalized Gottlieb groups and rational relative Gottlieb groups. The authors were the first ones who were interested in computation of these groups for certain cases.

Now, we turn to the case of homotopy theory. As we mentioned earlier, the Gottlieb groups play a well-known role in the homotopy theory of fibrations with fiber a CW-complex of finite type. Some general results about $G_*(X)$ are known but explicit computation is limited to sporadic examples. In fact, in [3] M. Golasiński and J. Mukai have determined $G_{n+k}(\mathbb{S}^n)$ for $k \leqslant 13$ except for the two-primary components in the cases: $k = 9$, $n = 53$

and $k = 11$, $n = 115$. Their result significantly extends result of Gottlieb from [5]. However, we have very little information in literature about the whole relative Gottlieb groups. In fact, M. H. Woo have showed that $G_n^{\mathrm{rel}}(\mathbb{S}^n, \mathbb{S}^{n-1}; i) \cong \mathbb{Z}$ if n is odd and $G_n^{\mathrm{rel}}(\mathbb{S}^n, \mathbb{S}^{n-1}; i) \cong \mathbb{Z} \oplus \mathbb{Z}$ if $n = 2, 4, 8$. Further, he proved that $G_k^{\mathrm{rel}}(\mathbb{S}^n, \mathbb{S}^{n-1}; i) = 0$ for $1 < k < n$, [8]. Though several authors [8, 12, 13] have studied the exactness of the G-sequence.

It would be most interesting to find the relative Gottlieb groups for the Hopf fibration: $\eta : \mathbb{S}^3 \to \mathbb{S}^2$, $\nu : \mathbb{S}^7 \to \mathbb{S}^4$, $\sigma : \mathbb{S}^{15} \to \mathbb{S}^8$, and their suspensions $\eta_n : \mathbb{S}^{n+1} \to \mathbb{S}^n$, $\nu_n : \mathbb{S}^{n+3} \to \mathbb{S}^n$ and $\sigma_n : \mathbb{S}^{n+7} \to \mathbb{S}^n$. As specific instances of this kind of problem, we offer the following which has been suggested by the referee.

Problem 7.1. Let $f : X \to Y$ be a map of CW-complexes of finite type. Determine the whole relative Gottlieb groups $G_*^{\mathrm{rel}}(Y, X; f)$.

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