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Centralizers of elements in Lie algebras of vector fields with polynomial coefficients

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Abstract. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $A = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring, and $R = \mathbb{K}(x_1, \ldots, x_n)$ the field of rational functions in n variables. Denote by $W_n = W_n(\mathbb{K})$ the Lie algebra of all $\mathbb{K}$-derivations on A (in case $\mathbb{C}$ it is the Lie algebra of all vector fields on $\mathbb{C}^n$ with polynomial coefficients). For a given $D \in W_n(\mathbb{K})$ the structure of the centralizer $C_{W_n(\mathbb{K})}(D)$ depends on the field of constants

$$\ker D = \{\phi \in R \mid D(\phi) = 0\}$$

(here we extend naturally every derivation D of A on the field R). The case $tr.\deg_{\mathbb{K}} \ker D \leqslant 1$ is studied, the structure of the subalgera $C_{W_n(\mathbb{K})}(D)$ is characterized, in particular it is proved that if $\ker D$ does not contain any non-constant polynomial, then $C_{W_n(\mathbb{K})}(D)$ is finite-dimensional over $\mathbb{K}$. Some results about centralizers of linear derivations in $W_n(\mathbb{K})$ are obtained.

Анотація. Нехай $\mathbb{K}$ – алгебраїчно замкнене поле характеристики нуль, $A = \mathbb{K}[x_1, \ldots, x_n]$ – кільце многочленів і $R = \mathbb{K}(x_1, \ldots, x_n)$ – поле раціональних функцій від n змінних. Позначимо через $W_n = W_n(\mathbb{K})$ алгебру Лі всіх $\mathbb{K}$-диференціювань на A (у випадку $\mathbb{C}$ це алгебра Лі всіх векторних полів на $\mathbb{C}^n$ з поліноміальними коефіцієнтами). Для заданого $D \in W_n(\mathbb{K})$ будова централізатора $C_{W_n(\mathbb{K})}(D)$ залежить від поля констант $\ker D = \{\phi \in R \mid D(\phi) = 0\}$ (тут ми природнім чином розширюємо кожне диференціювання D на A на поле R). Досліджено випадок, коли $tr.\deg_{\mathbb{K}} \ker D \leqslant 1$, охарактеризована будова підалгебри $C_{W_n(\mathbb{K})}(D)$. Зокрема доведено, що якщо $\ker D$ не містить несталих многочленів, то

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Keywords: Lie algebra, derivation, vector field, polynomial ring, centralizer

Ключові слова: Алгебра Лі, диференціювання, векторне поле, кільце многочленів, централізатор

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$C_{W_n(\mathbb{K})}(D)$ є скінченновимірним над $\mathbb{K}$. Також отримано деякі результати про централізатори лінійних диференціювань в $W_n(\mathbb{K})$.

Аннотация. Пусть $\mathbb{K}$ – алгебраически замкнутое поле характеристики нуль, $A = \mathbb{K}[x_1, \ldots, x_n]$ кольцо многочленов и $R = \mathbb{K}(x_1, \ldots, x_n)$ – поле рациональных функций от n переменных. Обозначим через $W_n = W_n(\mathbb{K})$ алгебру Ли всех $\mathbb{K}$-дифференцирований на A (в случае $\mathbb{C}$ это алгебра Ли всех векторниых полей на $\mathbb{C}^n$ с полиномиальными коэффициентами). Для данного $D \in W_n(\mathbb{K})$ строение централизатора $C_{W_n(\mathbb{K})}(D)$ зависит от поля констант $\ker D = \{\phi \in R \mid D(\phi) = 0\}$ (здесь мы естественным образом продолжаем каждое дифференцирование D на A на поле R). Изучен случай, когда $tr.\deg_{\mathbb{K}} \ker D \leqslant 1$, охарактеризована подалгебра $C_{W_n(\mathbb{K})}(D)$, в частности доказано, что если $\ker D$ не содержит непостоянных многочленов, то $C_{W_n(\mathbb{K})}(D)$ бесконечномерен над $\mathbb{K}$. Получены некоторые результаты о централизаторах линейных дифференцирований в $W_n(\mathbb{K})$.

1. INTRODUCTION

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero (without loss of generality one can assume that $\mathbb{K} = \mathbb{C}$, the field of complex numbers). Denote by $A = \mathbb{K}[x_1, \ldots, x_n]$ the polynonial ring and by $R = \mathbb{K}(x_1, \ldots, x_n)$ the field of rational functions in n variables.

Recall that a $\mathbb{K}$-linear map $D : A \to A$ is a $\mathbb{K}$-*derivation* (or simply a *derivation*) whenever

$$D(fg) = D(f)g + fD(g)$$

for all $f, g \in A$. In case $\mathbb{K} = \mathbb{C}$ every $\mathbb{C}$-derivation can be considered as a vector field on on $\mathbb{C}^n$ with polynomial coefficients. We will use this standard correspondence between (polynomial) vector fields and derivations on (polynomial) rings. Any derivation D on $A = \mathbb{K}[x_1, \ldots, x_n]$ can be uniquely extended to the derivation D on $R = \mathbb{K}(x_1, \ldots, x_n)$ (we use the same notation here) by the rule

$$D(f/g) = (D(f)g - fD(g)/g^2$$

for all $f, g \in A$, $g \neq 0$.

The Lie algebra $W_n(\mathbb{K})$ of all $\mathbb{K}$-derivations on A is of great interest because its finite dimensional subalgebras are closely connected with symmetries of differential equations (recall that any derivation D on A is of the form

$$D = f_1(x_1, \ldots, x_n)\frac{\partial}{\partial x_1} + \ldots + f_n(x_1, \ldots, x_n)\frac{\partial}{\partial x_n}$$

for some $f_i \in \mathbb{K}[x_1, \ldots, x_n]$, where $\frac{\partial}{\partial x_i}$ are partial derivatives on A).

Finite dimensional subalgebras of the Lie algebras $W_1(\mathbb{C})$ and $W_2(\mathbb{C})$ were classified by S. Lie [4] (more precisely Lie algebras of vector fields with analytical coefficients were described in [4]). Analogous problem for $W_3(\mathbb{C})$ is open, the problem of classifying all finite-dimensional Lie subalgebras of vector fields from $W_n(\mathbb{C})$, $n \geqslant 4$ is wild [1].

If $D \in W_n(\mathbb{K})$, then the centralizer $C_{W_n(\mathbb{K})}(D)$ is a subalgebra of $W_n(\mathbb{K})$ consisting of all vector fields commuting with D. An information about $C_{W_n(\mathbb{K})}(D)$ can be useful in many cases. For example, every vector field $D \in W_n(\mathbb{C})$, $D = \sum\limits_{i=1}^{n} f_i(x_1, \dots, x_n)\frac{\partial}{\partial x_i}$ defines an autonomous system of ODE:

$$\begin{cases} \frac{dx_1}{dt} = f_1(x_1, \dots, x_n) \\ \qquad \cdots \\ \frac{dx_n}{dt} = f_n(x_1, \dots, x_n) \end{cases} \tag{1.1}$$

with polynomial coefficients and information about $\ker D$ and $C_{W_n(\mathbb{K})}(D)$ can be very useful for searching solutions of (1.1) see, for example [5]. Given k commuting linearly independent over R vector fields on a smooth n-manifold M, one can construct a local coordinate system on M in which these vector fields are of the form $\frac{\partial}{\partial x_i}$, $i = 1, \dots, k$ (see, *e.g.* [3, Th. 9.46]). We study centralizers of elements $D \in W_n(\mathbb{K})$ in case when $\ker D$ (in the field $R = \mathbb{K}(x_1, \dots, x_n)$) is of transcendence degree $\leqslant 1$ over $\mathbb{K}$, *i.e.* any two rational functions f, g annihilated by D are algebraically dependent over $\mathbb{K}$.

In case $tr.\deg_{\mathbb{K}} \ker D = 0$ we have $\ker D = \mathbb{K}$ and then $C_{W_n}(D)$ is a vector space of dimension $\leqslant n$ over $\mathbb{K}$.

If $tr.\deg_{\mathbb{K}} \ker D = 1$, then by Gordan's theorem (see, *e.g.* [8]) either $\ker D = \mathbb{K}(p)$ or $\ker D = \mathbb{K}(\frac{p}{q})$, where p, q are irreducible polynomials that are algebraically independent over $\mathbb{K}$.

If $\ker D = \mathbb{K}(p)$, then the centralizer C is a module over the ring $\mathbb{K}[p]$ of rank k, $1 \leqslant k \leqslant n$ and C is either a Lie algebra over $\mathbb{K}[p]$ or it contains an ideal I of rank $k-1$ which is a Lie algebra over $\mathbb{K}[p]$ and $C = I + \mathbb{K}[p]T$ for some derivation $T \in C$ (Theorem 3.1).

In case $\ker D = \mathbb{K}(p/q)$ we have that

$$C = (\mathbb{K}(p/q)D + \ldots + \mathbb{K}(p/q)D_{k-1}) \cap W_n(\mathbb{K})$$

and C is finite-dimensional over $\mathbb{K}$ (Theorem 3.3).

We use standard notation. Every derivation $D \in W_n(\mathbb{K})$ can be uniquely written in the form

$$D = f_1(x_1, \dots, x_n)\frac{\partial}{\partial x_1} + \ldots + f_n(x_1, \dots, x_n)\frac{\partial}{\partial x_n}$$

for some $f_i \in A$. One can show that every nonzero derivation D can be written in the form $D = hD_0$, where D_0 is reduced, *i.e.* if $D_0 = h_1 D_1$ for some $D_1 \in W_n(\mathbb{K})$ and $h_1 \in A$ then $h_1 \in \mathbb{K}^*$. Denote by $\widetilde{W}_n(\mathbb{K})$ the Lie algebra of all $\mathbb{K}$-derivations of the field $R = \mathbb{K}(x_1, \ldots, x_n)$. It is obvious that $\widetilde{W}_n(\mathbb{K})$ is a vector space of dimension n over R (with the standard basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$) but not a Lie algebra over R.

A rational function $\varphi \in R = \mathbb{K}(x_1, \ldots, x_n)$ is called *closed* if the subfield $\mathbb{K}(\varphi)$ is algebraically closed in in the field R.

2. PRELIMINARY RESULTS ABOUT CENTRALIZERS

Lemma 2.1. *Let $D \in W_n(\mathbb{K})\backslash\{0\}$, F be the field of constants of D in R, and $C = C_{\widetilde{W}_n(\mathbb{K})}(D)$. Then either*

- *$C = C_{\widetilde{W}_n(\mathbb{K})}(D) = FD$, or*
- *$C = FD + FD_2 + \cdots + FD_k$ for some $D_2, \ldots, D_k \in C$ with $D, D_2, \ldots, D_k$ linearly independent over R.*

Proof. Note that C is a subalgebra of the Lie algebra $\widetilde{W}_n(\mathbb{K})$ over the field $\mathbb{K}$ and $D \in C$. Choose a basis $D, D_1, \ldots, D_k$ (this includes the case $k = 0$) for the vector space RC over the field R. Every $T \in C$ (note that $C \subseteq RC$) can be written in the form $T = rD + r_2 D_2 + \ldots + r_k D_k$ for some $r, r_i \in R$. But then the equality $[D, T] = 0$ implies $D(r_i) = 0, i = 1, \ldots, k$, *i.e.* $r_i \in \ker D = F$.

On the contrary, one can note that any element from $FD + \ldots + FD_k$ belongs to C. Therefore $C = FD + \ldots + FD_k$. If $F \subseteq \ker D_i$ for all $i \geqslant 2$, then C is not only k-dimensional vector space over F but also a Lie algebra over F. □

Corollary 2.2. *Under assumption of Lemma 2.1, if $F = \mathbb{K}$ then C is a k-dimensional Lie algebra over the field $\mathbb{K}$.*

Example 2.3. Let $D \in W_n(\mathbb{K})$ be a linear derivation,

$$D = f_1(x_1, \ldots, x_n)\tfrac{\partial}{\partial x_1} + \ldots + f_n(x_1, \ldots, x_n)\tfrac{\partial}{\partial x_n},$$

$f_i(x_1, \ldots, x_n) = D(x_i) = \sum_{j=1}^{n} a_{ij} x_j$. Assume also that the Jordan normal form of the matrix (a_{ij}) is diagonal

$$\begin{bmatrix} \lambda_1 & \ldots & 0 \\ & \ldots & \\ 0 & \ldots & \lambda_n \end{bmatrix},$$

where λ_i are linearly independent over $\mathbb{Z}$ eigenvalues of the matrix (a_{ij}). Then $C = C_{W_n(\mathbb{K})}(D)$ is of rank n over R and has dimension n over $\mathbb{K}$. Indeed, $\ker D = \mathbb{K}$ by [6, Theorem 10.1.2]. Let

$$L = \Big\{ \sum_{j=1}^{n} \mu_j x_j \tfrac{\partial}{\partial x_j} \mid \mu_j \in \mathbb{K} \Big\}.$$

One can easily see that $L \subseteq C$ and $\mathrm{rk}_R L = n$. Therefore $\mathrm{rk}_R C = n$ and $\dim_{\mathbb{K}} C = n$ by Lemma 2.1.

Lemma 2.4. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and L an algebraically closed subfield of the field $R = \mathbb{K}(x_1, \dots, x_n)$ with* $\mathrm{tr.deg}_{\mathbb{K}} L = 1$. *If L contains a non-constant polynomial from*

$$A = \mathbb{K}[x_1, \dots, x_n] \subset R,$$

then $L = \mathbb{K}(p)$ for some irreducible polynomial $p \in A$. If $L \cap A = \mathbb{K}$, then $L = \mathbb{K}(p/q)$ for some irreducible polynomials $p, q \in A$ which are algebraically independent over $\mathbb{K}$.

Proof. By Gordan's theorem (*e.g.* [8, Theorem 3]) we have that $L = \mathbb{K}(\varphi)$ for some rational function $\varphi \in R$.

First let $L \cap A = \mathbb{K}$. The by [7, Corollary 1], we have that $L = \mathbb{K}(\frac{p}{q})$ for some irreducible polynomials p, q which are algebraically independent over $\mathbb{K}$. Now let $L \cap A \neq \mathbb{K}$ and $r \in (L \cap A) \backslash \mathbb{K}$. Then $r = F(\frac{p}{q})$ or $r = F(p)$ for some rational function $F(t) \in \mathbb{K}(t)$:

$$F(t) = \frac{a_0 x^m + a_1 x^{m-1} + \dots + a_m}{b_0 x^n + b_1 x^{n-1} + \dots + b_n},$$

with $a_i, b_j \in \mathbb{K}$, $a_0 b_0 \neq 0$. If $r = F(p/q)$, then

$$F(p/q) = \frac{a_0 p^m + \ldots + a_m q^m}{b_0 p^n + \ldots + b_n q^n} q^{n-m}$$

and the numerator and denominator here are homogeneous polynomials in p and q of degree $\max\{m, n\}$. For simplicity assume that $n \geqslant m$ for simplicity. Then

$$r = F(p/q) = \frac{(\alpha_1 p + \beta_1 q) \ldots (\alpha_n p + \beta_n q)}{(\gamma_1 p + \delta_1 q) \ldots (\gamma_n p + \delta_n q)} \tag{2.1}$$

for some $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{K}$ since the ground field $\mathbb{K}$ is algebraically closed.

Note that the polynomials $\alpha_i p + \beta_i q$ and $\gamma_j p + \delta_j q$ are either coprime (when $\begin{vmatrix} \alpha_i & \beta_i \\ \gamma_j & \delta_j \end{vmatrix} \neq 0$) or proportional with a multiplier in $\mathbb{K}^\star$ when $\begin{vmatrix} \alpha_i & \beta_i \\ \gamma_j & \delta_j \end{vmatrix} = 0$.

Since the rational function $F(t)$ can be choosen irreducible, the equality (2.1) is impossible because its numerator and denominator are coprime and r is a non-constant polynomial. Thus the case $L = \mathbb{K}(p/q)$ is impossible and $L = \mathbb{K}(p)$ for an irreducible polynomial $p(x_1, \dots, x_n)$. □

Proposition 2.5. *Let $D_1 \in \widetilde{W}_n(\mathbb{K})$ be such a derivation of the field R that $F = \ker D_1$ in R is of transcendence degree 1 over $\mathbb{K}$. Then the centralizer*

$C = C_{\widetilde{W}_n(\mathbb{K})}(D_1)$ is a subalgebra of $\widetilde{W}_n(\mathbb{K})$ of $\mathrm{rk}_R C = k$*, $1 \leqslant k \leqslant n$, and*

$$C = FD_1 + FD_2 + \cdots + FD_k$$

for some $D_2, \dots, D_k \in C$. Moreover, either C is a Lie algebra over F of dimension k, or C contains an ideal of corank one over R which is a Lie algebra over F of dimension $k-1$.

Proof. By Gordan's theorem (*e.g.* [8, Theorem 3]) we have that $F = \mathbb{K}(\varphi)$ for some closed rational function $\varphi \in R$. Choose a basis $D_1, D_2, \dots, D_k$ of C over R. As $[D_1, D_i] = 0$, $i = 1, \dots, k$, we have $D_i(\ker D_1) \subseteq \ker D_1$. So $D_i(\varphi) = f_i(\varphi)$ for some rational functions $f_i(t), i = 1, \dots, k$.

If $f_1(t) = \cdots = f_n(t) = 0$, then $F \subseteq \ker D_i$ for $i = 1, \dots, k-1$. Therefore, $C = FD_1 + \ldots + FD_k$ is a k-dimensional Lie algebra over the field F.

Now suppose that $f_i(t) \neq 0$ for some $i, 2 \leqslant i \leqslant n$. Then one can easily prove that $f_i(\varphi) \neq 0$. Denote by $C_0 = \{T \in C \mid T(\varphi) = 0\}$ the annihilator of the element φ in C. Since $D_i(\ker D) \subseteq \ker D$, we see that C_0 is an ideal of C. We claim that $\mathrm{rk}_R C_0 = k-1$. Indeed, if $T, S \in C \backslash C_0$ then $T(\varphi) = g(\varphi)$ and $S(\varphi) = h(\varphi)$ for some nonzero rational functions $g(t)$ and $h(t)$. It now follows that $h(\varphi)T - g(\varphi)S \in C_0$ and therefore $\mathrm{rk}_R C/C_0 = 1$. Thus we have $\mathrm{rk}_R C_0 = k-1$. □

Next, we point out a series $D_2, \dots, D_n$ of derivations on the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ with centralizers $C_i = C_{W_n}(D_i)$ such that

$$\mathrm{rank}_R C_i = n - i + 1, \quad i = 2, \dots, n.$$

We use the known simple derivation from [6, Example 13.4.3].

Lemma 2.6. *Let $D_k = \frac{\partial}{\partial x_1} + (1 + x_1x_2)\frac{\partial}{\partial x_2} + \cdots (1 + x_{k-1}x_k)\frac{\partial}{\partial x_k}$ be a derivation of the polynomial ring $\mathbb{K}[x_1, \dots, x_n], 2 \leqslant k \leqslant n$. Then*

(1) $\ker D_k = \mathbb{K}[x_{k+1}, \dots, x_n]$ *for $k < n$ and* $\ker D_n = \mathbb{K}$;

(2) $$C_k = C_{W_n(\mathbb{K})}(D_k) = \mathbb{K}[x_{k+1}, \dots, x_n]D_k + \mathbb{K}[x_{k+1}, \dots, x_n]\frac{\partial}{\partial x_{k+1}} + \cdots + \mathbb{K}[x_{k+1}, \dots, x_n]\frac{\partial}{\partial x_n},$$

for $k < n$ and $C_n = \mathbb{K}D_n$.

In particular, $\mathrm{rk}_R(C_k) = n - k + 1$.

Proof. (1) The polynomial ring $A = \mathbb{K}[x_1, \dots, x_n]$ can be considered as the polynomial ring in variables $x_1, \dots, x_k$ over the ring $F := \mathbb{K}[x_{k+1}, \dots, x_n]$. By [6, Example 13.4.3] D_k is a simple derivation of the ring $F[x_1, \dots, x_k]$ (note that $F \subseteq \ker D_k$). Hence the kernel of D_k in $F[x_1, \dots, x_k]$ coincides with F. Therefore the kernel of the derivation D_k in the ring A coincides with $\mathbb{K}[x_{k+1}, \dots, x_n]$.

(2) Let $T \in C = C_{W_n(\mathbb{K})}$,

$$T = f_1 \tfrac{\partial}{\partial x_1} + \cdots + f_n \tfrac{\partial}{\partial x_n}.$$

Then the equality $[T, D_k] = 0$ implies equalities $D_k(f_1) = T(1) = 0$ and therefore $f_1 \in \mathbb{K}[x_{k+1}, \ldots, x_n]$,

$$\begin{aligned} &D_k(f_2) = x_1 f_2 + x_2 f_1, && \ldots, && D_k(f_k) = x_{k-1} f_k + x_k f_{k-1}, \\ &D_k(f_{k+1}) = 0, && \ldots, && D_k(f_n) = 0. \end{aligned}$$

The last $n - k$ equalities imply that

$$f_{k+1} \in \mathbb{K}[x_{k+1}, \ldots, x_n],\ \ldots,\ f_n \in \mathbb{K}[x_{k+1}, \ldots, x_n].$$

If $f_1 \neq 0$, then $f_1 D_k \in C_k$ and $T - f_1 D_k \in C_k$. Therefore without loss of generality one can assume that $f_1 = 0$. But then $D_k(f_2) = x_1 f_2$ which is possible only if $f_2 = 0$ because D_k is a simple derivation of the ring $F[x_1, \ldots, x_k]$. Repeating the arguments one can conclude that

$$f_3 = \cdots = f_k = 0.$$

The latter means that

$$T - f_1 D_k \in \mathbb{K}[x_{k+1}, \ldots, x_n] \tfrac{\partial}{\partial x_{k+1}} + \cdots + \mathbb{K}[x_{k+1}, \ldots, x_n] \tfrac{\partial}{\partial x_n}.$$

Taking into account the relation $f_1 \in \mathbb{K}[x_{k+1}, \ldots, x_n]$ we get the needed statement. □

In order to separate factors of a polynomial which belong to the kernel of a derivation we consider the following notions. Let $p \in \mathbb{K}[x_1, \ldots, x_n]$ be an irreducible polynomial. A polynomial $f = f(x_1, \ldots, x_n)$ will be called *p-free* if f is not divisible by any polynomial in p of positive degree. It can be easily shown that every polynomial $g \in \mathbb{K}[x_1, \ldots, x_n]$ can be written in the form $g = g_0 g_1$, where g_0 is a p-free polynomial and $g_1 = g_1(p)$ is a polynomial of p (this includes the case $g_1 = \text{const}$). The degree in p of the polynomial $g_1(p)$ will be called the *p-degree* of g and denoted by $\deg_p g$.

Let p and q be algebraically independent irreducible polynomials of the ring $\mathbb{K}[x_1, \ldots, x_n]$. A polynomial $f(x_1, \ldots, x_n] \in \mathbb{K}[x_1, \ldots, x_n)$ will be called *p-q-free* if f is not divisible by any homogeneous polynomial in p and q of positive degree. As earlier one can write every polynomial $g \in \mathbb{K}[x_1, \ldots, x_n]$ in the form $g_0 g_1$, where g_0 is a p-q-free polynomial and g_1 is a homogeneous polynomial in p, q. The (total) degree of g_1 in p, q will be called the *p-q-degree* of g and denoted by $\deg_{p-q} g$.

If D is a derivation on the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$, then D can be written in the form hD_0, where D_0 is an irreducible derivation on $\mathbb{K}[x_1, \ldots, x_n]$ and $h \in \mathbb{K}[x_1, \ldots, x_n]$. We will call D *p-free* if the polynomial h is p-free. We summarize all these remarks in the next statement

Lemma 2.7. *Let $D \in W_n(\mathbb{K})$ be a nonzero derivation. Then there exist unique (up to a factor from $\mathbb{K}^\star$) polynomials $f(p,q)$ and h such that*

$$D = f(p,q)hD_0,$$

where D_0 is a reduced derivation, $f(p,q)$ is a homogeneous polynomial in p, q and the polynomial h is p-q-free.

3. CENTRALIZERS OF ELEMENTS IN $W_n(\mathbb{K})$

Theorem 3.1. *Let D be a derivation of the ring $\mathbb{K}[x_, \dots, x_n]$ with the field of constants $F = \ker D$ in $R = \mathbb{K}(x_1, \dots, x_n)$ of the form $F = \mathbb{K}(p)$ for some irreducible polynomial p and let $C = C_{W_n(\mathbb{K})}(D)$. Then*

1. *If $\mathrm{rk}_R C = 1$, then $C = \mathbb{K}[p]D_0$ for some p-free derivation D_0 with $D = f(p)D_0$ for some $f(t) \in \mathbb{K}[t]$;*
2. *If $\mathrm{rk}_R C \geqslant 2$, then C is either a Lie algebra of rank k over the ring $\mathbb{K}[p]$ or C contains an ideal I of rank $k-1$ that is a Lie algebra over $\mathbb{K}[p]$ and $C = I + \mathbb{K}[p]S$ for some derivation $S \in C$.*

Proof. As noted above the derivation D can be written in the form

$$D = f(p)D_0,$$

where D_0 is a p-free derivation and the polynomial $f \in \mathbb{K}[t]$ is uniquely defined by D up to a nonzero multiplier in $\mathbb{K}^*$.

(1) First let $\mathrm{rk}_R C = 1$. Take an arbitrary element $T \in C$. Then $T = \varphi(p)D_0$ for some rational function $\varphi \in \mathbb{K}(t)$, where $\varphi(p) = g(p)/h(p)$ for some polyomials $g(t)$, $h(t) \in \mathbb{K}[t]$. Without loss of generality one can assume that $\varphi(t) = g(t)/h(t)$ is a reduced fraction. It follows from the equality $T = \varphi(p)D_0$ that $h(p)T = g(p)D_0$. Write D_0 and T in the form

$$D_0 = \sum_{i=1}^{n} P_i(x_1, \dots, x_n)\tfrac{\partial}{\partial x_i}, \qquad T = \sum_{j=1}^{n} Q_j(x_1, \dots, x_n)\tfrac{\partial}{\partial x_j},$$

where $P_i, Q_j \in \mathbb{K}[x_1, \dots, x_n]$. Suppose that the polynomial h is nonconstant. Since D_0 is p-free, at least one of the coefficients of D_0 is not a multiple of $h(p)$. Without loss of generality one can assume that P_1 is such a coefficient. Then it follows from the equality $h(p)T = g(p)D_0$ that $hQ_1 = gP_1$. Taking into account the equality $(g(p), h(p)) = 1$ we see that $h|P_1$ which gives a contradiction. Therefore $h \in \mathbb{K}^\star$ and

$$\varphi = g(x_1, \dots, x_n) \in \mathbb{K}[x_1, \dots, x_n].$$

But then $T = g(p)D_0$ and $C = \mathbb{K}[p]D_0$ since T was arbitrarily chosen.

(2) Let $\mathrm{rk}_R C = k \geqslant 2$. If for each $D_1 \in C$ we have that $D_1(F) = 0$, then it is easy to see that C is a Lie algebra of rank k over the ring $\mathbb{K}[p]$. Note that in this case C may not be a free $\mathbb{K}[p]$-module.

Suppose there exists an element $S \in C$ such that $S(F) \neq 0$. Then $S(p) \neq 0$. Choose S so that the p-degree of the polynomial $S(p)$ is minimal.

We claim that for each $T \in C$ the polynomial $T(p)$ is a multiple of $S(p)$. Indeed suppose $S(p) = v(p)$, $T(p) = u(p)$ for some polynomials $v(t), u(t) \in \mathbb{K}[t]$. Write $u(t) = v(t)q(t)+r(t)$ for some polynomials $q(t)$, $r(t)$, where $\deg r(t) < \deg v(t)$. Then $u(p) = v(p)q(p) + r(p)$ and $T - q(p)S \in C$. Since $(T - q(p)S)(p) = r(p)$ and $\deg_p r(p) < \deg_p S(p)$, we have by the choice of S that $r(p) = 0$ and $T-q(p)S$ annihilates the kernel $\ker D$. Denote by C_0 the subalgebra of C of all derivations annihilating $\mathbb{K}[p]$. Then as was shown above $T - q(p)S \in C_0$ and $C = C_0 + \mathbb{K}[p]S$. □

Corollary 3.2. *If $k = 2$ and $C(F) \neq 0$, then $C = \mathbb{K}[p]D_0 + \mathbb{K}[p]S$ is a free module of rank 2 over $\mathbb{K}[p]$.*

Theorem 3.3. *Let $D \in W_n(\mathbb{K})$ be a derivation with*

$$tr.\deg_{\mathbb{K}} \ker D = 1 \qquad and \qquad (\ker_R D) \cap A = \mathbb{K}.$$

Then

1) $\ker D = \mathbb{K}(p/q)$ *for some irreducible algebraically independent polynomials $p, q \in \mathbb{K}[x_1, \dots, x_n]$,*
2) *the derivation D is of the form $D = hf(p,q)D_0$ for some irreducible derivation D_0 and homogeneous in p, q polynomial f and a p-q-free polynomial h,*
3) *the centralizer $C = C_{W_n(\mathbb{K})}(D)$ is finite-dimensional over $\mathbb{K}$ being one of the following types:*
   - (a) *$C = \mathbb{K}[p,q]_m hD_0$, where $\mathbb{K}[p,q]_m$ is the linear space of homogeneous in p, q polynomials of degree $m = \deg_{p\text{-}q} f$, and in particular* $\dim_{\mathbb{K}} C = m + 1$;
   - (b) *$C = (\mathbb{K}(p/q)D + \mathbb{K}(p/q)D_2 + \cdots + \mathbb{K}(p/q)D_k) \cap W_n(\mathbb{K})$ for some elements $D_2, \dots, D_k$, $k \leqslant n$ in C with $D, D_2, \dots, D_k$ linearly independent over the field R.*

Proof. By Lemma 2.4 we have that $\ker D = \mathbb{K}(p/q)$ for some irreducible algebraically independent over $\mathbb{K}$ polynomials p, q. By Lemma 2.7 there exist unique (up to a nonzero factor from $\mathbb{K}$) polynomials $f(p,q)$ and h such that $D = f(p,q)hD_0$, where D_0 is a reduced derivation, $f(p,q)$ is a homogeneous polynomial at p, q and the polynomial h is p-q-free.

First, let $\mathrm{rk}_R C = 1$. Then for any $D_1 \in C$ we have that $D_1 = rD_0$ for some $r \in A$ (because D_0 is a reduced derivation). As mentioned above, $r = f_1h_1$ for some homogeneous polynomial $f_1(p,q)$ in p, q and a p-q-free polynomial h_1. By the choice of D_1 we have that

$$0 = [D, D_1] = [f_1 h_1 D_0, fhD_0].$$

The last relation implies the equality

$$D_0(fh/(f_1 h_1)) = 0.$$

By Lemma 2.4 $fh/(f_1h_1) = u(p, q)/v(p, q)$ for some homogeneous polynomials u, v in p, q with $\deg u = \deg v$. Hence

$$hfv = h_1 f_1 u,$$

where fv and f_1u are homogeneous in p, q and h, h_1 are p-q-free polynomials. Recall that the factorization of a polynomial as a product of a homogeneous in p, q and a p-q-free polynomial is unique up to a factor from $\mathbb{K}^\star$. Hence $h_1 = hc$, $c \in \mathbb{K}^*$, and $fv = c^{-1} f_1 u$. Then

$$\deg_{p-q} f = \deg_{p-q} f_1 = m.$$

This implies the relation

$$D = f_1 h_1 D_0 \in \mathbb{K}[p, q]_m hD_0.$$

Since D_1 was arbitrarily chosen in C, we have the inclusion

$$C \subseteq \mathbb{K}[p, q]_m hD_0.$$

It is easy to see that $\mathbb{K}[p, q]_m hD_0 \subseteq C$ and therefore $C = \mathbb{K}[p, q]_m hD_0$.

Further, let $\mathrm{rk}_R C = k \geqslant 2$. Choose a basis $D, D_2, \ldots, D_k$ of C over R. Then by Proposition 2.5

$$C = (\mathbb{K}(p/q)D + \mathbb{K}(p/q)D_2 + \ldots + \mathbb{K}(p/q)D_k) \cap W_n(\mathbb{K}).$$

We will show by induction on k that the centralizer $C = C_{W_n(\mathbb{K})}(D)$ is finite-dimensional over $\mathbb{K}$.

For $k = 1$ (*i.e.* in case $\mathrm{rk}_R C = 1$) this was proved above, so we may assume that $k \geqslant 2$.

Denote for convenience $D_1 = D$. Then every element D_i can be written in the form $D_i = \sum\limits_{j=1}^{n} P_{ij} \frac{\partial}{\partial x_j}$ for some polynomials $P_{ij} \in A$, $i = 1, \ldots, k$. Take an arbitrary element T of the centralizer C and write down it in the form $T = \sum\limits_{i=1}^{k} \alpha_i D_i$ for some rational functions $\alpha_i \in R$. On the other hand, the same derivation can be written in the standard form $T = \sum\limits_{j=1}^{n} Q_i \frac{\partial}{\partial x_i}$ for some polynomials $Q_1, \ldots, Q_n \in A$. Consider the derivations $D_1, \ldots, D_{k-1}, T$ and denote by (P'_{ij}) the polynomial matrix whose first $k-1$ rows consist of coefficients of derivations $D_1, \ldots, D_{k-1}$ and the k-th row is of the form $(Q_1, \ldots, Q_n)$, *i.e.* $P'_{ij} = P_{ij}$ and $P'_{kj} = Q_j$ for $i = 1, \ldots, k-1$, $j = 1, \ldots, n$.

Consider the minor $\delta = \delta_{i_1,\dots,i_k}$ on arbitrarily chosen columns $i_1, \dots, i_k$ of the matrix (P_{ij}) and the analogous minor $\mu = \mu_{i_1,\dots,i_k}$ on the same columns of the matrix (P'_{ij}). Since $T = \sum\limits_{i=1}^{k} \alpha_i D_i$, we have obviously the equality $\mu = \alpha_k \delta$.

Repeating the arguments from the proof of Lemma 2.4 one can show that there exist homogeneous polynomials u, v in p, q with $\deg_{p-q} u = \deg_{p-q} v$ such that $\alpha_k = u/v$. It follows from the equality $\mu = \alpha_k \delta$ (written in the form $v\mu = u\delta$) that $\deg_{p-q} \mu = \deg_{p-q} \delta$. Moreover, these polynomials have the same p-q-free part up to a factor from $\mathbb{K}^\star$ because of the equality $v\mu = u\delta$ mentioned above. We can assume that p-q-free parts of u and v are identical, denote their common value by h. Let $M_1, \dots, M_s$ be all the $(k \times k)$-minors of the matrix (P'_{ij}) enumerated in an arbitrary way, so $s = \binom{n}{k}$. Then they are polynomials from A. Let $m = m_i$ be the p-q-degree of the minor M_i and f_i the corresponding homogeneous polynomial which is a $p - q$-part of the minor M_i. We assign to the derivation T the sequence of homogeneous polynomials $\theta(T) = (f_1, \dots, f_s)$ of degrees $m_1, \dots, m_s$ correspondingly. Consider the map

$$\theta : C \to N = \mathbb{K}[p, q]_{m_1} \times \ldots \times \mathbb{K}[p, q]_{m_s},$$

where m_i are p-q-degree of the minor $M_i, i = 1, \dots, s$. The mapping θ is $\mathbb{K}$-linear and acts from C to N, note that $\dim_{\mathbb{K}} N < \infty$. Obviously $\ker \theta$ consists of such derivations T for which all the minors of order k are zeroes. But then

$$T \in (\mathbb{K}(p/q)D_1 + \ldots + \mathbb{K}(p/q)D_{k-1}) \cap W_n(\mathbb{K}) = C_{k-1}.$$

Therefore, $\dim C/C_{k-1} < \infty$. By inductive assumption the subspace C_{k-1} is finite dimensional over the field $\mathbb{K}$. Therefore $\dim_{\mathbb{K}} C < \infty$. $\square$

4. CENTRALIZERS OF SOME LINEAR DERIVATIONS

A derivation $D = \sum\limits_{i=1}^{n} P_i \frac{\partial}{\partial x_i}$ will be called *linear* if all the polynomial P_i are linear forms in n variables, *i.e.* $P_i = \sum\limits_{j=1}^{n} a_{ij} x_j, a_{ij} \in \mathbb{K}$. The linear derivation $D = \sum\limits_{i,j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_j}$ is determined by the square matrix (a_{ij}) of order n and if $D' = \sum\limits_{i,j=1}^{n} b_{ij} x_j \frac{\partial}{\partial x_j}$, then $[D, D']$ is linear and defined by the matrix $(c_{ij}) = [(a_{ij}), (b_{ij})]$. Therefore all the linear derivation form a subalgebra of $W_n(\mathbb{K})$ isomorphic to the general linear algebra $gl_n(\mathbb{K})$, which (for simplicity) will also be denoted by $gl_n(\mathbb{K})$

Let $D = \sum\limits_{i=1}^{n} a_{ij}x_j\frac{\partial}{\partial x_j} \in W_n(\mathbb{K})$ be a linear derivation. Then one can consider two centralizers:

$$C_0 = C_{gl_n(\mathbb{K})}(D) \qquad \text{and} \qquad C = C_{W_n(\mathbb{K})}(D),$$

Evidently, $C_0 \subseteq C$. The structure of the centralizer C_0 is well-known because it consists of all linear derivations defined by the matrices commuting with the matrix (a_{ij}). How to find the centralizer of a given matrix (a_{ij}) is a classical problem of linear algebra. It was solved many years ago (see, *e.g.* [2, Chapter VIII, §2]). Therefore it is interesting to study the case when $C = C_0$ because we will then have a complete description of the centralizer $C = C_{W_n(\mathbb{K})}(D)$.

In Theorem 4.2 we will present a necessary condition and a sufficient condition for a linear derivation D to satisfy the equality

$$C_{W_n(\mathbb{K})}(D) = C_{gl_n(\mathbb{K})}(D)$$

(unfortunately those conditions do not coincide).

Lemma 4.1. *Let $D = \sum\limits_{i=1}^{n} f_i\frac{\partial}{\partial x_i}$ and $T = \sum\limits_{i=1}^{n} g_i\frac{\partial}{\partial x_i}$ be two elements of $W_n(\mathbb{K})$, where $f_i = f_i(x_1, \ldots, x_n)$, $g_i = g_i(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n]$. Then the derivations D and T commute if and only if $D(g_i) = T(f_i)$ for all $i = 1, \ldots, n$.*

Proof. It is obvious that $DT = TD$ if and only if $DT(x_i) = TD(x_i)$ for all $i = 1, \ldots, n$. But $T(x_i) = g_i(x_1, \ldots, x_n)$ and $D(x_i) = f_i(x_1, \ldots, x_n)$ for all $i = 1, \ldots, n$. Hence $D(g_i) = T(f_i)$. □

Theorem 4.2. *Let $D = \sum\limits_{i,j=1}^{n} a_{ij}x_j\frac{\partial}{\partial x_i}$ be a linear derivation of the polynomial ring $K[x_1, \ldots, x_n]$, and $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the matrix (a_{ij}). Then the following statements hold:*

1. *If the eigenvalues $\lambda_1, \ldots, \lambda_n$ are linearly independent over $\mathbb{Z}$, then $C_{W_n(K)}(D) = C_{gl_n(K)}(D)$.*
2. *If $C_{W_n(K)}(D) = C_{gl_n(K)}(D)$, then the eigenvalues $\lambda_1, \ldots, \lambda_n$ are linearly independent over $\mathbb{N} \cup \{0\}$.*

Proof. (1) Suppose that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix (a_{ij}) are linearly independent over $\mathbb{Z}$. Take any $T \in C_{W_n(\mathbb{K})}(D)$,

$$T = \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i},$$

where $f_i \in A = \mathbb{K}[x_1, \ldots, x_n]$. Without loss of generality one may assume that the matrix (a_{ij}) is diagonal of the form

$$(a_{ij}) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

(the eigenvalues $\lambda_1, \ldots, \lambda_n$ are pairwise distinct, so the matrix (a_{ij}) is diagonalizable). In view of this assumption the derivation D is of the form

$$D = \sum_{i=1}^{n} \lambda_i x_i \tfrac{\partial}{\partial x_i}.$$

Evidently, $D(f_i) = T(\lambda x_i) = \lambda_i f_i$, *i.e.* the coefficients f_i of the derivation T are Darboux polynomials for D with cofactors λ_i, $i = 1, \ldots, n$. Moreover, $D(x_i) = \lambda_i x_i, i = 1, \ldots, n$. But then

$$D(f_i/x_i) = \frac{D(f_i)x_i - f_i D(x_i)}{x_i^2} = 0, \quad i = 1, \ldots, n,$$

i.e. the rational function f_i/x_i belongs to the kernel of D, $i = 1, \ldots, n$. Since all the eigenvalues $\lambda_1, \ldots, \lambda_n$ are linearly independent over $\mathbb{Z}$ it follows from [6, Theorem 10.1.2] that $f_i/x_i = \mu_i \in \mathbb{K}$, $i = 1, \ldots, n$. The latter means that

$$T = \sum_{i=1}^{n} \mu_i x_i \tfrac{\partial}{\partial x_i} \in gl_n(\mathbb{K})$$

and therefore $C_{W_n(K)}(D) = C_{gl_n(K)}(D)$.

(2) Suppose that

$$C_{W_n(K)}(D) = C_{gl_n(K)}(D).$$

This implies that $\ker D = \mathbb{K}$. Indeed, if $h \in \ker D \backslash \mathbb{K}$, then $hD \in C_{W_n(\mathbb{K})}(D)$ and the derivation hD is obviously nonlinear. Hence by [6, Theorem 10.1.1] the eigenvalues $\lambda_1, \ldots, \lambda_n$ are linearly independent over $\mathbb{N}_0$. □

Remark 4.3. Note that the derivation

$$D = x_1 \tfrac{\partial}{\partial x_1} + 2x_2 \tfrac{\partial}{\partial x_2}$$

on the polynomial ring $K[x_1, x_2]$ with eigenvalues $1, 2$ has nonlinear elements in its centralizer in $W_2(\mathbb{K})$, for example $x_1^2 \frac{\partial}{\partial x_2}$. So the condition (2) is not sufficient.

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