On closed weakly \( m \)-convex sets

Tetiana M. Osipchuk

Abstract. In the present work we study properties of generally convex sets in the \( n \)-dimensional real Euclidean space \( \mathbb{R}^n \), \( n > 1 \), known as weakly \( m \)-convex, \( m = 1, \ldots, n - 1 \). An open set of \( \mathbb{R}^n \) is called weakly \( m \)-convex if, for any boundary point of the set, there exists an \( m \)-dimensional plane passing through this point and not intersecting the given set. A closed set of \( \mathbb{R}^n \) is called weakly \( m \)-convex if it is approximated from the outside by a family of open weakly \( m \)-convex sets. A point of the complement of a set of \( \mathbb{R}^n \) to the whole space is called an \( m \)-nonconvexity point of the set if any \( m \)-dimensional plane passing through the point intersects the set. It is proved that any closed, weakly \((n - 1)\)-convex set in \( \mathbb{R}^n \) with non-empty set of \((n - 1)\)-nonconvexity points consists of not less than three connected components. It is also proved that the interior of a closed, weakly 1-convex set with a finite number of components in the plane is weakly 1-convex. Weakly \( m \)-convex domains and closed connected sets in \( \mathbb{R}^n \) with non-empty set of \( m \)-nonconvexity points are constructed for any \( n > 2 \) and any \( m = 1, \ldots, n - 2 \).

\textit{Анотація.} В роботі розглядаються властивості узагальнено опуклих множин \( n \)-вимірного дійсного евклідового простору \( \mathbb{R}^n \), \( n > 1 \), які називаються слабко \( m \)-опуклими, \( m = 1, \ldots, n - 1 \). Відкрита множина простору \( \mathbb{R}^n \) називається слабко \( m \)-опуклою, якщо через кожну точку межі множини проходить \( m \)-вимірна площина, яка дану множину не перетинає. Замкнена множина простору \( \mathbb{R}^n \) називається слабко \( m \)-опуклою, якщо вона апроксимується ззовні сім’єю \( m \)-закритих, слабко \( m \)-опуклих множин. Точка доповнення множини до всього простору \( \mathbb{R}^n \) називається точкою \( m \)-неопуклості множини, якщо довільна \( m \)-вимірна площина, яка містить цю точку, перетинає задану множину. Доведено, що довільна замкнена, слабко \((n - 1)\)-опукла множина з непорожньою множиною

2010 Mathematics Subject Classification: 32F17, 52A30

UDC 514.172

Keywords: convex set, closed set, \( m \)-convex set, weakly \( m \)-convex set, \( m \)-nonconvexity point of a set, real Euclidean space

Keywords: convex set, closed set, \( m \)-convex set, weakly \( m \)-convex set, \( m \)-nonconvexity point of a set, real Euclidean space

Ключові слова: опукла множина, замкнена множина, \( m \)-опукла множина, слабко \( m \)-опукла множина, точка \( m \)-неопуклості множини, дійсний евклідовий простір

DOI: http://dx.doi.org/10.15673/tmgc.v15i1.2139
1. Introduction

As is well known, a subset of the multidimensional real Euclidean space \( \mathbb{R}^n \) is called convex if, together with its two arbitrary points, it contains the entire segment connecting those points [6]. Moreover, the intersection of an arbitrary number of convex sets is again a convex set. The intersection of all convex sets containing a given set \( X \subset \mathbb{R}^n \) is called the convex hull of \( X \) [6] and is denoted by

\[
\text{conv } X = \bigcap_{K \supseteq X, K \text{ convex}} K.
\]

Consider some generalizations of the convexity notion. Recall that any \( m \)-dimensional affine subspace of the space \( \mathbb{R}^n \), \( 1 \leq m < n \), is called an \( m \)-dimensional plane, [9].

**Definition 1.1** ([11]). A subset \( E \subset \mathbb{R}^n \) is called \( m \)-convex with respect to a point \( x \in \mathbb{R}^n \setminus E \), \( 1 \leq m < n \), if there exists an \( m \)-dimensional plane \( L \) such that \( x \in L \) and \( L \cap E = \emptyset \).

**Definition 1.2** ([11]). A subset \( E \subset \mathbb{R}^n \) is called \( m \)-convex, \( 1 \leq m < n \), if it is \( m \)-convex with respect to every point \( x \in \mathbb{R}^n \setminus E \).

An intersection of an arbitrary number of \( m \)-convex sets is again an \( m \)-convex set, [10]. On the other hand, there exist convex sets that are not \( m \)-convex. An open triangle in the plane together with one of its vertices \( x \) is convex. However, one can not draw the straight line through a point inside the triangle’s sides with common vertex \( x \) which does not intersect the set. Thus, that set is not 1-convex. Conversely, there exist \( m \)-convex sets that are not convex. An example of such a set is the union of two open or closed non-overlapping rectangles in the plane which are symmetric with respect to the axis \( Ox \).

Properties of \( m \)-convex compacts in the space \( \mathbb{R}^n \), related to the estimating of cohomology groups, are investigated by Yuri B. Zelinskii in [11]. Properties of \( (n - 1) \)-convex subsets of \( \mathbb{R}^n \) are considered by V. L. Melnyk [7] and, under some additional conditions, by A. I. Gerasin [4,5].
particular, the topological classification of \((n-1)\)-convex sets in \(\mathbb{R}^n\), \(n \geq 2\), with smooth boundary is obtained in [7] and can be formulated as follows:

Each \((n-1)\)-convex subset of \(\mathbb{R}^n\), \(n \geq 2\), with smooth boundary is either convex, or consists of at most two unbounded connected components, or is the Cartesian product \(E^1 \times \mathbb{R}^{n-1}\) for some subset \(E^1 \subset \mathbb{R}\).

We shall use the following standard notations. For a subset \(G \subset \mathbb{R}^n\) let \(\overline{G}\) be its closure, \(\text{Int} G\) be its interior, and \(\partial G = \overline{G} \setminus \text{Int} G\) be its boundary.

**Definition 1.3 ([12]).** An open subset \(G \subset \mathbb{R}^n\) is weakly \(m\)-convex, \(1 \leq m < n\), if it is \(m\)-convex with respect to every point \(x \in \partial G\).

**Definition 1.4 ([1]).** A set \(A\) is approximated from the outside by a sequence of open sets \(\{A_k\}_{k \in \mathbb{N}}\), if \(A_{k+1}\) is contained in \(A_k\), and \(A = \bigcap_k A_k\).

It can be proved that each set approximated from the outside by a family of open sets is closed.

**Definition 1.5 ([2,12]).** A closed subset \(E \subset \mathbb{R}^n\) is weakly \(m\)-convex if it can be approximated from the outside by a family of open weakly \(m\)-convex sets.

Thus, any weakly \(m\)-convex set \(A\) is either open or closed. Among closed weakly \(m\)-convex sets there are also sets with empty interior:

\[
A = \overline{A} = \overline{A} \setminus \text{Int} A = \partial A.
\]

Obviously, every weakly \(m\)-convex set is weakly \(p\)-convex, \(p < m\). It is also easy to see that any open and convex subset \(E \subset \mathbb{R}^n\) is weakly \(m\)-convex, \(1 \leq m < n\). Indeed, for each boundary point of \(E\) there is a supporting hyperplane of the set [6]. Since \(E\) is open, the supporting hyperplane does not intersect \(E\). Thus, \(E\) is \((n-1)\)-convex and, therefore, \(m\)-convex, \(1 \leq m < n\).

Similarly, every closed convex subset of \(\mathbb{R}^n\) is weakly \(m\)-convex, \(1 \leq m < n\), since it can be approximated from the outside by a family of open convex sets homothetic to the given one. Moreover, there are open and closed weakly \(m\)-convex sets which are not convex.

The geometric and topological properties of weakly \(m\)-convex sets are investigated in [3]. In particular, the following proposition is proved in [3]:

*If a set \(E_1\) is weakly \(m\)-convex and a set \(E_2\) is weakly \(p\)-convex, \(p \leq m\), then their union \(E_1 \cap E_2\) is weakly \(p\)-convex.*

The properties of the class of generalized convex sets on Grassmannian manifolds which are closely related to the properties of the conjugate sets
(see [12, Definition 2]) are investigated in [12]. This class includes $m$-convex and weakly $m$-convex subsets of $\mathbb{R}^n$.

**Definition 1.6 ([8]).** A point $x \in \mathbb{R}^n \setminus E$ is an $m$-nonconvexity point of a subset $E \subset \mathbb{R}^n$ if each $m$-dimensional plane passing through $x$ intersects $E$.

The set of all $m$-nonconvexity points of a subset $E \subset \mathbb{R}^n$, $n \geq 2$, will be denoted by $(E)^\triangle_m$, $1 \leq m < n$. Thus, if $E$ is not $m$-semiconvex, then obviously $(E)^\triangle_m \neq \emptyset$. It will be convenient to denote $(E)^\triangle_1 := (E)^\triangle$, $E \subset \mathbb{R}^n$, $n \geq 2$.

Denote the classes of $m$-convex and weakly $m$-convex sets in $\mathbb{R}^n$, $n \geq 2$, by $\mathcal{C}_m^n$ and $\mathcal{WC}_m^n$, respectively. Every open set from the class $\mathcal{C}_m^n$ clearly belongs to the class $\mathcal{WC}_m^n$. The converse statement is not true. It turns out that the class $\mathcal{WC}_m^n \setminus \mathcal{C}_m^n$, $n \geq 2$, of open weakly $m$-convex but not $m$-convex sets is not empty for any $m = 1, \ldots, n - 1$ [2,8]. Moreover, the following lemma holds

**Lemma 1.7 ([2]).** Every open set of the class $\mathcal{WC}_{n-1}^n \setminus \mathcal{C}_{n-1}^n$ consists of at least three connected components.

The estimate of the number of components of the sets of the class $\mathcal{WC}_m^n \setminus \mathcal{C}_m^n$, $n \geq 3$, $1 \leq m < n - 1$, is different as proves the following

**Lemma 1.8 ([8]).** There exist domains in the space $\mathbb{R}^n$, $n \geq 3$, from the class $\mathcal{WC}_m^n \setminus \mathcal{C}_m^n$, $1 \leq m < n - 1$.

The following two lemmas are also proved in [8] and will be used for the proof of the main results of this paper.

**Lemma 1.9 ([8]).** Suppose a closed subset $E \subset \mathbb{R}^n$, $n \geq 2$, belongs to the class $\mathcal{WC}_m^n \setminus \mathcal{C}_m^n$, $1 \leq m < n$. Then, for any family of open, weakly $m$-convex sets $E_k^k$, $k \geq 1$, approximating $E$ from the outside, there exists an index $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$ the set $E_k^k$ of the family is not $m$-convex.

**Lemma 1.10 ([8]).** Let $E_p \subset \mathbb{R}^p$, $p \geq 2$, be an open or a closed set of the class $\mathcal{WC}_1^p \setminus \mathcal{C}_1^p$. Then $E := E_p \times \mathbb{R}^{n-p} \subset \mathbb{R}^n$, $n \geq 3$, belongs to the class $\mathcal{WC}_{n-p+1}^n \setminus \mathcal{C}_{n-p+1}^n$.

The examples of open and closed sets of the class $\mathcal{WC}_{n-1}^n \setminus \mathcal{C}_{n-1}^n$ with three and more connected components are constructed in [8] (see [8, Examples 1-4]). It is also proved in [8] that each compact set from the class $\mathcal{WC}_{n-1}^n \setminus \mathcal{C}_{n-1}^n$ consists of at least three connected components.
The present work proceeds the research of Yu. Zelinskii and his students by investigating the topological properties mainly of closed sets of the classes $\text{WC}^n_m \setminus \text{C}^n_m$, $n \geq 2$, $1 \leq m < n$. In particular, answers are given to some of the questions posed in [8]. Namely, in section 2 it is proved that not only compact sets but every closed set of the class $\text{WC}^n_{n-1} \setminus \text{C}^n_{n-1}$ consists of not less than three connected components. It is also proved that the interior of a closed, weakly 1-convex set with a finite number of components in the plane is weakly 1-convex. In section 3 domains and closed connected sets of the classes $\text{WC}^n_m \setminus \text{C}^n_m$, $n \geq 3$, $1 \leq m < n - 1$, are constructed.

The author is grateful to the anonymous Referee of the paper for carefully reading the manuscript and suggesting valuable improvements and remarks to the text.

2. TOPOLOGICAL PROPERTIES OF CLOSED SETS OF THE CLASS $\text{WC}^n_m \setminus \text{C}^n_m$, $n \geq 2$, $1 \leq m < n$

Given two points $x, y \in \mathbb{R}^n$, we will denote by $xy$ the interval between those points and by $|x - y|$ its length. Also, for $\varepsilon > 0$, let $U(y, \varepsilon) := \{x \in \mathbb{R}^n : |x - y| < \varepsilon\}$ be the $\varepsilon$-neighborhood of a point $y \in \mathbb{R}^n$.

**Lemma 2.1.** Let $E \subset \mathbb{R}^n$ be a closed, weakly $m$-convex subset consisting of $N$ connected components, $n \geq 2$, $1 \leq m < n$. Then $E$ can be approximated from the outside by a family of open, weakly $m$-convex sets $E^k$, $k \geq 1$, such that the number of components of each $E^k$ is not greater than $N$.

**Proof.** Since $E$ is weakly $m$-convex, there exists a family of open weakly $m$-convex sets $G^k$, $k \geq 1$, approximating $E$ from the outside. Suppose that each $E^k$, $k \geq 1$, consists only of the components of $G^k$ containing the points of $E$. Consider a point $y_k \in \partial E^k$. Then $y_k \in \partial G^k$. Since $G^k$ is open and weakly $m$-convex, there exists an $m$-dimensional plane $L_{y_k}$ passing through $y_k$ and such that $L_{y_k} \cap G^k \neq \emptyset$. Since $G^k \supseteq E^k$, it follows that $L_{y_k} \cap E^k \neq \emptyset$. Thus, each $E^k$, $k \geq 1$, is open, weakly $m$-convex, and consists of components the number of which is not greater than $N$.

Since $G^k \supseteq \overline{G^{k+1}} \supseteq \overline{E^{k+1}}$ and $\overline{E^{k+1}}$ is contained only in those components of $G^k$ which contain points of $E$, it follows that $E^k \supseteq \overline{E^{k+1}}$. Let us prove that $E = \bigcap_k E_k$.

Suppose $x \in \bigcap_k E_k$. Then $x \in E_k$ for any $k \geq 1$. Since $E_k \subset G_k$, we obtain that $x \in G_k$ for any $k \geq 1$. Therefore, $x \in \bigcap_k G_k = E$. Now let $x \in E$. Since $G_k \supseteq E$, $k \geq 1$, the point $x$ belongs to some component $G^0_k$ of $G_k$ for each $k \geq 1$. Hence $x \in G^0_k \subset E_k$, $k \geq 1$, which implies that $x \in \bigcap_k E_k$. 


Thus, $E$ is approximated from the outside by the family of open sets $E_k$, $k \geq 1$, due to Definition 1.4.

**Theorem 2.2.** Suppose a closed subset $E \subset \mathbb{R}^n$, $n \geq 2$, belongs to the class $\text{WC}_{m}^{n} \setminus \text{C}_{m}^{n}$, $1 \leq m < n$, and has exactly $N$ connected components. Then $E$ is approximated from the outside by a family of open sets $E^k$, $k \geq 1$, of the class $\text{WC}_{m}^{n} \setminus \text{C}_{m}^{n}$ such that the number of components of each $E^k$ is not greater than $N$.

**Proof.** By Lemma 2.1, $E$ is approximated from the outside by a family of open, weakly $m$-convex sets $G_k$, $k \geq 1$, such that the number of components of each $G_k$ is not greater than $N$. By Lemma 1.9, there exists an index $k_0$ such that every $G^k$ with $k \geq k_0$ belongs to the class $\text{WC}_{m}^{n} \setminus \text{C}_{m}^{n}$. Thus, $E$ is approximated from the outside by the family of sets $E_k := G^{k_0+(k-1)}$, $k \geq 1$, satisfying conditions of Theorem 2.2.

**Theorem 2.3.** Let $E \subset \mathbb{R}^n$ be a closed subset from the class $\text{WC}_{n-1}^{n} \setminus \text{C}_{n-1}^{n}$. Then $E$ consists of at least three components.

**Proof.** Suppose $E$ is connected. Then, by Theorem 2.2, it can be approximated from the outside by a family of domains $E^k$, $k \geq 1$, of the class $\text{WC}_{n-1}^{n} \setminus \text{C}_{n-1}^{n}$. But this contradicts Lemma 1.7. Thus, $E$ is disconnected.

Suppose $E$ consists of two components. Then, by Theorem 2.2, it can be approximated from the outside by a family of open sets $E^k$, $k \geq 1$, of the class $\text{WC}_{n-1}^{n} \setminus \text{C}_{n-1}^{n}$ and each of those sets consists of one or two components. This contradicts Lemma 1.7. Hence $E$ consists of more than two components. Examples 1-4 in [8] complete the proof.

**Definition 2.4.** The set of all points of the straight lines passing through a point $x \in \mathbb{R}^n \setminus A$ and intersecting a subset $A \subset \mathbb{R}^n$ is called the double cone of $A$ with respect to the point $x$ and is denoted by $D_x A$. We assume that $x \not\in D_x A$ whenever $A$ is open and $x \in D_x A$ otherwise.

**Definition 2.5.** The set of all points of the rays starting at a point $x \in \mathbb{R}^n \setminus A$ and passing through a subset $A \subset \mathbb{R}^n$ is called the cone of the set $A$ with respect to the point $x$ and is denoted by $C_x A$. We assume that $x \not\in C_x A$ whenever $A$ is open and $x \in C_x A$ otherwise.

**Lemma 2.6.** Let $E \subset \mathbb{R}^2$ be an open and convex subset and $x \in \mathbb{R}^2 \setminus E$. Then $C_x E$ is an open angle of value not greater than $\pi$.

**Proof.** Since $E$ is open and connected, $C_x E$ is also open and connected. Therefore, $C_x E$ is an open angle. Suppose its value is greater than $\pi$. Then there exists a line $\gamma(x)$ passing through $x$ and such that $\gamma(x) \setminus \{x\} \subset C_x E$. On closed weakly $m$-convex sets 55

Thus, $E$ is approximated from the outside by the family of open sets $E_k$, $k \geq 1$, due to Definition 1.4.

**Theorem 2.2.** Suppose a closed subset $E \subset \mathbb{R}^n$, $n \geq 2$, belongs to the class $\text{WC}_{m}^{n} \setminus \text{C}_{m}^{n}$, $1 \leq m < n$, and has exactly $N$ connected components. Then $E$ is approximated from the outside by a family of open sets $E^k$, $k \geq 1$, of the class $\text{WC}_{m}^{n} \setminus \text{C}_{m}^{n}$ such that the number of components of each $E^k$ is not greater than $N$.

**Proof.** By Lemma 2.1, $E$ is approximated from the outside by a family of open, weakly $m$-convex sets $G_k$, $k \geq 1$, such that the number of components of each $G_k$ is not greater than $N$. By Lemma 1.9, there exists an index $k_0$ such that every $G^k$ with $k \geq k_0$ belongs to the class $\text{WC}_{m}^{n} \setminus \text{C}_{m}^{n}$. Thus, $E$ is approximated from the outside by the family of sets $E_k := G^{k_0+(k-1)}$, $k \geq 1$, satisfying conditions of Theorem 2.2.

**Theorem 2.3.** Let $E \subset \mathbb{R}^n$ be a closed subset from the class $\text{WC}_{n-1}^{n} \setminus \text{C}_{n-1}^{n}$. Then $E$ consists of at least three components.

**Proof.** Suppose $E$ is connected. Then, by Theorem 2.2, it can be approximated from the outside by a family of domains $E^k$, $k \geq 1$, of the class $\text{WC}_{n-1}^{n} \setminus \text{C}_{n-1}^{n}$. But this contradicts Lemma 1.7. Thus, $E$ is disconnected.

Suppose $E$ consists of two components. Then, by Theorem 2.2, it can be approximated from the outside by a family of open sets $E^k$, $k \geq 1$, of the class $\text{WC}_{n-1}^{n} \setminus \text{C}_{n-1}^{n}$ and each of those sets consists of one or two components. This contradicts Lemma 1.7. Hence $E$ consists of more than two components. Examples 1-4 in [8] complete the proof.

**Definition 2.4.** The set of all points of the straight lines passing through a point $x \in \mathbb{R}^n \setminus A$ and intersecting a subset $A \subset \mathbb{R}^n$ is called the double cone of $A$ with respect to the point $x$ and is denoted by $D_x A$. We assume that $x \not\in D_x A$ whenever $A$ is open and $x \in D_x A$ otherwise.

**Definition 2.5.** The set of all points of the rays starting at a point $x \in \mathbb{R}^n \setminus A$ and passing through a subset $A \subset \mathbb{R}^n$ is called the cone of the set $A$ with respect to the point $x$ and is denoted by $C_x A$. We assume that $x \not\in C_x A$ whenever $A$ is open and $x \in C_x A$ otherwise.

**Lemma 2.6.** Let $E \subset \mathbb{R}^2$ be an open and convex subset and $x \in \mathbb{R}^2 \setminus E$. Then $C_x E$ is an open angle of value not greater than $\pi$.

**Proof.** Since $E$ is open and connected, $C_x E$ is also open and connected. Therefore, $C_x E$ is an open angle. Suppose its value is greater than $\pi$. Then there exists a line $\gamma(x)$ passing through $x$ and such that $\gamma(x) \setminus \{x\} \subset C_x E$. On closed weakly $m$-convex sets 55

Thus, $E$ is approximated from the outside by the family of open sets $E_k$, $k \geq 1$, due to Definition 1.4.
Let $\eta^1_x$ and $\eta^2_x$ be the complementary rays starting at $x$ in $\gamma(x)$. Then, by the definition of $C_x E$, there exist points $x^1 \in E \cap \eta^1_x$, $x^2 \in E \cap \eta^2_x$. Hence $E$ is not convex, since $x \in \overline{x^1x^2}$ and $x \notin E$. This gives a contradiction, and so the value of $C_x E$ is not greater than $\pi$. □

**Corollary 2.7.** Let $E \subset \mathbb{R}^2$ be an open and convex subset and $x \in \mathbb{R}^2 \setminus E$. Then $D_x E$ is the union of two vertical open angles of the same value $\leq \pi$.

**Theorem 2.8.** Let $E \subset \mathbb{R}^2$ be a closed subset having finitely many components and such that $\text{Int} E \neq \emptyset$. If $E$ is weakly 1-convex, then $\text{Int} E$ is weakly 1-convex.

**Proof.** Suppose $\text{Int} E$ is not weakly 1-convex. Then there exists a point $y \in \partial E$ of 1-nonconvexity of the interior $\text{Int} E$.

Let $E_i$, $i = 1, \ldots, k$, be the components of $\text{Int} E$. Denote

$$D_{i,j} := D_yE_i \cap D_yE_j, \quad i, j = 1, \ldots, k,$$

(see Figure 2.1a)). Since $y$ is a 1-nonconvexity point of $\text{Int} E$, for each $i \in \{1, \ldots, k\}$, there exist indices $j(i) \in \{1, \ldots, k\}$ such that $D_{i,j(i)} \neq \emptyset$. As the double cones $D_yE_i$, $i = 1, \ldots, k$, are open, we can reduce them so that the intersections of the reduced cones remain non empty. Denote by $\tilde{D}_yE_i$, $i = 1, \ldots, k$, the reduced double cones. Then $\tilde{D}_yE_i \subset D_yE_i$. Notice that the boundary of $\tilde{D}_yE_i$ consists of two straight lines, say $\gamma^1_i(y)$, $\gamma^2_i(y)$, passing through $y$, due to Corollary 2.7. Moreover, $\gamma^1_i(y), \gamma^2_i(y) \subset D_yE_i$. Thus, $\gamma^1_i(y) \cap E_i \neq \emptyset$, $\gamma^2_i(y) \cap E_i \neq \emptyset$ by Definition 2.4.

Take any points

$$x^1_i \in \gamma^1_i(y) \cap E_i, \quad x^2_i \in \gamma^2_i(y) \cap E_i, \quad i = 1, \ldots, k,$$
and choose any curves $\lambda_i \subset E_i$, $i = 1, \ldots, k$, connecting $x^1_i$, $x^2_i$. Then for any straight line $\gamma(y)$ passing through the point $y$, there exists $i \in \{1, \ldots, k\}$ such that $\gamma(y) \cap \lambda_i \neq \emptyset$.

Consider the function

$$d_j(x) = \inf_{x^0 \in \partial E_j} |x - x^0|, \quad x \in E_j, \ j = 1, \ldots, k.$$ 

It is continuous in the domain $E_j$, $j = 1, \ldots, k$. Therefore its restriction to the compact $\lambda_j$, $j = 1, \ldots, k$, reaches its minimum $d_j > 0$ on that compact, i.e.,

$$d_j = \min_{x \in \lambda_j} d_j(x), \quad j = 1, \ldots, k.$$ 

Put

$$d = \min_{j=1,\ldots,k} d_j > 0.$$ 

Then, for any point $x \in \lambda_j$, $j = 1, \ldots, k$, its neighborhood $U(x, d)$ is contained in $E$. Consider the neighborhood $U(y, d)$ of the point $y$, see Figure 2.1b). Since $E$ is weakly 1-convex, there exists a family of open, weakly 1-convex sets $G_k$, $k \geq 1$, approximating $E$ from the outside. Then there exists an index $k_0$ such that $\partial G_k \cap U(y, d) \neq \emptyset$ for all $k \geq k_0$. For each $k \geq k_0$, choose a point $z_k \in \partial G_k \cap U(y, d)$ and draw an arbitrary straight line $\eta(z_k)$ passing through $z_k$. Then the line $\eta(y)$ parallel to $\eta(z_k)$ and passing through $y$ intersects some curve $\lambda_q$, $q \in \{1, \ldots, k\}$, at some point $x_q$. Since $U(x_q, d) \subset E$ and $\eta(z_k) \cap U(x_q, d) \neq \emptyset$, it follows that $\eta(z_k) \cap E \neq \emptyset$. As $G_k \supset E$, $k \geq 1$, we also have that $\eta(z_k) \cap G_k \neq \emptyset$ for $k \geq k_0$.

Since the line $\eta(z_k)$ is arbitrary, the point $z_k \in \partial G_k$ is a 1-nonconvexity point of $G_k$ for all $k \geq k_0$, which gives a contradiction. Theorem 2.8 is proved.

The converse statement is not always true. An example of a closed, not weakly 1-convex set having weakly 1-convex interior can be constructed as follows. Consider an open convex subset $E \subset \mathbb{R}^2$ such that $\text{Int} \overline{E} = E$ and connect any two of its boundary points by a curve $\gamma \subset \mathbb{R}^2 \setminus \overline{E}$ (see Figure 2.2). Then the closed set $\gamma \cup \overline{E}$ is not weakly 1-convex and its interior is weakly 1-convex, since $\text{Int} (\gamma \cup \overline{E}) = E$.

![Figure 2.2](image-url)
3. CONNECTED SETS OF THE CLASS $\mathbf{WC}_{m}^{n}\setminus\mathbf{C}_{m}^{n}$, $n \geq 3$, $1 \leq m < n - 1$

The estimate of the number of components of the closed sets of the class $\mathbf{WC}_{m}^{n}\setminus\mathbf{C}_{m}^{n}$, $n \geq 3$, $1 \leq m < n - 1$, is expectedly the same as for the open sets of this class. To prove this, first, provide here the following:

**Lemma 3.1.** The closure of an open set $E$ of the class $\mathbf{WC}_{m}^{n}\setminus\mathbf{C}_{m}^{n}$, $n \geq 2$, is not $m$-convex, $1 \leq m < n$.

**Proof.** Since $E \in \mathbf{WC}_{m}^{n}\setminus\mathbf{C}_{m}^{n}$, there exists an $m$-nonconvexity point $x \in \mathbb{R}^{n}\setminus\overline{E}$ of the set $E$. Moreover, as $E \subset \overline{E}$, any $m$-dimensional plane passing through $x$ and intersecting $E$ intersects $\overline{E}$ as well. Thus, $x$ is an $m$-nonconvexity point of $\overline{E}$. $\square$

In [8] examples of open and closed sets of the class $\mathbf{WC}_{n-1}^{n}\setminus\mathbf{C}_{n-1}^{n}$, $n \geq 2$, were provided. We will construct here a closed set of the class $\mathbf{WC}_{1}^{2}\setminus\mathbf{C}_{1}^{2}$ in a slightly different way.

**Example 3.2.** Consider an open equilateral triangle $a_0b_0c_0$ with the orthocenter at the origin, and straight lines $\gamma^q_0$, $q = 1, 2, 3$, containing the sides of that triangle. Let $\gamma^q_t$, $q = 1, 2, 3$, $t \in (0, 1]$, be the straight lines not intersecting $a_0b_0c_0$, parallel to the respective lines $\gamma^q_0$, and such that the distance between $\gamma^q_0$ and $\gamma^q_t$ equals $t$, see Figure 3.1. Denote $a_t = \gamma^2_t \cap \gamma^3_t$, $b_t = \gamma^3_t \cap \gamma^1_t$, $c_t = \gamma^1_t \cap \gamma^2_t$, $t \in [0, 1]$.

Let $\alpha^1$ be the angle with vertex $a_1$, generated by $\gamma^2_1$, $\gamma^3_1$, and containing the open triangle $a_1b_1c_1$. Let $\alpha^2$ be the angle with vertex $b_1$, generated by $\gamma^3_1$, $\gamma^1_1$, and containing $a_1b_1c_1$. Finally, let $\alpha^3$ be the angle with vertex $c_1$, generated by $\gamma^1_1$, $\gamma^2_1$, and containing $a_1b_1c_1$. Inscribe an open trapezium...
G^q_1, q = 1, 2, 3, with height > 2 into the angle α^q such that the parallel sides of G^q_1 are parallel to the line γ^q_1 and

\[ \overline{G^q_1 \cap a_1b_1c_1} = \emptyset. \]

Let G^q_t \subset G^q_1, t \in [0, 1), q = 1, 2, 3, be the open trapezium the sides of which are parallel to the respective sides of G^q_1 and the distance between the sides of G^q_t and the respective sides of G^q_1 equals 1 - t. Then the open sets

\[ G_t = \bigcup_{q=1}^{3} G^q_t, \quad t \in [0, 1], \]

belong to the class WC^2_1 \setminus C^2_1. Indeed, for a fixed t \in [0, 1], any point of \( \partial G_t \) belongs to one of the sides of the trapeziums G^q_t, q = 1, 2, 3, and the straight line passing through this side does not intersect G_t by the construction. Moreover, (G_t) Δ = a_1b_1c_1 \neq \emptyset, t \in [0, 1].

In addition, the closed set \( \overline{G_0} \) belongs to the class WC^2_1 \setminus C^2_1 due to Lemma 3.1 and the fact that the sequence of sets \{G_{1/k}\}_{k \in \mathbb{N}} approximates \( \overline{G_0} \) from the outside.

**Theorem 3.3.** There exist closed connected sets in the space \( \mathbb{R}^n \), n ≥ 3, of the class WC^m_n \setminus C^m_n, 1 ≤ m < n - 1.

**Proof.** We will prove the theorem by presenting examples of appropriate sets. First, construct domains in the space \( \mathbb{R}^3 \) of the class WC^3_1 \setminus C^3_1 approximating from the outside a closed connected set of the same class. Consider the open sets G_0, G_{1/k}, k ≥ 1, of the class WC^2_1 \setminus C^2_1 constructed in Example 3.2. By the construction,

\[ (G_0) Δ \subset (G_{1/(k+1)}) Δ \subset (G_{1/k}) Δ \subset (G_1) Δ = a_1b_1c_1, \quad k \geq 1. \]

Further, for the convenience in the notations, we set

\[ 1/0 := 0. \]

Fix s > 1 and define the following sets:

\[ \tilde{E}^3_k := G_{1/k} \times [1/k - s, s - 1/k], \quad k \geq 0. \quad (3.1) \]

Let also P^2_k \subset \mathbb{R}^2 be the convex hull of G_{1/k}, k ≥ 0. For some τ_3 > 0, construct the following prisms:

\[ P^3_{l_k} := P^2_k \times (-1/k - \tau_3 - s, 1/k - s), \]
\[ P^3_{r_k} := P^2_k \times (s - 1/k, s + \tau_3 + 1/k), \quad k \geq 0. \quad (3.2) \]

Now consider the sets

\[ \tilde{E}^3_k := P^3_{l_k} \cup \tilde{E}^3_k \cup P^3_{r_k}, \quad k \geq 0. \]
They are 1-convex with respect to any point of $\partial \tilde{E}^3_k$ except the points of the respective triangles:

$$\tilde{R}l^2_k := \{(x_1, x_2, x_3) \in \partial \tilde{E}^3_k \mid (x_1, x_2) \in (G_{1/k})^\triangle, x_3 = 1/k - s\},$$

$$\tilde{R}r^2_k := \{(x_1, x_2, x_3) \in \partial \tilde{E}^3_k \mid (x_1, x_2) \in (G_{1/k})^\triangle, x_3 = s - 1/k\}.$$ 

We claim that

$$(\tilde{E}^3_k)^\triangle = (G_{1/k})^\triangle \times [1/k - s, s - 1/k], \quad k \geq 1. \quad (3.3)$$

The inclusion

$$(\tilde{E}^3_k)^\triangle \supset (G_{1/k})^\triangle \times [1/k - s, s - 1/k], \quad k \geq 1,$$

will be proved by contradiction. Suppose there exists a point

$$x^k \in (G_{1/k})^\triangle \times [1/k - s, s - 1/k]$$

and a straight line $\gamma(x^k)$ passing through $x^k$ and not intersecting $\tilde{E}^3_k$. Let $\gamma_0(x^k)$ be the orthogonal projection of $\gamma(x^k)$ onto the coordinate plane $x_1Ox_2$. Then the line $\gamma_0(x^k)$ passes through the point $x^k_0 \in (G_{1/k})^\triangle$ being the orthogonal projection of $x^k$ onto the coordinate plane $x_1Ox_2$. Therefore, $\gamma_0(x^k) \cap G_{1/k} \neq \emptyset$. Since $\gamma(x^k) \cap \tilde{E}^3_k = \emptyset$ by our assumption, we have that $\gamma(x^k) \cap Pl^3_k = \emptyset$ and $\gamma(x^k) \cap Pr^3_k = \emptyset$, whence

$$\gamma(x^k) \cap (y^k \times [1/k - s, s - 1/k]) \neq \emptyset$$

for any $y^k \in \gamma_0(x^k) \cap G_{1/k}$. Therefore $\gamma(x^k) \cap \tilde{E}^3_k \neq \emptyset$, which, together with $\tilde{E}^3_k \subset \tilde{E}^3_k$, implies that $\gamma(x^k) \cap \tilde{E}^3_k \neq \emptyset$. This gives a contradiction.

Conversely, we will show that if a point

$$z^k \in \mathbb{R}^n \setminus \left(\tilde{E}^3_k \cup (G_{1/k})^\triangle \times [1/k - s, s - 1/k]\right),$$

then $z^k \notin (\tilde{E}^3_k)^\triangle$, $k \geq 1$.

Let $L$ be a plane passing through $z^k$ parallel to the coordinate plane $x_1Ox_2$. Then the intersection $L \cap \tilde{E}^3_k$ is either

1) empty, or
2) congruent to $P^2_k$, or
3) congruent to $G_{1/k}$.

In the case 1) every straight line in $L$ passing through $z^k$ does not intersect $\tilde{E}^3_k$.

Consider the case 2). Since $P^2_k$ is convex, there exists a line in $L$ passing through $z^k$ and not intersecting $L \cap \tilde{E}^3_k$, and therefore not intersecting $\tilde{E}^3_k$. 

Finally, in the case 3), since \( z^k \notin (L \cap \tilde{E}_k^3) \subset \), there exists a line in \( L \) passing through \( z^k \) and not intersecting \( L \cap \tilde{E}_k^3 \). Therefore it also does not intersect \( \tilde{E}_k^3 \). This completes the proof of (3.3).

Let \( a'_k b'_k c'_k \supset a_1 b_1 c_1, k \geq 0, \) be the open triangles whose sides are parallel to the respective sides of \( a_1 b_1 c_1 \) and the distance between the sides of \( a'_k b'_k c'_k \) and the respective sides of \( a_1 b_1 c_1 \) equals \( 1 - \frac{1}{k} \). Then

\[
\begin{align*}
\quad & a'_1 b'_1 c'_1 = a_1 b_1 c_1, & a'_{k+1} b'_{k+1} c'_{k+1} = a'_k b'_k c'_k, \\
& a'_0 b'_0 c'_0 = a'_k b'_k c'_k \supset (G_1/k) \supset (G_0), & k \geq 1. 
\end{align*}
\]

(3.4)

Consider the triangles

\[
\begin{align*}
Rl_0^2 := \{ (x_1, x_2, x_3) \in \partial \tilde{E}_k^3 \mid (x_1, x_2) \in a'_k b'_k c'_k, x_3 = 1/k - s \}, \\
Rr_0^2 := \{ (x_1, x_2, x_3) \in \partial \tilde{E}_k^3 \mid (x_1, x_2) \in a'_k b'_k c'_k, x_3 = s - 1/k \}, & k \geq 0,
\end{align*}
\]

and some vector \( \vec{a}_3 \) generating an angle greater than \( 0 \) and less than \( \frac{\pi}{4} \) with the positive direction of the axis \( O x_3 \). Let \( Ll_0^3, Lr_0^3 \) be two oblique prisms with respective bases \( Rl_0^2, Rr_0^2 \) and generatrices parallel to the vector \( \vec{a}_3 \). Then, see Figure 3.2b,

\[
Ll_0^3 \supset Ll_{k+1}^3 \supset Ll_k^3, \quad Ll_0^3 \supset Lr_{k+1}^3 \supset Lr_k^3, \quad k \geq 1,
\]

Remove the closures of the prisms \( Ll_0^3, Lr_0^3 \) from the set \( \tilde{E}_k^3, k \geq 0, \) see Figure 3.2a). Then, due to (3.4), the sets

\[
E_k^3 := \tilde{E}_k^3 \setminus (Ll_k^3 \cup Lr_k^3), \quad k \geq 0,
\]

are weakly 1-convex domains. Moreover, we can choose \( s \) in (3.1) and \( \tau_3 \) in (3.2) large enough so that at least the origin becomes a 1-nonconvexity point of the sets \( E_k^3, k \geq 0. \) Indeed, since the origin \( O \) is a 1-nonconvexity point of \( \tilde{E}_k^3, k \geq 0, \) any line \( \gamma(O) \) passing through the origin intersects either \( Pl_0^3 \cup Pr_0^3 \) or \( \tilde{E}_k^3 \) for all \( k \geq 0. \)

Notice that the straight line \( \gamma(O) \) intersecting the prisms \( Pl_k^3, Pr_k^3 \) also intersects their bases

\[
\begin{align*}
Dl_0^2 := \{ (x_1, x_2, x_3) \in \partial \tilde{E}_k^3 \mid (x_1, x_2) \in P_k^2, x_3 = 1/k - s \}, \\
Dr_0^2 := \{ (x_1, x_2, x_3) \in \partial \tilde{E}_k^3 \mid (x_1, x_2) \in P_k^2, x_3 = s - 1/k \}, & k \geq 0.
\end{align*}
\]

Fix \( \vec{a}_3 \) and choose \( \tau_3 \) large enough so that the oblique prisms \( Dl_0^2, Dr_0^2 \) intersect side faces of the prisms \( Pl_k^3, Pr_k^3 \). The larger \( s \) is, the closer the directions of the lines \( \gamma(O) \) passing through the points of \( Dr_0^2 \subset Dl_0^2 \) and \( Dr_0^2 \subset Dl_0^2 \) are to the direction of the axis \( O x_3 \). Then fix \( \tau_3 \) and choose \( s \) large enough so that all lines \( \gamma(O) \) passing through the points of \( Dr_0^2 \)
and \(\overline{Rr^2_k}\) do not intersect the side faces of the prisms \(Pl^3_k\), \(Pr^3_k\). Hence, if 
\[\gamma(O) \cap (\overline{Pl^2_k} \cup \overline{Rr^2_k}) \neq \emptyset,\]
then 
\[\gamma(O) \cap (Pl^2_k \cup Ur^2_k) \neq \emptyset,\]
where 
\[U_l^2_k := \{(x_1, x_2, x_3) \in \partial\tilde{E}_k^3 \mid (x_1, x_2) \in P^2_k, x_3 = -1/k - \tau_3 - s\} \subset \partial Pl^3_k,\]
\[U_r^2_k := \{(x_1, x_2, x_3) \in \partial\tilde{E}_k^3 \mid (x_1, x_2) \in P^2_k, x_3 = s + \tau_3 + 1/k\} \subset \partial Pr^3_k.\]
Since 
\[(U_l^2_k \cup Ur^2_k) \cap (\overline{Ll^3_k} \cup \overline{Lr^3_k}) = \emptyset,\]
we have that the set 
\[(\gamma(O) \cap (Pl^3_k \cup Pr^3_k)) \setminus (\overline{Ll^3_k} \cup \overline{Lr^3_k})\]
is open with respect to its affine hull and, therefore, nonempty. If 
\[\gamma(O) \cap ((Dl^2_k \cup Dr^2_k) \setminus (\overline{Rl^2_k} \cup \overline{Rr^2_k})) \neq \emptyset,\]
then the set 
\[(\gamma(O) \cap (Pl^3_k \cup Pr^3_k)) \setminus (\overline{Ll^3_k} \cup \overline{Lr^3_k})\]
is open and nonempty again. Thus, any line \(\gamma(O)\) intersecting \(Pl^3_k \cup Pr^3_k\) intersects 
\[(Pl^3_k \cup Pr^3_k) \setminus (\overline{Ll^3_k} \cup \overline{Lr^3_k})\]
as well.

Define the smaller base of the trapezium \(G^q_1/k\) by \(S^q_1/k; \ k \geq 0, \ q = 1, 2, 3\), see Figure 3.1. Since, for every \(k \geq 0\), the straight line \(\gamma(O)\) in the coordinate plane \(x_1Ox_2\) intersects at least one of \(S^q_1/k, \ q = 1, 2, 3\), it follows from the construction that \(\gamma(O)\) intersecting \(\tilde{E}_k^3\) must also intersect both 
sets \(S^q_1/k \times (1/k - s, s - 1/k) \subset \partial\tilde{E}_k^3\) and \(\text{Int} \tilde{E}_k^3\) as well.

The larger \(s\) is, the closer the directions of the lines \(\gamma(O)\) passing through the points of 
\[(S^q_1/k \times (1/k - s, s - 1/k)) \cap (\overline{Ll^3_k} \cup \overline{Lr^3_k})\]
are to the direction of the axis \(Ox_3\). Then increase \(s\), if necessary, so that all lines \(\gamma(O)\) passing through the points of 
\[(S^q_1/k \times (1/k - s, s - 1/k)) \cap (\overline{Ll^3_k} \cup \overline{Lr^3_k})\]
intersect the prisms \(Pl^3_k\), \(Pr^3_k\) and therefore intersect 
\[(Pl^3_k \cup Pr^3_k) \setminus (\overline{Ll^3_k} \cup \overline{Lr^3_k})\].
If 
\[\gamma(O) \cap \left((S^q_1/k \times (1/k - s, s - 1/k)) \setminus (\overline{Ll^3_k} \cup \overline{Lr^3_k})\right) \neq \emptyset,\]
then 
\[(\gamma(O) \cap \text{Int} \tilde{E}_k^3) \setminus (\overline{Ll^3_k} \cup \overline{Lr^3_k})\]
is open with respect to its affine hull and, therefore, nonempty. Hence, all other lines passing through the origin and intersecting \( \tilde{E}_k^3 \) intersect \( \tilde{E}_k^3 \setminus (Ll_k^3 \cup Lr_k^3) \) as well.

Thus, \( O \) is a 1-nonconvexity point of \( E_k^3 \), \( k \geq 0 \), which means that \( (E_k^3)^\triangle \neq \emptyset \), \( k \geq 0 \). Moreover, since \( E_k^3 \subset \tilde{E}_k^3 \), we see that

\[
(E_k^3)^\triangle \subset (\tilde{E}_k^3)^\triangle, \quad k \geq 0. \tag{3.5}
\]

Thus, the domains \( E_k^3 \subset \mathbb{R}^3 \), \( k \geq 0 \), belong to the class \( \text{WC}_1^3 \setminus \mathcal{C}_1^3 \). Moreover, the closure \( \overline{E}_0^3 \) of the set \( E_0^3 \) is approximated from the outside by the family of the domains \( E_k^3 \), \( k \geq 1 \), see Figure 3.2b). However, \( \overline{E}_0^3 \) is not 1-convex due to Lemma 3.1. Hence, the closed and connected set \( \overline{E}_0^3 \) belongs to the class \( \text{WC}_1^3 \setminus \mathcal{C}_1^3 \).

![Diagram](image_url)

**Figure 3.2.**

We will now construct domains in the space \( \mathbb{R}^4 \) of the class \( \text{WC}_1^4 \setminus \mathcal{C}_1^4 \) approximating from the outside a closed connected set of the same class. Consider the sets

\[
\tilde{E}_k^4 := E_k^3 \times [1/k - s, s - 1/k], \quad k \geq 0
\]

Let \( P_k^3 \subset \mathbb{R}^3 \) be the convex hull of the set \( E_k^3 \), \( k \geq 0 \). Fix \( \tau_4 > 0 \) and construct the following prisms:

\[
\begin{align*}
Pl_k^4 & := P_k^3 \times (-1/k - \tau_4 - s, 1/k - s), \\
Pr_k^4 & := P_k^3 \times (s - 1/k, s + \tau_4 + 1/k), \quad k \geq 0.
\end{align*}
\tag{3.6}
\]
Now consider the sets
\[ \tilde{E}_k^4 := P_l^4 \cup \tilde{E}_k^4 \cup P_r^4, \quad k \geq 0. \]
They are 1-convex with respect to any point of the sets
\[ \tilde{R}_k^3 := \{(x_1, x_2, x_3, x_4) \in \partial \tilde{E}_k^4 \mid (x_1, x_2, x_3) \in (E_k^3)^{\Delta}, x_4 = 1/k - s\}, \]
\[ \tilde{R}_k^3 := \{(x_1, x_2, x_3, x_4) \in \partial \tilde{E}_k^4 \mid (x_1, x_2, x_3) \in (E_k^3)^{\Delta}, x_4 = s - 1/k\}. \]
Moreover,
\[ (\tilde{E}_k^4)^{\triangle} = (E_k^3)^{\triangle} \times [1/k - s, s - 1/k], \quad k \geq 0. \]
Construct the prisms
\[ L_k^3 := a_k^b b_k^c c_k \times [1/k - s, s - 1/k], \quad k \geq 0. \]
Then, due to (3.3), (3.4), and (3.5),
\[ L_k^3 \supset (E_k^3)^{\triangle} \supset (E_k^3)^{\triangle}, \quad k \geq 0. \]
(3.7)
Now consider the following sets:
\[ R_l_k^3 := \{(x_1, x_2, x_3, x_4) \in \partial \tilde{E}_k^4 \mid (x_1, x_2, x_3) \in L_k^3, x_4 = 1/k - s\}, \]
\[ R_r_k^3 := \{(x_1, x_2, x_3, x_4) \in \partial \tilde{E}_k^4 \mid (x_1, x_2, x_3) \in L_k^3, x_4 = s - 1/k\}, \quad k \geq 0. \]
Since \( L_{k+1}^3 \supset L_k^3 \), we have that \( R_l_{k+1}^3 \supset R_l_k^3 \) and \( R_r_{k+1}^3 \supset R_r_k^3 \). Moreover, due to (3.7),
\[ R_l_k^3 \supset \tilde{R}_k^3, \quad R_r_k^3 \supset \tilde{R}_k^3. \]
(3.8)
Consider some vector \( \overrightarrow{a}_k^3 \) generating an angle greater than 0 and less than \( \frac{\pi}{2} \) with the positive direction of the axis \( Ox_4 \). This provides that two oblique prisms \( L_l_k^3, L_r_k^3 \) with respective bases \( R_l_k^3, R_r_k^3 \) and generatrices parallel to the vector \( \overrightarrow{a}_k^3 \) are such that
\[ L_l_0^3 \supset L_l_{k+1}^3 \supset L_l_k^3, \quad L_r_0^3 \supset L_r_{k+1}^3 \supset L_r_k^3, \quad k \geq 1. \]
Remove the closures of the prisms \( L_l_k^3 \) and \( L_r_k^3 \) from the set \( \tilde{E}_k^4 \), \( k \geq 0 \). Then, by (3.8), the obtained sets
\[ E_k^4 := \tilde{E}_k^4 \setminus (L_l_k^3 \cup L_r_k^3), \quad k \geq 0, \]
are weakly 1-convex domains. Moreover, choose \( s \) in (3.1) and \( \tau_4 \) in (3.6) large enough so that the origin is a 1-nonconvexity point of the sets \( E_k^4 \), \( k \geq 0 \).
Then the domains \( E_k^4 \subset \mathbb{R}^4, \quad k \geq 0 \), belong to the class \( WC_{1}^4 \setminus C_{1}^4 \). Also the closure \( \overline{E}_{0}^4 \) of the set \( E_{0}^4 \) is approximated from the outside by the family of the domains \( E_k^4, \quad k \geq 1 \). Moreover, \( \overline{E}_{0}^4 \) is not 1-convex by Lemma 3.1. Thus, the closed and connected set \( \overline{E}_{0}^4 \) belongs to the class \( WC_{1}^4 \setminus C_{1}^4 \).
Extending the process of constructing the sets $E^n_k$, $k \geq 1$, and $E^n_0$ to the spaces $\mathbb{R}^n$, $n > 4$, using the sets $E^{n-1}_k$, $E^{n-1}_0$ by the induction, one constructs domains and closed connected sets of the class $\text{WC}^{n,1}_1 \setminus \text{C}^{n,1}_1$ for any $n \geq 3$. Then, by Lemma 1.10, the domains

$$E^{n-m+1}_k \times \mathbb{R}^{m-1} \subset \mathbb{R}^n, \quad n \geq 3, \quad 1 \leq m < n - 1, \quad k \geq 1,$$

and the closed connected sets

$$E^{n-m+1}_0 \times \mathbb{R}^{m-1} \subset \mathbb{R}^n, \quad n \geq 3, \quad 1 \leq m < n - 1,$$

belong to the class $\text{WC}^{n,m}_m \setminus \text{C}^{n,m}_m$. \hfill \Box

**References**


Received: October 31, 2021, accepted: June 5, 2022.

Tetiana M. Osipchuk

**Institute of Mathematics of NASU**, 3, Tereschenkivska st., Kyiv, 01024 Ukraine

Email: osipchuk.tania@gmail.com