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The properties of 2-CNF of the mutually dual and self-dual T_0-topologies on the finite set and the calculation of T_0-topologies of a certain weight

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Abstract. The problem of counting non-homeomorphic topologies as well as all topologies on an n-element set is still open. The topologies with the weight $k > 2^{n-1}$, where k is the number of the elements of the topology on an n-element set, which are called close to the discrete topology have been studied completely. Moreover R. Stanley in 1971, M. Kolli in 2007 and in 2014 have been found the number of T_0-topologies on an n-element set with weights $k \geqslant 7 \cdot 2^{n-4}$, $k \geqslant 3 \cdot 2^{n-3}$, and $k \geqslant 5 \cdot 2^{n-4}$ respectively.

In the present paper we investigate T_0-topologies using the topology vector, being an ordered set of the nonnegative integers that define the minimal neighborhoods of the elements of the given finite set, and also using the special form of 2-CNF of Boolean function. In 2021 the authors found the form of the vector of T_0-topologies with $k \geqslant 5 \cdot 2^{n-4}$ and the values $k \in [5 \cdot 2^{n-4}, 2^{n-1}]$, for which there are no T_0-topologies with the weight k. The method of describing of T_0-topologies using the special form of 2-CNF of Boolean function is used for the identification of the mutually dual and self-dual T_0-topologies, and the properties of such 2-CNF Boolean function are used for counting T_0-topologies with the weight $25 \cdot 2^{n-6}$.

Анотація. Питання про загальну кількість негомеоморфних топологій, а також про кількість всіх топологій на n-елементній множині залишається відкритим. Топології з вагою $k > 2^{n-1}$, де k – число відкритих множин в топології на n-елементній множині, які називаються близькими до дискретної топології, були повністю вивчені. Крім того, у роботах Stanley 1971 р. та Kolli 2007 р. та 2014 р. знайдено кількість T_0-топологій на n-елементній множині з вагами $k \geqslant 7 \cdot 2^{n-4}$, $k \geqslant 3 \cdot 2^{n-3}$ та $k \geqslant 5 \cdot 2^{n-4}$ відповідно.

Keywords: topology weight, topology vector, dual topologies, minimal neighborhood, T_0-topologies

Ключові слова: вага топології, вектор топології, дуальна топологія мінімальний окіл T_0-топології

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В даній роботі T_0-топології досліджуються за допомогою вектору топології – впорядкованого набору невід'ємних цілих чисел, які визначають мінімальні околи елементів заданої скінченної множини, а також за допомогою 2-КНФ булевої функції. У роботі 2021 року нами було знайдено вигляд векторів T_0-топологій з $k \geqslant 5 \cdot 2^{n-4}$ і значення $k \in [5 \cdot 2^{n-4}, 2^{n-1}]$, для яких не існує T_0-топологій з вагою k. Метод описання T_0-топологій за допомогою 2-КНФ булевої функції використовується для дослідження взаємно двоїстих та самодвоїстих T_0-топологій, а також для підрахунку кількості T_0-топологій з вагою $25 \cdot 2^{n-6}$.

1. INTRODUCTION

Methods of the graph theory, partially ordered sets, Boolean functions, the homotopy topology, and others are used in the modern studies of the topologies on finite sets.

Distinguishing of topologies from all possible subsets of a given finite set can be performed with the help of the computer although this method is not effective for the sets with the sufficiently large number of the elements. In the Online Encyclopedia of the Integer Sequences one can find the number of all topologies on n-element sets for $n = \overline{0, 18}$, as well as the number of the topologies up to homeomorphisms for $n = \overline{0, 16}$, [1]. These computer-generated data are important for testing of the results obtained by the other methods.

At present, there is no explicit formula for the determination of the total number of non-homeomorphic topologies as well as the number of all topologies on an n-element set, so this question is open. The following formula relates the number $T(n)$ of all topologies on an n-element set with the number $\tilde{T}(m)$ of all T_0-topologies on m-element sets:

$$T(n) = \sum_{m=1}^{n} S(n,m) \cdot \tilde{T}(m), \tag{1.1}$$

where $S(n,m)$ are the Stirling numbers of the second kind, which are well known objects in the discrete mathematics and are given by:

$$S(n,m) = \frac{1}{m!}[m^n - C_m^1(m-1)^n + C_m^2(m-2)^n + \ldots + (-1)^{m-1}C_m^{m-1}1^n],$$

where $S(n,m) = 0$, if $m > n$ or $m = 0$, and $S(0,0) = 0$, $S(n,n) = 1$.

For the first time the formula (1.1) was published in J. W. Evans, F. Harary, M. S. Lynn [3].

The weight of the topology can take integer values from 2 to 2^n. The topologies with the weight $k > 2^{n-1}$ are called *close to the discrete topology* and they are completely described in [6, 7, 9].

In 1971, R. Stanley published the result concerning the number of T_0-topologies of weight $k \geqslant 7 \cdot 2^{n-4}$ on an n-element set. The number of all topologies with the weight $k \geqslant 3 \cdot 2^{n-3}$ has been found in the article M. Kolli [5] in 2007, and later in 2014 he also found the number of all T_0-topologies with the weight $k \geqslant 5 \cdot 2^{n-4}$, [6].

In [8], T_0-topologies have been studied using the topology vector, being an ordered set of the nonnegative integers that define the minimal neighborhoods of the elements of the given finite set. Also in that paper the topologies on an n-element set, which are compatible with topologies close to discrete ones on an $(n-1)$-element set, have been considered. The form of the vector of T_0-topologies with $k \geqslant 5 \cdot 2^{n-4}$ and the values $k \in [5 \cdot 2^{n-4}, 2^{n-1}]$ for which there are no T_0-topologies with the weight k have been found.

There is a number of subproblems in the complex unsolved problem of counting all topologies on an n-element set, the solution of which requires one of the mentioned methods. One of them is an identification of the mutually dual and self-dual T_0-topologies. The method of the describing of T_0-topologies using certain special form of 2-CNF of Boolean function is convenient for the solution of this problem.

Structure of the paper. Section 2 contains the definition of T_0-topologies on a finite set using 2-CNF of Boolean function. In Section 3 we investigate properties of 2-CNF of Boolean functions, which specify mutually dual and self-dual T_0-topologies, and in Section 4 we apply the obtained results to counting the number of T_0-topologies of the weight $25 \cdot 2^{n-6}$. The main result of the article is proved in Section 4.

2. DEFINITION OF T_0-TOPOLOGIES ON A FINITE SET WITH THE HELP OF 2-CNF OF BOOLEAN FUNCTIONS

Consider all possible combinations of the subsets of a set X. For each combination there is a single Boolean function, which has this combination as its truth set, and vice versa. There is a question which Boolean function corresponds to the combination, that forms the topology on the set X. We will find the form of CNF of such function. The concepts of weakly negative, weakly positive, bijunctive Boolean functions have been defined in [4], and the classes of such functions have been denoted by WN, WP and Bi accordingly.

Definition 2.1. *A Boolean function $f(x_1, x_2, \ldots, x_n)$ is called*

- **weakly negative function**, *if there is a conjunctive normal form for it in which each disjunction contains no more than one variable without negation,* *namely has the form*

$$(x_{i_1}^{\alpha} \vee \bar{x}_{i_2} \vee \ldots \vee \bar{x}_{i_k})(x_{j_1}^{\beta} \vee \bar{x}_{j_2} \vee \ldots \vee \bar{x}_{j_k}) \ldots (x_{t_1}^{\gamma} \vee \bar{x}_{t_2} \vee \ldots \vee \bar{x}_{t_k}),$$

where $\alpha, \beta, \ldots, \gamma$ *are Boolean constants.*

- **weakly positive function**, *if there is a conjunctive normal form for it in which each disjunction contains no more than one variable with the negation, namely has the form*

  $$(x_{i_1}^{\alpha} \vee x_{i_2} \vee \ldots \vee x_{i_k})(x_{j_1}^{\beta} \vee x_{j_2} \vee \ldots \vee x_{j_k}) \ldots (x_{t_1}^{\gamma} \vee x_{t_2} \vee \ldots \vee x_{t_k}),$$

  where $\alpha, \beta, \ldots, \gamma$ *are Boolean constants.*

- **bijunctive function**, *if there is a conjunctive normal form for it in which each disjunction contains exactly two variables, namely has the form*

  $$(x_{i_1}^{\alpha_1} \vee x_{i_2}^{\alpha_2})(x_{j_1}^{\beta_1} \vee x_{j_2}^{\beta_2}) \ldots (x_{t_1}^{\gamma_1} \vee x_{t_2}^{\gamma_2}),$$

  where $\alpha_i, \beta_i, \ldots, \gamma_i$ *are Boolean constants. This conjunctive normal form is called* 2*-CNF of Boolean function.*

The following theorem has been proved in [2].

Theorem 2.2 ([2]). *A subset* $\tau \subset 2^X$ *of an* n*-element ordered set*

$$X = (x_1, x_2, \ldots, x_n)$$

form the topology on X *if and only if the Boolean function* $f(x_1, x_2, \ldots, x_n)$*, corresponding to this set, belongs to the intersection* $Bi \cap WP \cap WN$*.*

Notice that the topology with 2-CNF is not uniquely defined. In fact a Boolean function from the intersection $Bi \cap WP \cap WN$ may have several different 2-CNF. Nevertheless, there is a single (up to the designation of the variables) maximal 2-CNF for each Boolean function.

Definition 2.3. *A* 2*-CNF of a Boolean function which defines a topology is called* **maximal** *whenever it has the following property:*

- *if a pair of disjunctions* $x_i \vee \bar{x_j}$ *and* $x_j \vee \bar{x_k}$ *belongs to* 2*-CNF, then the disjunction* $x_i \vee \bar{x_k}$ *also belongs to* 2*-CNF, where* $i, j, k = \overline{1, n}$*.*

Let τ be a topology on a finite set $X = \{x_1, \ldots, x_n\}$. Then the collection $B = \{\varnothing, M_1, \ldots, M_n\}$, in which M_i is the minimal neighborhood of the point x_i, constitute a base of some the topology τ, which will be called the *minimal base.*

Theorem 2.4. *Let* τ *be the topology on an* n*-element set* X *with the minimal base* $B = \{\varnothing, M_1, \ldots, M_n\}$*. If* $M_k = \{x_k, x_{i_1}, \ldots, x_{i_s}\}$*, then there* *is a 2-CNF of Boolean function that specifies the topology τ that contains the conjunction of the form $\varphi_k = (\bar{x_k} \vee x_{i_1}) \wedge \ldots \wedge (\bar{x_k} \vee x_{i_s})$.*

In particular, if $M_k = \{x_k\}$, then $\varphi_k \equiv 1$. Conversely, if some 2-CNF of Boolean function defining the topology τ has the form

$$\varphi = \tilde{\varphi} \wedge (\bar{x_k} \vee x_{i_1}) \wedge \ldots \wedge (\bar{x_k} \vee x_{i_s}),$$

where 2-CNF $\tilde{\varphi}$ does not contain $\bar{x_k}$, then $\{x_k, x_{i_1}, \ldots, x_{i_s}\} \subseteq M_k$.

Proof. Renumber the elements of the set X so that there is a minimal neighborhood $M_k = \{x_k, x_1, \ldots, x_s\}$. Then the corresponding Boolean function equals 1 on the vectors

$$(1, 1, \ldots, 1, \alpha_{s+1}, \ldots, \alpha_{k-1}, 1, \alpha_{k+1}, \ldots, \alpha_n).$$

In [4] such vectors are called *executive.* On all these vectors

$$\varphi_k = (\bar{x_k} \vee x_1) \wedge \ldots \wedge (\bar{x_k} \vee x_s) = 1,$$

that is φ_k belongs to some 2-CNF.

Conversely, suppose $\varphi_k = (\bar{x_k} \vee x_1) \wedge \ldots \wedge (\bar{x_k} \vee x_s)$ belongs to some 2-CNF of the Boolean function and it is not presented in other disjunctions $\bar{x_k}$. Obviously $\varphi_k = 1$ on the sets $(1, 1, \ldots, 1, \alpha_{s+1}, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_n)$, among which only the sets $(1, 1, \ldots, 1, \alpha_{s+1}, \ldots, \alpha_{k-1}, 1, \alpha_{k+1}, \ldots, \alpha_n)$ correspond to the subsets that contain x_k. Hence $\{x_k, x_1, \ldots, x_s\} \subseteq M_k$. □

Theorem 2.5. *Let $B = \{\varnothing, M_1, \ldots, M_n\}$ be the minimal base of the topology τ and φ_k be the conjunction corresponding to the minimal neighborhood M_k, $k = \overline{1, n}$. Then the conjunctive normal form $\varphi = \wedge_{k=1}^{n} \varphi_k$ is the maximal 2-CNF.*

Proof. We will show that if a 2-CNF φ is written on the minimal base $B = \{\varnothing, M_1, \ldots, M_n\}$, then φ is maximal. Suppose that both disjunctions $x_i \vee \bar{x_j}$ and $x_j \vee \bar{x_k}$ are included in φ. Then due to Theorem 2.4 this means that $x_i \in M_j$ and $x_j \in M_k$. From the definition of the minimal neighborhood we have the inclusions $M_i \subset M_j$ and $M_j \subset M_k$, whence $M_i \subset M_k$, and in particular, $x_i \in M_k$. Therefore by Theorem 2.4 2-CNF φ contains the disjunction $x_i \vee \bar{x_k}$. Then by Definition 2.3, 2-CNF φ is maximal. □

Corollary 2.6. *A maximal 2-CNF of Boolean function defining a topology on a finite set, uniquely determines the minimal neighborhoods of all of its points.*

Proof. Let $\varphi = \wedge_{k=1}^{n} \varphi_k$ be a maximal 2-CNF. Let also K be set containing x_k and all the variables which are in disjunction with $\bar{x_k}$. Then by Theorem 2.4 we have the inclusion $K \subseteq M_k$. The inverse inclusion $M_k \subseteq K$

follows from the maximality of φ. Since $x_i \in M_k$, the disjunction $x_i \vee \bar{x_k}$ belongs to φ which means that $x_i \in K$. Therefore $K = M_k$. If φ does not contain the disjunctions with $\bar{x_k}$, then $M_k = \{x_k\}$. □

Example 2.7. For the maximal 2-CNF

$$\varphi = (x_1 \vee \bar{x_3})(x_2 \vee \bar{x_4})(x_1 \vee \bar{x_5})(x_2 \vee \bar{x_5})(x_3 \vee \bar{x_5})$$

of the topology on a five-element set the minimal base of the topology has the form $B = \{\varnothing, \{x_1\}, \{x_2\}, \{x_1, x_3\}, \{x_2, x_4\}, \{x_1, x_2, x_3, x_5\}\}$.

Corollary 2.8. *Each Boolean function $\varphi(x_1, \dots, x_n)$ that specifies the topology on the set $X = \{x_1, \dots, x_n\}$ has a unique maximal 2-CNF.*

3. 2-CNF OF THE MUTUALLY DUAL AND SELF-DUAL TOPOLOGIES

Definition 3.1. *Let $C\tau$ be the set of the complements of the elements of a topology τ. Then the topologies τ and $C\tau$, as well as the corresponding 2-CNF f_τ and $f_{C\tau}$, are called mutually dual.*

Theorem 3.2. *Suppose that a 2-CNF f_τ specifies a topology τ on a set $X = \{x_1, \dots, x_n\}$. Then the 2-CNF, obtained from f by the substitutions*

$$x_i \leftrightarrow \bar{x}_i, \ i = \overline{1, n},$$

specifies the topology $C\tau$.

Proof. Consider any disjunction $x_i \vee \bar{x_j}$ which is a part of f_τ. This disjunction equals 1 on

$$(\alpha_1, \dots, \alpha_{i-1}, 1, \alpha_{i+1}, \dots, \alpha_j, \dots, \alpha_n)$$

and

$$(\alpha_1, \dots, \alpha_{i-1}, 0, \alpha_{i+1}, \dots, \alpha_{j-1}, 0, \alpha_{j+1}, \dots, \alpha_n).$$

After the above replacements, they will change to

$$(\bar{\alpha}_1, \dots, \bar{\alpha}_{i-1}, 0, \bar{\alpha}_{i+1}, \dots, \bar{\alpha}_j, \dots, \bar{\alpha}_n)$$

and

$$(\bar{\alpha}_1, \dots, \bar{\alpha}_{i-1}, 1, \bar{\alpha}_{i+1}, \dots, \bar{\alpha}_{j-1}, 1, \bar{\alpha}_{j+1}, \dots, \bar{\alpha}_n),$$

respectively, while $x_i \vee \bar{x_j}$ will change to $\bar{x_i} \vee x_j$. Then the disjunction $\bar{x_i} \vee x_j$ will be equal to 1 on those new sets. Notice that

$$(\alpha_1, \dots, \alpha_{i-1}, 1, \alpha_{i+1}, \dots, \alpha_j, \dots, \alpha_n)$$

defines the subset $A \in \tau$, then

$$(\bar{\alpha}_1, \dots, \bar{\alpha}_{i-1}, 0, \bar{\alpha}_{i+1}, \dots, \bar{\alpha}_j, \dots, \bar{\alpha}_n)$$

defines its complement $CA \in C\tau$. Similarly,

$$(\alpha_1, \dots, \alpha_{i-1}, 0, \alpha_{i+1}, \dots, \alpha_{j-1}, 0, \alpha_{j+1}, \dots, \alpha_n)$$

defines the subset $V \in \tau$, while

$$(\bar{\alpha}_1, \ldots, \bar{\alpha}_{i-1}, 1, \bar{\alpha}_{i+1}, \ldots, \bar{\alpha}_{j-1}, 1, \bar{\alpha}_{j+1}, \ldots, \bar{\alpha}_n)$$

defines the subset $CV \in C\tau$. Thus, f_τ transforms into 2-CNF which specifies the dual topology $C\tau$. □

Definition 3.3. *Two 2-CNF f_1 and f_2 of the topology on the set X are called* **equivalent** *if there exists a bijection $\varphi : X \to X$ such that $f_2 = \varphi(f_1)$. If a 2-CNF f_τ is equivalent to $f_{C\tau}$, then f_τ and $f_{C\tau}$, as well as corresponding homeomorphic topologies τ and $C\tau$, will be called* **self-dual***.*

Example 3.4. Let $f_\tau = (x_1 \vee \bar{x}_2)(x_1 \vee \bar{x}_5)(x_2 \vee \bar{x}_5)$. We construct 2-CNF of the dual topology $f_{C\tau} = (\bar{x}_1 \vee x_2)(\bar{x}_1 \vee x_5)(\bar{x}_2 \vee x_5)$. The bijection

$$x_1 \leftrightarrow x_5, \qquad x_2 \leftrightarrow x_2, \qquad x_3 \leftrightarrow x_3, \qquad x_4 \leftrightarrow x_4,$$

is a self-equivalence of τ, i.e. τ is a self-dual topology.

We will also describe T_0-topologies using the topology vector. Let τ be a topology on a finite set X and $a \in X$.

Definition 3.5. *The number $ind_\tau(a)$ of the elements distinct from a in its minimum neighborhood M_a is called* **the index of the point (element)** *$a \in X$ in the topology τ,* [9].

Definition 3.6. *The non-descending sequence of the indices of all elements will be called* **the topology vector***. The vector of the topology τ will be denoted by $\nu(\tau) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.* **The index of the topology** *τ is the sum of the indices of all elements of the set X in the vector of this topology,* [9].

Theorem 3.7 ([9])**.** *The sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of nonnegative integers is the vector of some T_0-topology on an n-element set if and only if it satisfies the following conditions:*

a) $\alpha_i \leqslant \alpha_{i+1}$, $i = \overline{1, n-1}$,
b) $\alpha_i \leqslant i - 1$, $i = \overline{1, n}$.

4. The enumeration and the calculation of all labeled T_0-topologies with the weight $25 \cdot 2^{n-6}$

The purpose of this section is to show the use of the vector and 2-CNF of T_0-topologies for their enumeration and calculation on the example of T_0-topologies with the weight $25 \cdot 2^{n-6}$.

In [6, Table 6] it is shown that in the class $25 \cdot 2^{n-6}$ there are 7 nonlabeled T_0-topologies, and in [5] it is shown that the number of all labeled T_0-topologies of this class at $n > 6$ is equal to $\frac{n+14}{24}(n)_6 + \frac{1}{24}(n)_7$.

Theorem 4.1. *A* T_0*-topology has the weight* $25 \cdot 2^{n-6}$ *if and only if its vector has one of the following forms:*

1) $(0, \dots, 0, 1, 5)$*, whenever either a)* $M_{n-1} \subset M_n$ *or b)* $M_{n-1} \cap M_n$ *is a one-element set;*
2) $(0, \dots, 0, 2, 2)$*, if* $M_{n-1} \cap M_n = \varnothing$*;*
3) $(0, \dots, 0, 1, 1, 1, 1, 2)$ *if* $\bigcap_{m=n-4}^{n-1} M_m = \{y\}$ *is a one-element set and either a)* $M_{n-1} \subset M_n$ *or b)* $\bigcap_{m=n-4}^{n} M_m = \{y\}$*.*
4) $(0, \dots, 0, 1, 1, 2)$*, if the intersection* $M_{n-2} \cap M_{n-1}$ *is a one-element set and* $\bigcap_{m=n-2}^{n} M_m = \varnothing$*;*
5) $(0, \dots, 0, 1, 1, 1, 1)$*, if* $M_{n-3} \cap M_{n-2} = \{y\}$ *and* $M_{n-1} \cap M_n = \{z\}$ *are one-element sets with* $y \neq z$*.*

Thus pairs T_0*-topologies with the vectors*

- $(0, \dots, 0, 1, 5)$ *and* $(0, \dots, 0, 1, 1, 1, 1, 2)$*;*
- $(0, \dots, 0, 2, 2)$ *and* $0, \dots, 0, 1, 1, 1, 1)$*;*

are mutually dual, and T_0*-topology with the vector* $(0, \dots, 0, 1, 1, 2)$ *is self-dual.*

Proof. We will show that the weight of T_0-topologies for each of these vectors is equal to $25 \cdot 2^{n-6}$.

In [8, Theorem 2] it was proven that the topologies with the vectors $(0, \dots, 0, 1, 5)$ for the cases a) and b), as well as with the vector $(0, \dots, 0, 2, 2)$, under the condition $M_{n-1} \cap M_n = \varnothing$, have weight $25 \cdot 2^{n-6}$. These topologies induce close to discrete topologies on a set $X = (x_1, x_2, \dots, x_{n-1})$.

Let us show that the topologies with the vector $(0, \dots, 0, 1, 5)$ in the case of $M_{n-1} \subset M_n$ are mutually dual to the topologies with the vector $(0, \dots, 0, 1, 1, 1, 1, 2)$ whenever the intersection $\bigcap_{m=n-4}^{n-1} M_m$ is a one-element set and $M_{n-1} \subset M_n$. In order to do that, denote one of the homeomorphic topologies with the vector $(0, \dots, 0, 1, 5)$ by the symbol τ and suppose that its minimal base have the form

$$B_1 = \bigl\{\varnothing, \{x_1\}, \dots, \{x_{n-2}\}, \{x_1, x_{n-1}\}, \{x_1, x_2, x_3, x_4, x_{n-1}, x_n\}\bigr\}.$$

Then the corresponding maximum 2-CNF has the form

$$f_1 = (x_1 \vee \bar{x}_{n-1})(x_1 \vee \bar{x}_n)(x_2 \vee \bar{x}_n)(x_3 \vee \bar{x}_n)(x_4 \vee \bar{x}_n)(x_{n-1} \vee \bar{x}_n).$$

After replacement $x_i \to \bar{x}_i$ and $\bar{x}_j \to x_j$, we obtain the topology $C\tau$ and the corresponding 2-CNF

$$\tilde{f}_1 = (x_{n-1} \vee \bar{x}_1)(x_n \vee \bar{x}_1)(x_n \vee \bar{x}_2)(x_n \vee \bar{x}_3)(x_n \vee \bar{x}_4)(x_n \vee \bar{x}_{n-1}).$$

Evidently, the bijection

$$x_1 \leftrightarrow x_n, \quad x_{n-1} \leftrightarrow x_{n-1}, \quad x_2 \leftrightarrow x_{n-4}, \quad x_3 \leftrightarrow x_{n-3}, \quad x_4 \leftrightarrow x_{n-2}$$

reduces it to the equivalent 2-CNF

$$f_2 = (x_1 \vee \bar{x}_{n-4})(x_1 \vee \bar{x}_{n-3})(x_1 \vee \bar{x}_{n-2})(x_1 \vee \bar{x}_{n-1})(x_1 \vee \bar{x}_n)(x_{n-1} \vee \bar{x}_n).$$

Thus, the minimum base of the topology $C\tau$ has the form

$$\begin{aligned} B_2 = \Big\{ &\varnothing, \{x_1\}, \dots, \{x_{n-5}\}, \\ &\{x_1, x_{n-4}\}, \{x_1, x_{n-3}\}, \{x_1, x_{n-2}\}, \{x_1, x_{n-1}\}, \{x_1, x_{n-1}, x_n\} \Big\}, \end{aligned}$$

its vector is $(0, \dots, 0, 1, 1, 1, 1, 2)$, and the conditions $\bigcap\limits_{m=n-4}^{n-1} M_m = \{x_1\}$ and $M_{n-1} \subset M_n$ hold. Thus, the topologies are mutually dual.

Similarly, we can show that

- the topology having vector $(0, \dots, 0, 1, 5)$ and such that the intersection $M_{n-1} \cap M_n$ consists of one point, and
- the topology having vector $(0, \dots, 0, 1, 1, 1, 1, 2)$ and such that the intersection $\bigcap\limits_{m=n-4}^{n} M_m$ consists of one point,

are mutually dual, and also that the topologies with vectors $(0, \dots, 0, 2, 2)$ and $(0, \dots, 0, 1, 1, 1, 1)$ are also mutually dual. This leads to the conclusion that the topologies with vectors $(0, \dots, 0, 1, 1, 1, 1, 2)$ and $(0, \dots, 0, 1, 1, 1, 1)$, under these conditions, also have weight $25 \cdot 2^{n-6}$.

It remains to consider the topologies τ with the vector $(0, \dots, 0, 1, 1, 2)$ under the conditions $M_{n-2} \cap M_{n-1}$ is one-element set while the intersection $\bigcap\limits_{m=n-2}^{n} M_m = \varnothing$. It follows from the idea described in [8], that all such topologies have the same weight

$$\begin{aligned} |\tau| &= 2^{n-2} + 2^{n-4} + 2^{n-5} + 2^{n-6}(2^2 - 1) \\ &= 2^{n-6}(2^4 + 2^2 + 2 + 3) = 25 \cdot 2^{n-6}. \end{aligned}$$

Let us show that such a topology is self-dual. The minimal base of one of the homeomorphic topologies has the form

$$B = \{\varnothing, \{x_1\}, \dots, \{x_{n-3}\}, \{x_1, x_{n-2}\}, \{x_1, x_{n-1}\}, \{x_2, x_3, x_n\}\},$$

and the corresponding to it 2-CNF is

$$f_\tau = (x_1 \vee \bar{x}_{n-2})(x_1 \vee \bar{x}_{n-1})(x_2 \vee \bar{x}_n)(x_3 \vee \bar{x}_n).$$

Therefore the 2-CNF of the dual T_0-topology is

$$f_{C\tau} = (\bar{x}_1 \vee x_{n-2})(\bar{x}_1 \vee x_{n-1})(\bar{x}_2 \vee x_n)(\bar{x}_3 \vee x_n).$$

On easily seen that the bijection

$$x_1 \leftrightarrow x_n, \qquad x_2 \leftrightarrow x_{n-2}, \qquad x_3 \leftrightarrow x_{n-1}$$

is a self-homeomorphism of τ, so τ is self-dual.

In order to prove the inverse statement, count the number of all labeled T_0-topologies with the specified vectors. In [8], it was shown that the number of the labeled topologies with the vectors $(0, \dots, 0, 1, 5)$ and $(0, \dots, 0, 2, 2)$ equals to

$$\tfrac{7}{12} \cdot (n)_6 + \tfrac{1}{12} \cdot (n)_7,$$

where

$$(n)_m = n(n-1) \cdots (n-m+1)$$

is the symbol of Pochhammer. Then the number of dual to them will be the same.

Similarly, the number of labeled T_0-topologies with vector

$$(0, \dots, 0, 1, 1, 2)$$

is

$$C_n^{n-3} \cdot C_3^2 \cdot (n-3) \cdot C_{n-4}^2 = \tfrac{1}{4} \cdot (n)_6.$$

Hence the total number of topologies with weight $25 \cdot 2^{n-6}$ is thus

$$\begin{aligned}
&\tfrac{7}{12} \cdot (n)_6 + \tfrac{1}{12} \cdot (n)_7 + \tfrac{1}{4} \cdot (n)_6 = \\
&\quad = \tfrac{7}{12} \cdot (n)_6 + \tfrac{1}{24} \cdot (n)_7 + \tfrac{1}{24} \cdot (n)_7 + \tfrac{1}{4} \cdot (n)_6 = \\
&\quad = \tfrac{10}{12} \cdot (n)_6 + \tfrac{1}{24} \cdot (n)_6 \cdot (n-6) + \tfrac{1}{24} \cdot (n)_7 = \\
&\quad = \tfrac{n+14}{24} \cdot (n)_6 + \tfrac{1}{24} \cdot (n)_7.
\end{aligned}$$

Since this number coincides with that obtained in Kolli's work, it means that there are no other topologies with such weight. ☐

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