

On tensor products of nuclear operators in Banach spaces

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Abstract. The following result of G. Pisier contributed to the appearance of this paper: if a convolution operator $\star f : M(G) \rightarrow C(G)$, where G is a compact abelian group, can be factored through a Hilbert space, then f has the absolutely summable set of Fourier coefficients. We give some generalizations of the Pisier's result to the cases of factorizations of operators through the operators from the Lorentz-Schatten classes $S_{p,q}$ in Hilbert spaces both in scalar and in vector-valued cases. Some applications are given.

Анотація. Появі статті сприяв наступний результат Ж. Пізье: якщо оператор згортки $\star f : M(G) \rightarrow C(G)$, де G – компактна абелева група, можна факторизувати через гільбертовий простір, то f має абсолютно підсумований набір коефіцієнтів Фур'є. В роботі наведено деякі узагальнення результату Пізье на випадки факторизації операторів через оператори з класів Лоренца-Шаттена $S_{p,q}$ у гільбертових просторах як у скалярному, так і у векторнозначному випадках. Стаття також містить деякі застосування отриманих результатів.

1. INTRODUCTION

One of the most remarkable works of the 20th century on functional analysis is the dissertation of Alexander Grothendieck, [6]. Topological tensor products, nuclear operators, approximation and bounded approximation properties, factorization of nuclear operators through Hilbert spaces and through operators from Schatten classes, applications to problems of distribution of eigenvalues, the theory of nuclear spaces and specific examples of the application of the theory and much more – here is an incomplete list of issues considered there (mostly for the first time).

2010 Mathematics Subject Classification: 47A80, 47B37

Keywords: factorization, Lorentz-Schatten class, nuclear operator

Ключові слова: факторизація, клас Лоренца-Шаттена, ядерний оператор

DOI: <http://dx.doi.org/10.15673/tmgc.v14i3.2083>

In this note, we will be interested in problems related to the possibility of factorizations of various types of nuclear operators through operators in Hilbert spaces and their applications, in particular, in the theory of tensor products of Banach spaces.

Studying the behavior of the eigenvalues of nuclear operators, Grothendieck based, in particular, on the possibility of factorizations of these operators through the operators of Schatten classes S_p , then applying the theory of operators in Hilbert spaces. Moreover, as is now clear, he obtained exact results in the scales of those spaces of operators that he studied. Let us give some examples of A. Grothendieck's ideas in applications of this technique (which, in particular, contributed to the appearance of this note).

Recall that an operator $T : X \rightarrow Y$ in Banach spaces is called *nuclear* if it can be represented in the following form:

$$Tx = \sum_{n=1}^{\infty} \mu_n \langle x'_n, x \rangle y_n, \quad \text{for } x \in X, \tag{1.1}$$

where $(x_n) \in X, (y_n) \in Y$ are bounded sequences, $\mu := (\mu_n)$ is an absolutely summable sequence of complex numbers (it is clear that they can be considered real and non-negative). Every nuclear operator T can be factorized in the following way:

$$T = AB : X \xrightarrow{B} l^2 \xrightarrow{A} Y, \tag{1.2}$$

where A, B are bounded operators. Indeed, it is enough to put

$$Bx := \{\sqrt{\mu_n} \langle x'_n, x \rangle\} \in l^2, \quad A(\alpha_n) := \sum \alpha_n \sqrt{\mu_n} y_n$$

for $(\alpha_n) \in l^2$. Let now $X = Y$. Consider, along with the operator AB , the operator $BA : l^2 \rightarrow X \rightarrow l^2$. If e_k denotes k -th orth in l^2 , then $Ae_k = \sqrt{\mu_k} y_k$ and

$$\langle BAe_k, e_m \rangle = \left\langle \sum_n \sqrt{\mu_n} \langle x'_n, \sqrt{\mu_k} y_k \rangle e_n, e_m \right\rangle = \sqrt{\mu_m} \langle x'_m, \sqrt{\mu_k} y_k \rangle,$$

i.e. $\langle BAe_k, e_m \rangle = \sqrt{\mu_m} \langle x'_m, y_k \rangle \sqrt{\mu_k}$ and thus $\sum_k \|BAe_k\|^2 < \infty$. Therefore, BA is the Hilbert-Schmidt operator. Further, since the (complete) sequence of eigenvalues of the operator BA (counted with their algebraic multiplicities) belongs to l^2 , the complete sequence of all eigenvalues of the nuclear operator $T = AB$ also belongs to l^2 .

As A. Grothendieck [6] noted, this result is sharp, and the continuous on the unit circle Carleman function [2], whose Fourier coefficients belong to l^2 and do not belong to any space l_p for $p < 2$, gives an example of this. Indeed, the convolution operator with this function considered in the space of all continuous functions on the circle \mathbb{T} is nuclear, and the complete set

of its eigenvalues coincides with the sequence of Fourier coefficients of the Carleman function.

At the same time, this example shows that for an arbitrary nuclear operator in Banach spaces, the factorization through a continuous operator in a Hilbert space (through an operator of the class S_∞ ; here, through the identity operator) is the best possible among the factorizations of operators through operators of the Schatten classes S_p . Indeed, otherwise (e.g., in the case of a possible factorization through the S_p -operator with $p \in (0, \infty)$), the above reasoning with the Hilbert-Schmidt operator would allow us to conclude that the eigenvalues of the nuclear operators belong to some l^q for $q < 2$ (namely, for q from the relation $\frac{1}{2} + \frac{1}{p} = \frac{1}{q}$; we use the inclusion $S_p \circ S_2 \subset S_q$).

Note, however, that each nuclear operator can be factorized through a compact operator in Hilbert space. To see this, it is enough to split off from the non-negative sequence μ a piece (sequence) $\nu := (\nu_n)$ so as to obtain the relations $\nu_n \rightarrow \infty$ and $\sum \mu_n \nu_n < \infty$. Correcting the definitions of the operators A and B in (1.2), respectively, we obtain the factorization of T through a diagonal operator in l^2 with a diagonal tending to zero (for example, through $\Delta := (\nu_n)$).

The situation is somewhat more complicated with the so-called p -nuclear operators for $0 < p < 1$ (considered for the first time by A. Grothendieck, but under the name “*Opérateurs de puissance p . ème sommable*”). The definition of a p -nuclear operator is similar to the above definition of a nuclear operator, but in the relation (1.1) we consider the sequence (μ_n) from the space l^p .

Each p -nuclear operator $T : X \rightarrow Y$ admits a factorization of the form (1.2). Moreover, one can factorize T not only through a compact operator, but also through an operator from the class S_q for the exponent q with $\frac{1}{q} = \frac{1}{p} - 1$, see [6, Chapter. 2, p. 11]. As above, this implies the corresponding result on the distribution of the eigenvalues of p -nuclear operators:

the eigenvalues of p -nuclear operators belong to l_q , where $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$.

It should be noted, however, that this result is not final. In the scale of spaces l_p , it is exact (see, for example, [12]). But if we go to the scale of Lorentz spaces $l_{p,r}$, then the exact result looks like this ([12]):

for any $p \in (0, 1)$ the sequence of any p -nuclear operators belongs to $l_{q,p}$, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$.

In the first case, A. Grothendieck’s remark on the sharpness of his statement on the eigenvalues of nuclear operators (using the example of Carleman) is unimprovable in the scale of Lorentz spaces. Perhaps the use of the

Carleman function confirms this fact, but I do not know this, and I did not check if this function belongs to any Lorentz space $l_{2,s}$ for $s < 2$. Therefore, let us use, e.g., the Kahan-Katznelson-de Leeuw theorem [3]: for any sequence $a \in l_2(\mathbb{Z})$ there exists a function $F_a \in C(\mathbb{T})$ such that for all $j \in \mathbb{Z}$, $|\hat{F}_a(j)| \geq |a_j|$.

To obtain the exactness of Grothendieck's theorem (in the scale of Lorentz sequence spaces), it suffices to take any sequence $a \in l_2(\mathbb{Z}) \setminus \bigcup_{s < 2} l_{2,s}$ and the corresponding function F_a and consider, following Grothendieck, the convolution operator with this function in the space $C(\mathbb{T})$. Note that a simpler example could be used, namely the example from the book [12, Example 2, b.14, p.107], in which the sharpness (in the scale of Lorentz sequence spaces) of the Grothendieck's theorem is explicitly obtained: for any sequence $\sigma \in l^2$ there exists a nuclear operator T such that the sequence $\sigma \in l^2$ is a subsequence of the sequence the eigenvalues of the operator T . Finally, we can use the stronger (final) result of R. Kaiser and J. Rutherford [10]: any sequence $a \in l^2$ is exactly a sequence of eigenvalues of some nuclear operator (zero terms of sequences are not considered).

All that was said above shows that for a nuclear operator to be factorized through a compact operator (S_∞ -operator) in a Hilbert space is the best that one can get answering a question

“What are the exponents $p, q \in (0, \infty]$ for which every nuclear operator can be factored through an $S_{p,q}$ -operator?”

Here $S_{p,q}$ is the Lorentz-Schatten class, see below.

In the second case, where we consider the p -nuclear (or more generally, (p, q) -nuclear) operators, we can use the same factorization ideas as above for the case $p = 1$. It can be seen that every p -nuclear operator T can be factored through an operator from $S_q(H)$, where $\frac{1}{q} = \frac{1}{p} - 1$ (so, $q = \infty$ if $p = 1$). It follows from such a factorization that the eigenvalue sequence of T belongs to l_r with $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. In the scale S_p of Schatten classes the last result is best possible.

The same can be said about the factorizations of p -nuclear operators through an S_q -operators. However, if we consider the scale s of Lorentz sequence spaces $l_{r,s}$ and $S_{r,s}$ of operators of Lorentz-Schatten classes, then the questions are not so clear. Recall that an operator $T \in L(X, Y)$ is said to be (r, s) -nuclear, where $0 < r, s \leq 1$, if it admits a representation (1.1) with $(\mu_n) \in l_{r,s}$.

On the one hand, H. König showed in [11], that the eigenvalues of the p -nuclear operators ($0 < p < 1$) belong to the Lorentz space $l_{r,p}$, where $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$ and this result is sharp (see [12, p. 126]).

On the other hand, this eigenvalues result do not give a possibility to find out whether the above result on the factorization of T through an S_q -operator is the best one in the scale $S_{r,s}$. We need to proceed in another way and we will use a result of G. Pisier [17].

This result helps to get the sharpness in such factorizations in the scale $S_{r,s}$ and also, among other things, gives us a possibility to get one more negative answer to the question considered in [18] on the product of two nuclear operators (see below).

G. Pisier has shown that if a convolution operator

$$\star f : M(G) \rightarrow C(G),$$

where G is a compact abelian group, $M(G) = C(G)^*$ and $f \in C(G)$, can be factored through a Hilbert space, then f has the absolutely summable set of Fourier coefficients. It is clear that the condition “*the convolution operator ... can be factorized through a Hilbert space*”

$$\star f : M(G) \rightarrow H \rightarrow C(G)$$

is the same as the condition “*the operator $\star f$ can be factored through a bounded (or compact) operator U in a Hilbert space*”:

$$\star f : M(G) \rightarrow H \xrightarrow{U} H \rightarrow C(G).$$

We are going to generalize this result, so let us give some notes about it.

Let $S(H)$ be an ideal in the algebra $L(H)$ of all bounded operators in H (e.g., the ideal of compact operators). What is the condition on the set $\{\hat{f}(\gamma)\}$ that gives a possibility to factorize the operator $\star f$ through an operator from $S(H)$:

$$\star f : M(\mathbb{T}) \rightarrow H \xrightarrow{U \in S} H \rightarrow C(\mathbb{T})?$$

We present some generalizations of the result of G. Pisier, giving answers to the question for the ideals $S_{p,q}(H)$ of operators from the Lorentz-Schatten classes (operators, whose singular numbers are in the Lorentz sequence space $l_{p,q}$) and for general compact abelian groups G :

$$\star f : M(G) \rightarrow H \xrightarrow{S_p(H)} H \rightarrow C(G), \quad f \in C(G).$$

Moreover, we will consider even convolution operators in vector-valued cases, generalizing the result of G. Pisier and a result of P. Saab [20, Theorem 4.2], where it was shown that the Pisier’s techniques in the scalar case can be extended to the vector-valued case (factorizations of a vector valued convolutions through Hilbert spaces).

Shortly on the content of the paper.

In Section 2, we give notation and preliminaries (function spaces on compact abelian groups, tensor products and some operator ideals, integral

and 2-absolutely summing operators, Lorentz-Schatten classes of operators on Hilbert spaces).

Section 3 is devoted to the Pisier's result that was mentioned in Introduction. We prove some generalizations of the Pisier's theorem to the cases of $S_{p,q}$ -factorizations of operators for the scalar cases. Some applications are given.

In Section 4 we give some generalizations of the Pisier's theorem to the cases of $S_{p,q}$ -factorizations of operators in vector-valued case. We will generalize also a mentioned above result of P. Saab [20, Theorem 4.2]. At the end of the section, we consider the factorizations through $S_{p,q}$ -operators linear mappings between tensor products of several Banach spaces.

2. PRELIMINARIES

2.1. General notation. All the spaces X, Y, Z, W etc. are Banach, x, x_n, y, y_k etc. are elements of spaces X, Y, \dots respectively. All linear mappings (operators) are continuous; as usual, X^*, X^{**} are Banach duals (to X), and x', x'', \dots (or y', \dots) are the functionals on X, X^*, \dots (or on Y, \dots). By π_Y we denote the natural isometric injection of Y into its second dual. If $x \in X, x' \in X^*$ then $\langle x, x' \rangle = \langle x', x \rangle = x'(x)$. Also $L(X, Y)$ stands for the Banach space of all linear bounded operators from X to Y . We always consider the space X as the subspace $\pi_X(X)$ of its second dual X^{**} .

2.2. Analysis on groups. We refer to [19] on general topics of this subsection and to [4] for the information on vector-valued function spaces and vector measures.

Let G be a compact abelian group, m be a Haar measure on G (i.e. the unique translation invariant normalized regular Borel measure), Γ be the dual group of G , i.e. the group of all characters on G (multiplicative continuous complex functions γ so that $|\gamma(t)| = 1$ for all $t \in G$). Note that Γ is discrete. Denote by $C(G)$ the Banach space of all continuous (complex-valued) functions on G with the natural uniform norm: if $\varphi \in C(G,)$ then

$$\|\varphi\|_\infty := \sup_{t \in G} |\varphi(t)|.$$

Let also $L_p(G)$, $1 \leq p < \infty$, be the Banach space of all (m -equivalent classes of) absolutely p -summable Borel functions on G ,

$$\|\varphi\|_p := \left(\int_G |\varphi|^p dm \right)^{\frac{1}{p}} < \infty \text{ for } \varphi \in L_p(G),$$

and $M(G)$ be the Banach space of all (complex-valued) finite regular Borel measures on G with the variation norm $|\mu|(G)$ (or, what is the same, with the norm induced from $C^*(G)$ by the Riesz Representation Theorem).

If $f \in L_p(G)$, $1 \leq p < \infty$, and $\mu \in M(G)$, then

$$f \star \mu(g) := \int_G f(g-h) d\mu(h) \text{ for } g \in G,$$

$$\|f \star \mu\|_p \leq \|f\|_p \|\mu\|.$$

If $f \in C(G)$, then

$$f \star \mu(g) \in C(G), \quad \|f \star \mu\|_\infty \leq \|f\|_\infty \|\mu\|.$$

If $f \in L_1(G)$, and $\mu \in M(G)$, then the Fourier transform of f and μ are defined by

$$\hat{f}(\gamma) := \int_G \overline{\gamma(h)} f(h) dm(h) \text{ for } \gamma \in \Gamma;$$

(maps $L_1(G) \rightarrow C_0(\Gamma)$, $\|\hat{f}\|_\infty \leq \|f\|_1$) and

$$\hat{\mu}(\gamma) := \int_G \overline{\gamma(h)} d\mu(h) \text{ for } \gamma \in \Gamma.$$

Here $C_0(\Gamma)$ is the subspace of $C(\Gamma)$, consisting of all functions which vanish at infinity.

2.3. Tensor products and integral operators. We refer to [4, 6, 7, 21] on tensor products of Banach spaces and to [16, 5] for the information on p -absolutely summing operators.

For Banach spaces X, Y , denote by $F(X, Y)$ the linear subspace of the space $L(X, Y,)$ consisting of all finite rank operators. Algebraic tensor product $X^* \otimes Y$ will be identify with the linear space $F(X, Y)$: every tensor element $z := \sum_{n=1}^N x'_n \otimes y_n$ can be considered as an operator

$$\tilde{z}(\cdot) := \sum_{n=1}^N \langle x'_n, \cdot \rangle y_n.$$

Also, $X \otimes Y$ can be considered as a subspace of the vector space $F(X^*, Y)$ (namely, as the vector space of all linear weak*-to-weak continuous finite rank operators). We can identify also the tensor product (in a natural way) with a corresponding subspace of $F(Y^*, X)$. If $X = W^*$, then $W^* \otimes Y^{**}$ is identified with $F(X, Y^{**})$ (or with $F(Y^*, X^*)$).

The projective norm of an element $z \in X \otimes Y$ is defined as

$$\|z\|_\wedge := \inf \left\{ \sum_{n=1}^N \|x_n\| \|y_n\| : z = \sum_{n=1}^N x_n \otimes y_n, (x_n) \subset X, (y_n) \subset Y \right\}.$$

The completion of the normed space $(X \otimes Y, \|\cdot\|_\wedge)$ is called the *projective tensor product* of Banach spaces X and Y and denoted by $X \widehat{\otimes} Y$. Every element can be written in the form

$$z = \sum_{n=1}^{\infty} x_n \otimes y_n \quad \text{with} \quad \sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \infty. \tag{2.1}$$

Note, that $X \widehat{\otimes} Y = Y \widehat{\otimes} X$ under natural identification. Every element z of $X \widehat{\otimes} Y$ generates an operator $\tilde{z} : X^* \rightarrow Y$: If z has a representation (2.1), then

$$\tilde{z}(x') := \sum_{n=1}^{\infty} \langle x_n, x' \rangle y_n.$$

A linear functional “trace” is defined on each tensor product $z \in X^* \otimes X$: if $z = \sum_{n=1}^N x'_n \otimes x_n$, then $\text{trace}(z) = \sum_{n=1}^N \langle x'_n, x_n \rangle$ and the last sum does not depend on a representation of z . This functional has a unique extension to the completion $X^* \widehat{\otimes} X$ and its value at an element $z \in X^* \widehat{\otimes} X$ is denoted again by $\text{trace}(z)$. If $z = \sum_{n=1}^{\infty} x_n \otimes x_n$, then $\text{trace}(z) = z = \sum_{n=1}^{\infty} \langle x'_n, x_n \rangle$.

A dual space of the tensor product $X \widehat{\otimes} Y$ is $L(Y, X^*)$ with duality defined by

$$\langle T, z \rangle := \text{trace}(T \circ z) = \sum_{n=1}^{\infty} \langle x_n, T y_n \rangle, \quad z \in X \widehat{\otimes} Y, T \in L(Y, X^*).$$

Here, $T \circ z$ is an element $\sum_{n=1}^{\infty} x_n \otimes T y_n \in X \widehat{\otimes} X^*$. In particular,

$$(X^* \widehat{\otimes} Y)^* = L(Y, X^{**}) = L(X^*, Y^*).$$

If $A \in L(X, W)$, $B \in L(Y, G)$ and $x \otimes y \in X \otimes Y$, then a linear map $A \otimes B : X \otimes Y \rightarrow W \otimes G$ is defined by $A \otimes B((x \otimes y)) := Ax \otimes By$ (and then extended by linearity). Since $\widetilde{A \otimes B}(z) = B \tilde{z} A^*$ for $z \in X \otimes Y$, we can use notation $B \circ z \circ A^* \in W \otimes G$ for $A \otimes B(z)$.

There is another natural norm on the tensor product $X \otimes Y$, namely, the norm induced from $L(X^*, Y)$, that is the uniform norm. The completion of $X \otimes Y$ with respect to this norm coincides with the closure of $X \otimes Y$ in $L(X^*, Y)$, is denoted by $X \widetilde{\otimes} Y$ and is called the *injective tensor product* of X and Y . In particular, the injective tensor product $X^* \widetilde{\otimes} Y$ is exactly the closure of all finite rank operators in $L(X, Y)$ and contained in the Banach space $K(X, Y)$ of all compact operators from X to Y .

The dual space to $X \widetilde{\otimes} Y$ can be identify with so-called integral operators from Y to X^* . We will use the following definition of an integral operator in Banach spaces. An operator $T : Z \rightarrow W$ is said to be *integral* (they say

also “integral in the sense of Pietsch”) if there exist a compact space K , a probability measure $\mu \in C^*(K)$ and two bounded operators $A : Z \rightarrow C(K)$ and $B : L_1(K, \mu) \rightarrow W$ so that T admits the following factorization:

$$T = BjA : Z \xrightarrow{A} C(K) \xrightarrow{j} L_1(K, \mu) \xrightarrow{B} W$$

where j is a natural inclusion. With the norm $i(T) := \inf \|A\| \|B\|$ the space $I(Z, W)$ of all integral operators is Banach.

For any $z = \sum_{n=1}^N x_n \otimes y_n \in X \otimes Y$ and $V \in I(Y, X^*)$ the composition $V \circ z$ belongs to $X \otimes X^*$ and $\|V \circ z\|_{\wedge} \leq \|\tilde{z}\| i(V)$. Thus V generates a linear continuous map from $X \tilde{\otimes} Y$ into $X \hat{\otimes} X^*$, the trace of $V \circ A$ is well defined for every $A \in X \tilde{\otimes} Y$ and $|\text{trace}(V \circ A)| \leq \|A\| i(V)$. The linear continuous functional $\text{trace}(V \circ \cdot)$ defines a duality between the spaces $X \tilde{\otimes} Y$ and $I(Y, X^*)$ and the last space is the dual to the injective tensor product $X \tilde{\otimes} Y$ with respect to this duality.

Let us mention that the above norms in $\hat{\otimes}$ and in $\tilde{\otimes}$ are the greatest and least crossnorms respectively (see, e.g. [4, p. 221]). Projective and injective tensor products of several Banach spaces can be defined by induction.

Two important notion in connection with the just introduced notions: a Banach space X is said to have

- the *approximation property* if for every Banach space Y the natural mapping $Y^* \hat{\otimes} X \rightarrow L(Y, X)$ is injective;
- the *metric approximation property* if for every Banach space Y the natural mapping $Y^* \hat{\otimes} X \rightarrow I(Y, X^{**})$ is an isometric embedding.

Such spaces as $L_p(\mu), C(K), M(G) = C^*(G)$ and all their duals have the metric approximation property [6].

2.4. 2-Absolutely summing operators. An operator $T : X \rightarrow Y$ is said to be 2-absolutely summing if there is a constant $C > 0$ such that for every finite sequence $(x_n)_1^N \subset X$ one has

$$\sum_{n=1}^N \|Tx_n\|^2 \leq C \sup_{\|x'\| \leq 1} \left| \sum_{n=1}^N \langle x_n, x' \rangle \right|^2.$$

Denote by $\Pi_2(X, Y)$ the space of all such operators; it is Banach with a norm $\pi_2(T) := \inf C$.

Here are few examples of such operations:

- 1) An inclusion $j : C(K) \rightarrow L_2(K, \mu)$, where μ is a probability measure on a compact set K .
- 2) $\Pi_2(H, H) = S_2(H, H)$.
- 3) Any operator from $C(K)$ to $M(K) = C^*(K)$ is 2-absolutely summing.

2.5. Lorentz-Schatten classes of operators in Hilbert spaces. The Lorentz-Schatten class $S_{p,q}$, $0 < p, q < \infty$, considered for the first time by H. Triebel in [22], can be defined in the following way. Let U be a compact operator in a Hilbert space H and (s_n) is the sequence of its singular numbers (see, e.g., [15, 2.1.13]). An operator U belongs to the space $S_{p,q}(H)$, if $(s_n) \in l_{p,q}$, (see, e.g., [15, 2.11.15]). The space $S_{p,q}(H)$ has a natural quasi-norm

$$\sigma_{p,q}(U) = \|(s_n)\|_{p,q} = \left(\sum_{n=1}^{\infty} n^{(q/p)-1} s_n^q \right)^{1/q}.$$

If $p = q$, then $S_{p,p}$ coincides with the class S_p (with a quasi-norm σ_p). Let us mention that, for $p, q \in (0, 1]$, we have the equality $N_{p,q}(H) = S_{p,q}(H)$ (see, e.g., [8]) and the inclusions $S_{p,q} \subset S_{p',q'}$, if $0 < p < \infty$ and $0 < q \leq q' < \infty$ or $0 < p < p' < \infty, 0 < q, q' < \infty$ (see [22, Lemma 2]) and

$$S_{p,q} \circ S_{p',q'} \subset S_{s,r}, \quad \frac{1}{p} + \frac{1}{p'} = \frac{1}{s}, \quad \frac{1}{q} + \frac{1}{q'} = \frac{1}{r}.$$

Moreover, if $V \in S_{p,q}$ and $U \in S_{p',q'}$, then $\sigma_{s,r}(UV) \leq 2^{\frac{1}{s}} \sigma_{p',q'}(U) \sigma_{p,q}(V)$, (see [14, p.155]). In the case where $p = q, p' = q'$, one has the constant 1 instead of $2^{\frac{1}{s}}$ in the last inequality, [9], [15, p.128], [1, p.262].

Examples of the $S_{p,q}$ -operators are the diagonal operators D in l_2 with the diagonals (d_n) from $l_{p,q}$; in such cases we write $D = (d_n)$.

Given two complex Hilbert spaces H_1 and H_2 , we denote by $H_1 \otimes_2 H_2$ the completion of the tensor product $H_1 \otimes H_2$ with respect to the natural scalar product.

3. ON A PISIER'S RESULT

In this section we are going to prove some generalizations of the Pisier's theorem mentioned in Introduction to the cases of $S_{p,q}$ -factorizations of operators for scalar cases. Some applications are given.

Definition 3.1. An operator $T \in L(X, Y)$ is said to be (s, r) -nuclear, where $0 < r, s \leq 1$, if it admits a representation

$$Tx = \sum_{n=1}^{\infty} \mu_n \langle x'_n, x \rangle y_n, \quad \text{for } x \in X, \tag{3.1}$$

where $(x_n) \in X, (y_n) \in Y, \|x_n\| \leq 1, \|y_n\| \leq 1$ and $(\mu_n) \in l_{s,r}$. We put $\nu_{s,r}(T) := \inf \|(\mu_n)\|_{l_{s,r}}$, where the infimum is taken over all possible factorizations of T in the form (3.1).

It is clear that we can assume that μ_n are real and non-negative. With the quasi-norm $\nu_{s,r}$, the space $N_{s,r}(X, Y)$ of all (s, r) -nuclear operator from X to Y is a complete quasi-normed space.

Proposition 3.2. *If $T \in N_{s,r}(X, Y)$ ($0 < s, r \leq 1$), then T can be factored through an operator from $S_{q,v}(H)$, where $\frac{1}{q} = \frac{1}{s} - 1$ and $\frac{1}{v} = \frac{1}{r} - 1$. Moreover, $\gamma_{S_{q,v}}(T) \leq \nu_{s,r}(T)$.*

Proof. The operator $T : X \rightarrow Y$ admits the following factorization:

$$T : X \xrightarrow{W} l_\infty \xrightarrow{\Delta} l_1 \xrightarrow{V} Y, \tag{3.2}$$

where $\|V\| = \|W\| = 1$ and Δ is a diagonal operator with the diagonal $(d_n) \in l_{s,r}$. Indeed, it is enough to put $Wx := (\langle x'_k, x \rangle)$, $V(\alpha_n) := \sum \alpha_n y_n$ and $\Delta(\beta_n) := (d_n \beta_n)$ (where $d_n := \mu_n$). Rewrite the factorization (3.2) as follows:

$$T : X \xrightarrow{W} l_\infty \xrightarrow{\Delta_1} l_2 \xrightarrow{\Delta_0} l_2 \xrightarrow{\Delta_2} l_1 \xrightarrow{V} Y, \tag{3.3}$$

where $\Delta_1 := (\sqrt{n^{r/s-1} d_n^r})$, $\Delta_2 := (\sqrt{n^{r/s-1} d_n^r})$, and $\Delta_0 := (n^{1-r/s} d_n^{1-r})$.

Suppose that $\varepsilon > 0$ and in the factorization (3.2), $\|V\| = \|W\| = 1$ and $\|(d_n)\|_{l_{s,r}} \leq (1 + \varepsilon)\nu_{s,r}(T)$. Then

$$\begin{aligned} \|\Delta_2\| &= \|\Delta_1\| \leq \pi_2(\Delta_1) \leq \|\sqrt{n^{r/s-1} d_n^r}\|_{l_2} \\ &= \|n^{r/s-1} d_n^r\|_{l_1}^{\frac{1}{2}} \leq [(1 + \varepsilon)\nu_{s,r}(T)]^{r/2}. \end{aligned}$$

Also $\Delta_0 \in S_{q,v}(l_2)$, where $\frac{1}{q} = \frac{1}{s} - 1$ and $\frac{1}{v} = \frac{1}{r} - 1$. Moreover, since $\frac{1}{q} - \frac{1}{v} = \frac{1}{s} - \frac{1}{r}$, $1 - r = \frac{r}{v}$, and

$$\frac{v}{q} - 1 + v - \frac{vr}{s} = v \left(\frac{1}{s} - \frac{1}{r} + r \left(\frac{1}{r} - \frac{1}{s} \right) \right) = v(1 - r) \left(\frac{1}{s} - \frac{1}{r} \right) = r \left(\frac{1}{s} - \frac{1}{r} \right),$$

we have

$$\begin{aligned} \sigma_{q,v}(\Delta_0) &= \left(\sum n^{\frac{v}{q}-1} [n^{1-\frac{r}{s}} d_n^{1-r}]^v \right)^{\frac{1}{v}} \\ &= \left(\sum [n^{\frac{1}{s}-\frac{1}{r}} d_n]^r \right)^{\frac{1}{v}} \leq [(1 + \varepsilon)\nu_{s,r}(T)]^{\frac{r}{v}}. \quad \square \end{aligned}$$

We are going to show that this result is sharp. For this we need the following first generalization of the Pisier result, mentioned in Introduction.

Theorem 3.3. *Let $f \in C(G)$, $0 < q, s \leq 1$ and $\frac{1}{r} = \frac{1}{s} - 1$, $\frac{1}{p} = \frac{1}{q} - 1$. Consider a convolution operator $\star f : M(G) \rightarrow C(G)$. The set of Fourier coefficients \hat{f} belongs to $l_{s,q}$ if and only if the operator $\star f$ can be factored through a Lorentz-Schatten $S_{r,p}$ -operator in a Hilbert space.*

Proof. 1) Suppose there exists $U \in S_{r,p}(H)$ such that

$$\star f = AUB : M(G) \xrightarrow{B} H \xrightarrow{U} H \xrightarrow{A} C(G).$$

If $j : C(G) \hookrightarrow M(G)$ is a natural injection, then the Fourier coefficients of f are the eigenvalues of the operator $AUBj : C(G) \rightarrow M(G) \rightarrow C(G)$. Consider a diagram

$$C(G) \xhookrightarrow{j} M(G) \xrightarrow{B} H \xrightarrow{U} H \xrightarrow{A} C(G) \xhookrightarrow{j} M(G) \xrightarrow{B} H.$$

The operators $AUBj$ and $BjAU$ have the same sequences of eigenvalues. Since $B \in \Pi_2(M(G), H)$ $j : C(G) \hookrightarrow L_2(G) \hookrightarrow M(G) \in \Pi_2(C(G), M(G))$ and $U \in S_{r,p}$, we get that

$$BjAU \in S_{r,p} \circ S_1 \subset S_{s,q}, \tag{3.4}$$

where $\frac{1}{s} = 1 + \frac{1}{r}$, $\frac{1}{q} = 1 + \frac{1}{p}$. Therefore, the eigenvalues of $AUBj$ belongs to $l_{s,q}$. So $\{\hat{f}(\gamma)\} \in l_{s,q}$.

2) Suppose that $\{\hat{f}(\gamma)\} \in l_{s,q}$, where $\frac{1}{s} = 1 + \frac{1}{r}$, $\frac{1}{q} = 1 + \frac{1}{p}$. Let $\{c_{\gamma_n} = |\hat{f}(\gamma_n)|\}$ be the non-increasing rearrangement of the sequence $\{\hat{f}(\gamma)\}$. Consider the operators $B : M(G) \rightarrow L_2(G)$, $U : L_2(G) \rightarrow L_2(G)$, and $A : L_2(G) \rightarrow C(G)$, defined by

$$\begin{aligned} B\mu &:= \sum_n \hat{\mu}(\gamma_n) n^{q/(2s) - \frac{1}{2}} c_{\gamma_n}^{q/2} \gamma_n, \\ U\varphi &:= \sum_\gamma \hat{\varphi}(\gamma_n) n^{1-q/s} c_{\gamma_n}^{1-q} \gamma_n, \\ A\psi &:= \sum \hat{\psi}(\gamma_n) \operatorname{sign} \hat{f}(\gamma_n) n^{q/(2s) - \frac{1}{2}} c_{\gamma_n}^{q/2} \gamma_n \end{aligned}$$

They are well defined since the series

$$\sum \hat{\psi}(\gamma_n) \operatorname{sign} \hat{f}(\gamma_n) n^{q/(2s) - \frac{1}{2}} c_{\gamma_n}^{q/2} \gamma_n$$

is absolutely convergent and the sequence $\{n^{1-q/s} c_{\gamma_n}^{1-q}\}$ is bounded (notice that $c_{\gamma_n} = o(1/n)$ and $q - q/s \leq 0$).

We have that

$$\star f = AUB : M(G) \xrightarrow{B} l_2 \xrightarrow{U} l_2 \xrightarrow{A} C(G),$$

where A, B are bounded and U is from $S_{r,p}(l_2)$. □

Corollary 3.4. *Let $f \in C(G)$, $0 < q, s \leq 1$ and $\frac{1}{r} = \frac{1}{s} - 1$, $\frac{1}{p} = \frac{1}{q} - 1$. For the convolution operator $\star f : M(G) \rightarrow C(G)$ the following are equivalent:*

- (1) $\star f$ can be factored through a Lorentz-Schatten $S_{r,p}$ -operator;
- (2) $\hat{f} \in l_{s,q}$;
- (3) $\star f \in N_{s,q}(M(G), C(G))$.

Proof. (3) \implies (1) is valid for any (s, q) -nuclear operator (Proposition 3.2).
 (2) \implies (3). It is enough to consider the diagram

$$\star f = AUB : M(G) \xrightarrow{B} l_\infty \xrightarrow{\Delta} l_1 \xrightarrow{A} C(G),$$

where $B\mu := \{\hat{\mu}(\gamma)\}$, $\Delta\{a_\gamma\} := \{|\hat{f}(\gamma)|a_\gamma\}$, $A\{b_\gamma\} := \sum_\gamma \text{sign}\hat{f}(\gamma)b_\gamma\gamma$. \square

It follows from the above corollary that the result of Proposition 3.2 is sharp.

We now give an application to the products of two nuclear operators. A. Grothendieck [6] proved that the eigenvalue sequence of a product of two nuclear operators is absolutely summable. The following corollary shows that the result is sharp in the scale $l_{r,s}$.

Corollary 3.5. *There exist two nuclear operators T_1 and T_2 whose product has the eigenvalues in $l_1 \setminus \bigcup_{s < 1} l_{1,s}$. Namely, let $f \in C(G)$ with $\hat{f} \in l_1 \setminus \bigcup_{s < 1} l_{1,s}$, $T_1 = \star f : M(G) \rightarrow C(G)$ and $T_2 : C(G) \rightarrow M(G)$ be an inclusion. Then T_2T_1 is such an operator.*

Proof. By Corollary 3.4, the operator T_1 is nuclear. By using a natural factorization of T_1 through a diagonal operator from l_∞ to l_1 , we can represent T_1 as a product kt_1 of a nuclear operator $t_1 : M(G) \rightarrow l_1$ and a compact operator $k : l_1 \rightarrow C(G)$. Since T_2 is an integral operator, the product operator T_1k is nuclear. Thus, $T_2T_1 = (T_2k)t_1$. The complete sequence of the eigenvalues of T_2T_1 coincides with \hat{f} . \square

Remark 3.6. Of course, the assertion of the corollary follows from the results on nuclear operators described in the beginning of Introduction as well as implicitly from the Pisier’s theorem. Also, note that Corollary 3.4 gives us one more proof of the sharpness of the Grothendieck’s theorem about eigenvalues of nuclear operators as well as his factorization result (take $p = q = 1$ and f with $\hat{f} \in l_1 \setminus \bigcup_{s < 1} l_{1,s}$).

4. VECTOR-VALUED CASE

In this section we are going to give some generalizations of the Pisier’s theorem mentioned in Introduction to the cases of $S_{p,q}$ -factorizations of operators for vector-valued cases. We will generalize also a result of P. Saab [20, Theorem 4.2], where it was shown that the Pisier’s techniques in the scalar case can be extended to the vector-valued case (factorizations of a vector valued convolutions through Hilbert spaces). In the end of the section, we consider the factorizations through $S_{p,q}$ -operators linear mappings between tensor products of several Banach spaces.

Let $f \in C(G)$ and $T \in L(X, Y)$. All operators under considerations are supposed to be not identically zero.

Denote by $M(G, X)$ the Banach space of all regular Borel X -valued measures of bounded variation, $C(G, X)$ the Banach space of all continuous X -valued functions defined on G equipped with the supremum norm.

Note that

$$M(G) \otimes X \subset M(G) \widehat{\otimes} X \subset M(G, X),$$

where $\widehat{\otimes}$ is the projective tensor product.

Define the following map $T_f := T \circ \star f : M(G, X) \rightarrow C(G, Y)$ by

$$T_f(\bar{\mu})(s) = \int_G f(s - t) dT\bar{\mu}(t), \quad \bar{\mu} \in M(G, X).$$

Theorem 4.1. *Let $f \in C(G)$, $0 < r, s < \infty$. Consider a convolution operator $\star f : M(G) \rightarrow C(G)$ and an operator $T : X \rightarrow Y$. If the operator*

$$T_f : M(G, X) \rightarrow C(G, Y)$$

can be factored through an $S_{r,s}$ -operator, then the operators $\star f$ and T possess the same property. The same is true for the case where $r = s = \infty$ (or $q_1 = q_2 = 1$).

Proof. We use partially (in the first part of the proof) an idea from [20]. One may assume that $T \neq 0$ and $f \neq 0$. Let $T_f = BUA$, where

$$A \in L(M(G, X), H), \quad U \in S_{r,s}(H), \quad B \in L(H, C(G, Y)).$$

Fix a point $s_0 \in G$ for which $f(s_0) \neq 0$ and define the operators

$$\begin{aligned} i : X &\rightarrow M(G, X), & ix &= \delta_e \otimes x, \\ j : C(G, Y) &\rightarrow Y, & jh &= h(s_0)/f(s_0), \end{aligned}$$

where e is the neutral element of G . For $s \in G$

$$\begin{aligned} (T_f ix)(s) &= \int f(s - t) dT(\delta_e \otimes x)(t) \\ &= \int_G f(s - t)Tx d\delta_e(t) = f(s)Tx \in C(G, Y). \end{aligned} \tag{4.1}$$

Hence, $jT_f ix = f(s_0)Tx/f(s_0) = Tx$ or $T = jBUAi$.

Now, let $k : M(G) \rightarrow M(G, X)$ be defined by $k\mu = \mu \otimes x_0$, where x_0 is such that $\|Tx_0\| = 1$. Then

$$BUAk\mu(M(G)) =: C_1 \subset C(G, Y), \quad UAk(M(G)) = H_1 \subset H.$$

Denote by P an orthogonal projector from H onto H_1 and by R the composition Bl , where $l : H_1 \rightarrow H$ is the identity injection. We have a diagram:

$$M(G) \xrightarrow{k} M(G, X) \xrightarrow{A} H \xrightarrow{U} H \xrightarrow{P} H_1 \xrightarrow{R} C_1 \subset C(G, Y).$$

If $\mu \in M(G)$, then

$$RPUAk\mu = T_f k\mu = T_f(\mu \otimes x_0) = Tx_0 \int_G f(\cdot - t) d\mu(t) = Tx_0 f * \mu.$$

Therefore

$$C_1 = \{Tx_0 f * \mu : \mu \in M(G)\} \subset C(G) \otimes \text{span}\{Tx_0\} \subset C(G, Y).$$

Take a functional $y' \in Y^*$ with $\langle y', Tx_0 \rangle = \|y'\| = 1$ and define an operator $V : C(G) \otimes \text{span}\{Tx_0\} \rightarrow C(G)$ putting $V(h \otimes y) = h\langle y', y \rangle$ for $y \in \text{span}\{Tx_0\}$. Then, for $\mu \in M(G)$ we have that

$$VRPUAk\mu = V(f * \mu \otimes Tx_0) = f * \mu.$$

Thus, the convolution operator $\star f$ is factorized through an operator from $S_{r,s}$. □

Remark 4.2. It is clear from the proof that the condition “the operator $T_f : M(G, X) \rightarrow C(G, Y)$ ” can be replaced by the condition “the restricted operator $T_f : M(G) \hat{\otimes} X \rightarrow C(G, Y)$ ”.

Corollary 4.3. *Let $f \in C(G)$, $0 < p \leq \infty$. Consider a convolution operator $\star f : M(G) \rightarrow C(G)$ and an operator $T : X \rightarrow Y$. If the operator*

$$T_f : M(G, X) \rightarrow C(G, Y)$$

can be factored through an S_p -operator, then the operators $\star f$ and T possess the same property.

In a particular case when $X = Y$, $T = \text{id}_X$, and $p := p_1 = p_2 = \infty$, we get the result of E. Saab [20]:

Corollary 4.4. *Let $f \in C(G)$. Consider an operator $T : X \rightarrow Y$ and a convolution operator $\star f : M(G) \rightarrow C(G)$. If the operator*

$$T_f : M(G, X) \rightarrow C(G, Y)$$

can be factored through a Hilbert space then $\hat{f} \in l_1$ and $X \cong H$.

We will prove now a general theorem:

Theorem 4.5. *Let $0 < s \leq r < \infty$, $T_i \in L(X_i, Y_i)$ for $i = 1, 2, \dots, m$. If the operators T_i can be factored through the $S_{r,s}$ -operators, then the tensor product*

$$T := T_1 \otimes T_2 \otimes \dots \otimes T_m : X_1 \hat{\otimes} X_2 \hat{\otimes} \dots \hat{\otimes} X_m \rightarrow Y_1 \tilde{\otimes} Y_2 \tilde{\otimes} \dots \tilde{\otimes} Y_m$$

possesses the same property.

Proof. It is enough to consider the case of the product of two operators. Let

$$T_i : X_i \xrightarrow{A_i} H_i \xrightarrow{U_i} H \xrightarrow{B_i} Y_i$$

be the factorizations of operators T_i , $i = 1, 2$. Here

$$A_i \in L(X_i, H_i), \quad B_i \in L(H_i, Y_i), \quad U_i \in S_{s,r}(H_i), \quad i = 1, 2.$$

Let $U_i := \sum_{n=1}^{\infty} s_n^i e_n^i \otimes f_n^i$, where (e_n^i) and (f_n^i) are orthonormal systems in corresponding Hilbert spaces and $(s_n^i) \in l_{s,r}$ are the sequences of singular numbers of the operators U_i , $i = 1, 2$. Then tensor product $U_1 \otimes U_2$ has the following representation:

$$\begin{aligned} U_1 \otimes U_2(h_1 \otimes h_2) &= \sum_{n=1}^{\infty} s_n^1(h_1, e_n^1) f_n^1 \otimes \sum_{k=1}^{\infty} s_k^2(h_2, e_k^2) f_k^2 \\ &= \sum_{k,n} s_n^1 s_k^2(h_1 \otimes h_2, e_n^1 \otimes e_k^2) f_n^1 \otimes f_k^2. \end{aligned} \tag{4.2}$$

The last series is convergent in $H_1 \otimes_2 H_2$ since it follows from the conditions on r, s that, e.g., $U_1 \otimes U_2 \in S_{2r}$ (we have $\sum_{k,n} (s_n^1)^{2r} (s_k^2)^{2r} < \infty$). Therefore, the sequence $(s_n^1 s_k^2)_{k,n}$ is the sequence of all singular numbers of $U_1 \otimes U_2$. Thus,

$$U_1 \otimes U_2 = \sum_{k,n} s_n^1 s_k^2 (e_n^1 \otimes e_k^2) \otimes (f_n^1 \otimes f_k^2).$$

Since the sequences (s_n^1) and (s_k^2) belong to $l_{s,r}$ and $r \leq s$, their product $(s_n^1 s_k^2)$ is also in $l_{s,r}$ by the O’Neil’s theorem (see [13, Theorem 7.7]).

Now, consider the mappings $A_1 \otimes A_2$ and $B_1 \otimes B_2$. The first one acts from the projective tensor product $X_1 \widehat{\otimes} X_2$ to the projective tensor product $H_1 \widehat{\otimes} H_2$. The second one maps $H_1 \widetilde{\otimes} H_2$ to $Y_1 \widetilde{\otimes} Y_2$. Denoting by φ and ψ the canonical injections $H_1 \widehat{\otimes} H_2 \rightarrow H_1 \otimes_2 H_2$ and $H_1 \otimes_2 H_2 \rightarrow H_1 \widetilde{\otimes} H_2$ respectively, we obtain a factorization of $T_1 \otimes T_2$ through an $S_{r,s}$ -operator $T = B_1 \otimes B_2 \psi U_1 \otimes U_2 \varphi A_1 \otimes A_2$:

$$T : X_1 \widehat{\otimes} X_2 \xrightarrow{A_1 \otimes A_2} H_1 \widehat{\otimes} H_2 \xrightarrow{\varphi} H_1 \otimes_2 H_2 \xrightarrow{U_1 \otimes U_2} H_1 \otimes_2 H_2 \xrightarrow{B_1 \otimes B_2} Y_1 \widetilde{\otimes} Y_2. \quad \square$$

For the case where one of the space is $M(G)$ we can get a more general result. Recall that if the space X has the RN property, then

$$M(G, X) = M(G) \widehat{\otimes} X.$$

This statement is rather simple. Let $\bar{\mu} \in M(G, X)$. If $X \in RN$, then there is a function $\bar{f} \in L^1(G, |\bar{\mu}|; X)$ such that $\bar{\mu}(E) = \int_E \bar{f} d|\bar{\mu}|$ for every Borel set E . Identifying the space $L^1(G, |\bar{\mu}|; X)$ with a subspace of $M(G, X)$ in a natural way, we see that $\bar{\mu} \in L^1(G, |\bar{\mu}|; X) = L^1(G, |\bar{\mu}|) \widehat{\otimes} X$, (see [4]).

Thus, it follows from the theorem above that if $0 < s \leq r < \infty$ and $X \in RN$, then the possibility of factorization through $S_{s,r}$ -operators of the operators $\star f : M(G) \rightarrow C(G)$ and $T : X \rightarrow Y$ implies the possibility of such a factorization for the operator $T_f : M(G, X) \rightarrow C(G, Y)$. However, we can prove such a theorem without any assumption on the Banach space X .

Below we will use the following simple fact: if an operator $S : Z \rightarrow W$ in Banach spaces can be factored as

$$S : Z \xrightarrow{L} H \xrightarrow{V} H \xrightarrow{M} W$$

and $W_0 := \overline{S(Z)} \subset W$, then there is an operator $M_0 : H \rightarrow W_0$ such that S has the factorization

$$S : Z \xrightarrow{L} H \xrightarrow{V} H \xrightarrow{M_0} W_0 \xrightarrow{j} W,$$

where j is an inclusion. Indeed, consider the subspace $H_0 := \overline{VL(Z)} \subset H$, and take an orthonormal projector $P : H \rightarrow H_0$. Put $M_0 := M|_{H_0}PVL$.

Theorem 4.6. *Let $f \in C(G)$, $0 < s \leq r < \infty$. Consider the convolution operator $\star f : M(G) \rightarrow C(G)$ and an operator $T : X \rightarrow Y$. If the operators $\star f$ and T can be factored through the $S_{r,s}$ -operators then the operators*

$$T_f : M(G, X) \rightarrow C(G, Y)$$

possesses the same property.

Proof. Denote the restriction of the operator T_f onto $M(G) \widehat{\otimes} X$ by \tilde{T}_f . Then

$$M(G, X) = I(C(G), X) \text{ and } (X^* \tilde{\otimes} C(G))^* = I(C(G), X^{**}) \supset M(G, X).$$

By Theorem 4.5, the restricted operator $\tilde{T}_f : M(G) \widehat{\otimes} X \rightarrow C(G, Y)$ can be factored through a $S_{r,s}$ -operator. Then the dual operator

$$\tilde{T}_f^* : I(Y, M(G)) = (C(G) \tilde{\otimes} Y)^* \rightarrow L(X, C(G)^{**})$$

can be factored through an $S_{r,s}$ -operator, (note that we used the equality $C(G, Y) = C(G) \tilde{\otimes} Y$). But

$$Y^* \widehat{\otimes} M(G) \subset I(Y, M(G)),$$

since the space $C(G)$ and all of its duals have the metric approximation property. Therefore, \tilde{T}_f^* maps $Y^* \widehat{\otimes} M(G)$ into $X^* \tilde{\otimes} C(G)$ (apply definition of T_f) and its restriction to the first tensor product can be factored through a $S_{r,s}$ -operator.

Let τ be a restriction of \widetilde{T}_f^* to the subspace $Y^* \widehat{\otimes} M(G)$. Consider the dual operator τ^* :

$$\tau^* : I(C(G), X^{**}) \rightarrow L(M(G), Y^{**}).$$

Since $I(C(G), Z) = \Pi_1(C(G), Z)$ for any Banach space Z and the ideal Π_1 of 1-absolutely summing operators is injective, the space $M(G, X)$ can be naturally identify with a subspace of $I(C(G), X^{**})$ and the restriction of τ^* to this subspace is nothing that T_f . Hence T_f can be factored through a $S_{r,s}$ -operator. \square

Corollary 4.7. *Let $f \in C(G)$, $0 < s \leq r < \infty$. Consider the convolution operator $\star f : M(G) \rightarrow C(G)$ and operators $T_k : X_k \rightarrow Y_k$, $k = 1, \dots, n$. If the operators $\star f$ and T_k can be factored through the $S_{r,s}$ -operators, then the corresponding operator*

$$T_f : M(G, \widehat{\otimes}_{k=1}^n X_k) \rightarrow C(G, \widetilde{\otimes}_{k=1}^n Y_k)$$

possesses the same property.

Proof. Apply Theorem 4.5 to $X = \widehat{\otimes}_{k=1}^n X_k$ and to the tensor product of the operators T_k . Then apply Theorem 4.6. \square

Corollary 4.8. *Let $f_k \in C(G)$, $k = 1, \dots, n$, $0 < s \leq r < \infty$. Consider the convolution operators $\star f_k : M(G) \rightarrow C(G)$ and operators $T_k : X_k \rightarrow Y_k$, $k = 1, \dots, n$. If the operators $\star f_k$ and T_k can be factored through the $S_{r,s}$ -operators then the corresponding operator*

$$T_f : \widehat{\otimes}_{k=1}^n M(G, X_k) \rightarrow \widetilde{\otimes}_{k=1}^n C(G, Y_k)$$

possesses the same property.

Proof. By Theorem 4.6, for every k the operator

$$T_{f_k} : M(G, X_k) \rightarrow C(G, X_k)$$

can be factored through an $S_{r,s}$ -operator. Then by Theorem 4.5, the operator T_f possesses the same property. \square

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Received: June 21, 2021, accepted: August, 25, 2021.

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