

# On symmetry reduction and some classes of invariant solutions of the $(1 + 3)$ -dimensional homogeneous Monge-Ampère equation

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**Abstract.** We study the relationship between structural properties of the two-dimensional nonconjugate subalgebras of the same rank of the Lie algebra of the Poincaré group  $P(1, 4)$  and the properties of reduced equations for the  $(1 + 3)$ -dimensional homogeneous Monge-Ampère equation. In this paper, we present some of the results obtained concerning symmetry reduction of the equation under investigation to identities. Some classes of the invariant solutions (with arbitrary smooth functions) are presented.

**Анотація.** Вивчається взаємозв'язок між структурними властивостями двовимірних неспряжених підалгебр того самого рангу алгебри Лі групи Пуанкаре  $P(1, 4)$  і властивостями редукованих рівнянь для  $(1 + 3)$ -вимірного однорідного рівняння Монжа-Ампера. У цій роботі представлено деякі отримані результати, що стосуються симетрійної редукції досліджуваного рівняння до тотожностей. Наведено деякі класи інваріантних розв'язків (з довільними гладкими функціями).

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## 1. INTRODUCTION

In many cases mathematical models of the processes of real world can be written with the help of partial differential equations (PDEs). Among of these equations there are a lot of PDEs with non trivial symmetry groups. To investigate of this type equations we can use the classical Lie-Ovsiannikov method [27, 31, 30, 29] (see, also, the references therein). The application of this approach allow us, in particular, to perform the symmetry reductions of the equations under study and to construct classes of exact solutions.

However, it turned out that in order to efficiently apply the classical Lie-Ovsiannikov method for PDEs with non-trivial symmetry groups we had to solve a pure algebraic problem of describing all nonconjugate (nonsimilar) subalgebras of the Lie algebras of symmetry groups of the equations under investigation. More details on this theme can be found in [31, 30] (see, also, the references therein).

In 1975, Patera, Winternitz, and Zassenhaus [32] proposed a general method for describing the nonconjugate subalgebras of Lie algebras with nontrivial ideals.

In 1984, Grundland, Harnad, and Winternitz [20] pointed out that the reduced equations, obtained with the help of nonconjugate subalgebras of the same ranks of the Lie algebras of the symmetry groups of some PDEs, were of different types. They also investigated the similar phenomenon. The confirmation of this conclusion can be found in [7, 5, 6, 28, 9, 19, 10–14] (see, also, the references therein).

The results obtained cannot be explained using only the rank of nonconjugate subalgebras of the Lie algebras of the symmetry groups of PDEs under investigation.

To try to explain some of the differences in the properties of the reduced equations for PDEs with nontrivial symmetry groups, we recently suggested to investigate the relationship between the structural properties of nonconjugate subalgebras of the same rank of the Lie algebras of the symmetry groups of those PDEs and the properties of the corresponding reduced equations [10]. To realise of this suggestion we need to solve pure algebraic problem of classifying of all nonconjugate subalgebras of the Lie algebras of symmetry groups of the equations under investigation into classes of isomorphic ones.

A solution a lot of problems of the geometry, string theory, geometrical optics, elastic theories of shallow shells, optimal transportation, one-dimensional gas dynamics, meteorology and oceanography, etc. has reduced to the investigation of the Monge-Ampère equations in the spaces of different dimensions and different types. Some details on this theme can

be found in [33, 1, 34, 37, 38, 24, 21, 26, 39, 22, 2, 15, 40, 25, 36, 23, 35] (see, also, the references therein).

This paper is devoted to the study the relationship between structural properties of the low-dimensional ( $\dim L \leq 3$ ) nonconjugate subalgebras of the same rank of the Lie algebra of the Poincaré group  $P(1,4)$  and the properties of reduced equations for the  $(1+3)$ -dimensional homogeneous Monge-Ampère equation.

At the present time, the relationship has been investigated between the structural properties of the two-dimensional nonconjugate subalgebras of the same rank of the Lie algebra of the Poincaré group  $P(1,4)$  and the properties of reduced equations for the  $(1+3)$ -dimensional homogeneous Monge-Ampère equation.

We have obtained the following types of the reduced equations:

- identities,
- partial differential equations.

In this paper, we plan to present some of our results, which are obtained on the way of symmetry reduction of the  $(1+3)$ -dimensional homogeneous Monge-Ampère equation to identities.

## 2. LIE ALGEBRA OF THE POINCARÉ GROUP $P(1,4)$ AND ITS NONCONJUGATE SUBALGEBRAS

The group  $P(1,4)$  is a group of rotations and translations of the five-dimensional Minkowski space  $M(1,4)$ . It is the smallest group, which contains, as subgroups, the extended Galilei group  $\tilde{G}(1,3)$  [18] (the symmetry group of classical physics) and the Poincaré group  $P(1,3)$  (the symmetry group of relativistic physics).

Lie algebra of the group  $P(1,4)$  is generated by 15 bases elements

$$M_{\mu\nu} = -M_{\nu\mu}, \quad (\mu, \nu = 0, 1, 2, 3, 4), \quad P_\mu, \quad (\mu = 0, 1, 2, 3, 4),$$

which satisfy the commutation relations:

$$\begin{aligned} [P_\mu, P_\nu] &= 0, & [M_{\mu\nu}, P_\sigma] &= g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu, \\ [M_{\mu\nu}, M_{\rho\sigma}] &= g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho}, \end{aligned}$$

where  $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$ ,  $g_{\mu\nu} = 0$ , if  $\mu \neq \nu$ .

In this paper, we consider the following representation [16] of the Lie algebra of the group  $P(1,4)$ :

$$\begin{aligned} P_0 &= \frac{\partial}{\partial x_0}, & P_1 &= -\frac{\partial}{\partial x_1}, & P_2 &= -\frac{\partial}{\partial x_2}, & P_3 &= -\frac{\partial}{\partial x_3}, \\ P_4 &= -\frac{\partial}{\partial u}, & M_{\mu\nu} &= x_\mu P_\nu - x_\nu P_\mu, & x_4 &\equiv u. \end{aligned}$$

In what follows, we will use the next bases elements:

$$\begin{aligned} G &= M_{04}, & L_1 &= M_{23}, & L_2 &= -M_{13}, & L_3 &= M_{12}, \\ P_a &= M_{a4} - M_{0a}, & C_a &= M_{a4} + M_{0a}, & (a &= 1, 2, 3), \\ X_0 &= \frac{1}{2}(P_0 - P_4), & X_k &= P_k \quad (k = 1, 2, 3), & X_4 &= \frac{1}{2}(P_0 + P_4). \end{aligned}$$

Nonconjugate subalgebras of the Lie algebra of the group  $P(1, 4)$  have been described in the papers [3, 4, 17].

The Lie algebra of the extended Galilei group  $\tilde{G}(1, 3)$  is generated by the following bases elements:

$$L_1, \quad L_2, \quad L_3, \quad P_1, \quad P_2, \quad P_3, \quad X_0, \quad X_1, \quad X_2, \quad X_3, \quad X_4.$$

The classification of all nonconjugate subalgebras of the Lie algebra of the group  $P(1, 4)$  of dimensions  $\leq 3$  was performed in [8].

### 3. ON CLASSIFICATION OF SYMMETRY REDUCTIONS USING TWO-DIMENSIONAL NONCONJUGATE SUBALGEBRAS OF THE LIE ALGEBRA OF THE POINCARÉ GROUP $P(1, 4)$

In this section, we consider the homogeneous Monge-Ampère equation in the space  $M(1, 3) \times R(u)$ :

$$\det(u_{\mu\nu}) = 0,$$

where

$$u = u(x), \quad x = (x_0, x_1, x_2, x_3) \in M(1, 3), \quad u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu},$$

for  $\mu, \nu = 0, 1, 2, 3$ . Here, and in what follows,  $M(1, 3)$  is a four-dimensional Minkowski space, and  $R(u)$  is a real number axis of the depended variable  $u$ .

In 1983, Fushchich and Serov [16] studied symmetry properties and constructed some classes of exact solutions for the multidimensional Monge-Ampère equation. It follows from that paper that the Lie algebra of the symmetry group of the equation under consideration contains, as subalgebra, the Lie algebra of the group  $P(1, 4)$ . As we mentioned before, the results of the classification of all the low-dimensional ( $\dim L \leq 3$ ) nonconjugate subalgebras of the Lie algebra of the group  $P(1, 4)$  could be found in [8]. We will present below some of the results obtained.

**3.1. Lie algebras of type  $2A_1$ .** Taking into account some invariants of two-dimensional nonconjugate subalgebras, we constructed the ansatzes reducing the  $(1 + 3)$ -dimensional homogeneous Monge-Ampère equation to identities.

1)  $\langle L_3 - P_3 \rangle \oplus \langle X_4 \rangle$ .

The first step is to construct a functional basis of the invariants for this subalgebra. In order to realize it we have solved the following system of PDEs:

$$\begin{cases} (L_3 - P_3)\omega(x_0, x_1, x_2, x_3, u) = 0, \\ (X_4)\omega(x_0, x_1, x_2, x_3, u) = 0. \end{cases}$$

The solutions of this system are

$$\begin{aligned} \omega_1 &= (x_1^2 + x_2^2)^{1/2}, \\ \omega_2 &= \arctan \frac{x_1}{x_2} + \frac{x_3}{x_0 + u}, \\ \omega_3 &= x_0 + u. \end{aligned}$$

From these invariants we construct an ansatz as follows

$$x_0 + u = \varphi(\omega_1, \omega_2), \quad \omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \arctan \frac{x_1}{x_2} + \frac{x_3}{x_0 + u}.$$

The second step is to calculate the second order derivatives  $u_{\mu\nu}$  ( $\mu, \nu = 0, 1, 2, 3$ ). The third step is to substitute the derivatives into the equation under investigation ((1+3)-dimensional homogeneous Monge-Ampère equation). This substitution reduces the equation under investigation to the identity of the form  $0 = 0$ . It means that the ansatz is a solution of the following (1+3)-dimensional homogeneous Monge-Ampère equation:

$$x_0 + u = \varphi \left( \sqrt{x_1^2 + x_2^2}, \arctan \frac{x_1}{x_2} + \frac{x_3}{x_0 + u} \right),$$

where  $\varphi$  is an arbitrary smooth function.

Since the method is the same for all of the subalgebras, we will omit the interim details and provide with the final results only.

2)  $\langle L_3 + \lambda G, \lambda > 0 \rangle \oplus \langle X_3 \rangle$ .

In this case we have the following ansatz:

$$\begin{aligned} (x_1^2 + x_2^2)^{1/2} &= \varphi(\omega_1, \omega_2), \\ \omega_1 &= (x_0^2 - u^2)^{1/2}, \quad \omega_2 = \ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2}, \end{aligned}$$

and the solution of the (1+3)-dimensional homogeneous Monge-Ampère equation has the form

$$(x_1^2 + x_2^2)^{1/2} = \varphi \left( x_0^2 - u^2, \ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} \right),$$

where  $\varphi$  is an arbitrary smooth function.

3)  $\langle G \rangle \oplus \langle X_1 \rangle$ .

The ansatz is

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_2, \quad \omega_2 = x_3,$$

and the solution of the (1 + 3)-dimensional homogeneous Monge-Ampère equation has the form

$$(x_0^2 - u^2)^{1/2} = \varphi(x_2, x_3),$$

where  $\varphi$  is an arbitrary smooth function.

4)  $\langle G + \alpha X_2, \alpha > 0 \rangle \oplus \langle X_1 \rangle$ .

The ansatz is

$$x_3 = \varphi(\omega_1, \omega_2), \quad \omega_1 = (x_0^2 - u^2)^{1/2}, \quad \omega_2 = x_2 - \alpha \ln(x_0 + u),$$

and the solution of the (1 + 3)-dimensional homogeneous Monge-Ampère equation has the form

$$x_3 = \varphi(x_0^2 - u^2, x_2 - \alpha \ln(x_0 + u)),$$

where  $\varphi$  is an arbitrary smooth function.

5)  $\langle L_3 + \frac{1}{2}(P_3 + C_3) \rangle \oplus \langle X_0 + X_4 \rangle$ .

The ansatz is

$$\begin{aligned} \arctan \frac{x_1}{x_2} - \arctan \frac{x_3}{u} &= \varphi(\omega_1, \omega_2), \\ \omega_1 &= (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (u^2 + x_3^2)^{1/2}, \end{aligned}$$

and the solution of the (1 + 3)-dimensional homogeneous Monge-Ampère equation has the form

$$\arctan \frac{x_1}{x_2} - \arctan \frac{x_3}{u} = \varphi\left(\sqrt{x_1^2 + x_2^2}, \sqrt{u^2 + x_3^2}\right),$$

where  $\varphi$  is an arbitrary smooth function.

6)  $\langle L_3 + \frac{\lambda}{2}(P_3 + C_3), 0 < \lambda < 1 \rangle \oplus \langle X_0 + X_4 \rangle$ .

The ansatz is

$$\begin{aligned} \lambda \arctan \frac{x_1}{x_2} - \arctan \frac{x_3}{u} &= \varphi(\omega_1, \omega_2), \\ \omega_1 &= (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (u^2 + x_3^2)^{1/2}, \end{aligned}$$

and the solution of the (1 + 3)-dimensional homogeneous Monge-Ampère equation has the form

$$\lambda \arctan \frac{x_1}{x_2} - \arctan \frac{x_3}{u} = \varphi\left(\sqrt{x_1^2 + x_2^2}, \sqrt{u^2 + x_3^2}\right),$$

where  $\varphi$  is an arbitrary smooth function.

7)  $\langle L_3 + \alpha(X_0 + X_4), \alpha > 0 \rangle \oplus \langle X_4 \rangle$ .

The ansatz is

$$\begin{aligned} x_0 + u - \alpha \arctan \frac{x_1}{x_2} &= \varphi(\omega_1, \omega_2), \\ \omega_1 &= x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}, \end{aligned}$$

and the solution of the (1 + 3)-dimensional homogeneous Monge-Ampère equation has the form

$$u = \varphi \left( x_3, \sqrt{x_1^2 + x_2^2} \right) + \alpha \arctan \frac{x_1}{x_2} - x_0,$$

where  $\varphi$  is an arbitrary smooth function.

8)  $\langle L_3 + \alpha X_3, \alpha > 0 \rangle \oplus \langle X_0 + X_4 \rangle$ .

The ansatz is

$$x_3 + \alpha \arctan \frac{x_1}{x_2} = \varphi(\omega_1, \omega_2), \quad \omega_1 = u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$

and the solution of the (1 + 3)-dimensional homogeneous Monge-Ampère equation has the form

$$x_3 + \alpha \arctan \frac{x_1}{x_2} = \varphi \left( u, \sqrt{x_1^2 + x_2^2} \right),$$

where  $\varphi$  is an arbitrary smooth function.

9)  $\langle L_3 + 2X_4 \rangle \oplus \langle X_3 \rangle$ .

The ansatz is

$$\begin{aligned} x_0 - u + 2 \arctan \frac{x_2}{x_1} &= \varphi(\omega_1, \omega_2), \\ \omega_1 &= x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2}, \end{aligned}$$

and the solution of the (1 + 3)-dimensional homogeneous Monge-Ampère equation has the form

$$x_0 - u + 2 \arctan \frac{x_2}{x_1} = \varphi \left( x_0 + u, \sqrt{x_1^2 + x_2^2} \right),$$

where  $\varphi$  is an arbitrary smooth function.

10)  $\langle L_3 - P_3 + 2\alpha X_0, \alpha \neq 0 \rangle \oplus \langle X_4 \rangle$ .

The ansatz is

$$\begin{aligned} x_0 + u - 2\alpha \arctan \frac{x_1}{x_2} &= \varphi(\omega_1, \omega_2), \\ \omega_1 &= (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = (x_0 + u)^2 + 4\alpha x_3, \end{aligned}$$

and the solution of the (1 + 3)-dimensional homogeneous Monge-Ampère equation has the form

$$u = \varphi \left( \sqrt{x_1^2 + x_2^2}, (x_0 + u)^2 + 4\alpha x_3 \right) + 2\alpha \arctan \frac{x_1}{x_2} - x_0,$$

where  $\varphi$  is an arbitrary smooth function.

$$11) \langle P_3 \rangle \oplus \langle X_1 \rangle.$$

The ansatz is

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_2, \quad \omega_2 = x_0 + u,$$

and the solution of the (1 + 3)-dimensional homogeneous Monge-Ampère equation has the form

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(x_2, x_0 + u),$$

where  $\varphi$  is an arbitrary smooth function.

$$12) \langle P_3 - X_2 \rangle \oplus \langle X_1 \rangle.$$

The ansatz is

$$x_2 - \frac{x_3}{x_0 + u} = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0 + u, \quad \omega_2 = (x_0^2 - x_3^2 - u^2)^{1/2},$$

and the solution of the (1 + 3)-dimensional homogeneous Monge-Ampère equation has the form

$$x_2 - \frac{x_3}{x_0 + u} = \varphi(x_0 + u, \sqrt{x_0^2 - x_3^2 - u^2}),$$

where  $\varphi$  is an arbitrary smooth function.

$$13) \langle P_3 - 2X_0 \rangle \oplus \langle X_4 \rangle.$$

The ansatz is

$$(x_0 + u)^2 + 4x_3 = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_1, \quad \omega_2 = x_2,$$

and the solution of the (1 + 3)-dimensional homogeneous Monge-Ampère equation has the form

$$(x_0 + u)^2 + 4x_3 = \varphi(x_1, x_2),$$

where  $\varphi$  is an arbitrary smooth function.

$$14) \langle P_3 - 2X_0 \rangle \oplus \langle X_1 \rangle.$$

The ansatz is

$$\begin{aligned} \frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u) + x_0 - u &= \varphi(\omega_1, \omega_2), \\ \omega_1 = x_2, \quad \omega_2 &= (x_0 + u)^2 + 4x_3, \end{aligned}$$

and the solution of the  $(1 + 3)$ -dimensional homogeneous Monge-Ampère equation has the form

$$\frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u) + x_0 - u = \varphi(x_2, (x_0 + u)^2 + 4x_3),$$

where  $\varphi$  is an arbitrary smooth function.

15)  $\langle L_3 \rangle \oplus \langle X_4 \rangle$ .

The ansatz is

$$(x_1^2 + x_2^2)^{1/2} = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0 + u, \quad \omega_2 = x_3,$$

and the solution of the  $(1 + 3)$ -dimensional homogeneous Monge-Ampère equation has the form

$$(x_1^2 + x_2^2)^{1/2} = \varphi(x_0 + u, x_3),$$

where  $\varphi$  is an arbitrary smooth function.

16)  $\langle L_3 + \alpha X_3, \alpha > 0 \rangle \oplus \langle X_4 \rangle$ .

The ansatz is

$$x_3 + \alpha \arctan \frac{x_1}{x_2} = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$

and the solution of the  $(1 + 3)$ -dimensional homogeneous Monge-Ampère equation has the form

$$x_3 + \alpha \arctan \frac{x_1}{x_2} = \varphi(x_0 + u, (x_1^2 + x_2^2)^{1/2}),$$

where  $\varphi$  is an arbitrary smooth function.

17)  $\langle P_3 - X_1 \rangle \oplus \langle X_4 \rangle$ .

The ansatz is

$$x_1 - \frac{x_3}{x_0 + u} = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_2, \quad \omega_2 = x_0 + u,$$

and the solution of the  $(1 + 3)$ -dimensional homogeneous Monge-Ampère equation has the form

$$x_1 - \frac{x_3}{x_0 + u} = \varphi(x_2, x_0 + u),$$

where  $\varphi$  is an arbitrary smooth function.

**3.2. Lie algebras of type  $A_2$ .** Taking into account some invariants of two-dimensional nonconjugate subalgebras, we constructed the ansatzes, which reduced the  $(1 + 3)$ -dimensional homogeneous Monge-Ampère equation to identities.

$$1) \langle -G - \frac{1}{\lambda}L_3, X_4, \lambda > 0 \rangle.$$

The ansatz is

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_3, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$$

and the solution of the (1 + 3)-dimensional homogeneous Monge-Ampère equation has the form

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = \varphi(x_3, (x_1^2 + x_2^2)^{1/2}),$$

where  $\varphi$  is an arbitrary smooth function.

$$2) \langle -G - \alpha X_1, X_4, \alpha > 0 \rangle.$$

The ansatz is

$$x_1 - \alpha \ln(x_0 + u) = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_2, \quad \omega_2 = x_3,$$

and the solution of the (1 + 3)-dimensional homogeneous Monge-Ampère equation has the form

$$x_1 - \alpha \ln(x_0 + u) = \varphi(x_2, x_3),$$

where  $\varphi$  is an arbitrary smooth function.

$$3) \langle -\frac{1}{\lambda}(L_3 + \lambda G + \alpha X_3), X_4, \alpha > 0, \lambda > 0 \rangle.$$

The ansatz is

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = \varphi(\omega_1, \omega_2), \\ \omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = x_3 + \alpha \arctan \frac{x_1}{x_2},$$

and the solution of the (1 + 3)-dimensional homogeneous Monge-Ampère equation has the form

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = \varphi((x_1^2 + x_2^2)^{1/2}, x_3 + \alpha \arctan \frac{x_1}{x_2}),$$

where  $\varphi$  is an arbitrary smooth function.

#### 4. CONCLUSIONS

We study the relationship between the structural properties of the low-dimensional ( $\dim L \leq 3$ ) nonconjugate subalgebras of the same rank of the Lie algebra of the Poincaré group  $P(1, 4)$  and the properties of the reduced equations for the (1 + 3)-dimensional homogeneous Monge-Ampère equation.

At the present time, the relationship has been investigated between the structural properties of the two-dimensional nonconjugate subalgebras of the same rank of the Lie algebra of the group  $P(1, 4)$  and the properties of reduced equations for the (1 + 3)-dimensional homogeneous Monge-Ampère equation. We have obtained the following types of the reduced equations:

- identities,
- partial differential equations.

In this paper, we have presented some of our results, which were obtained on the way of symmetry reduction of the  $(1 + 3)$ -dimensional homogeneous Monge-Ampère equation to identities. More detailed: we provided some classes of the invariant solutions (with arbitrary smooth functions) for the equation under investigation. Those classes were divided into two subclasses: subclasses invariant under nonconjugate subalgebras of the Lie algebra of the group  $P(1, 4)$  of the type  $2A_1$  and subclasses invariant under nonconjugate subalgebras of the Lie algebra of the group  $P(1, 4)$  of the type  $A_2$ . To do that we have used the classification of all nonconjugate subalgebras of the Lie algebra of the group  $P(1, 4)$  of dimensions  $\leq 3$  which was performed in [8].

From the results obtained it follows that the reductions to identities can be obtained by using some subalgebras of the following types:  $2A_1$  and  $A_2$ .

It should be noted that the ansatzes (non-singular manifolds in the space  $M(1, 3) \times R(u)$ , invariant with respect to the corresponding subalgebras) are classes of the invariant solutions (with arbitrary smooth functions) of the  $(1 + 3)$ -dimensional homogeneous Monge-Ampère equation.

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