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On symplectic invariants of planar 3-webs

Konovenko Nadiia

Abstract. The classical web geometry [1,3,4] studies invariants of foliation families with respect to pseudogroup of diffeomorphisms. Thus for the case of planar 3-webs the basic semi invariant is the Blaschke curvature, [2]. It is also curvature of the Chern connection [4] that are naturally associated with a planar 3-web. In the present paper we investigate invariants of planar 3-webs with respect to group of symplectic diffeomorphisms. We found the basic symplectic invariants of planar 3-webs that allow us to solve the symplectic equivalence problem for planar 3-webs in general position. The Lie-Tresse theorem, [5], gives the complete description of the field of rational symplectic differential invariants of planar 3-webs. We also give normal forms for homogeneous 3-webs, i.e. 3-webs having constant basic invariants.

Анотація. В роботі вивчаються класи еквівалентності плоских 3-тканин відносно дії симплектичної псевдогрупи дифеоморфізмів площини. Знайдено поле раціональних диференціальних симплектичних інваріантів, яке далі використано для встановлення критеріїв симплектичної еквівалентності 3-тканин.

1. Normalization of 3-webs

Let $\mathbf{D} \subset \mathbb{R}^2$ be a connected and simply connected domain in the plane equipped with the symplectic structure given by the differential 2-form $\Omega = dx \wedge dy$ in the standard coordinates on the plane.

Recall that a 3-*web* in $\mathbf{D}$ is a family of three foliations being in general position. We will assume that these foliations are integral curves of differential 1-forms ω_i, $i = 1, 2, 3$, and write

$$W_3 = \langle \omega_1, \omega_2, \omega_3 \rangle ,$$

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where $\omega_i \in \Omega^1(\mathbf{D})$ are such differential 1-forms that $\omega_i \wedge \omega_j \neq 0$ in $\mathbf{D}$, if $i \neq j$.

Note that if $f_i \in \mathcal{C}^\infty(\mathbf{D})$ are smooth functions which do not vanish at all points of $\mathbf{D}$, then the differential forms $f_i\omega_i$, $i = 1, 2, 3$, define the same 3-web. The following statement is well known and is easy, but for completeness we will present a short proof.

Lemma 1.1. *There exist unique everywhere non-zero functions $f_i \in \mathcal{C}^\infty(\mathbf{D})$, $i = 1, 2, 3$, such that $a_3 > 0$ and the forms*

$$\omega_i' = f_i\omega_i, \quad i = 1, 2, 3,$$

satisfy the following identities:

$$\omega_1' + \omega_2' + \omega_3' = 0, \qquad\qquad \omega_1' \wedge \omega_2' = \Omega. \tag{1.1}$$

Proof. Since for every $x \in \mathbf{D}$ the cotangent vectors $\omega_1(x)$ and $\omega_2(x)$ are pairwise linearly independent in the contangent space $T_x^*\mathbf{D}$, one can find unique everywhere non-zero functions $a_1, a_2, b \in C^\infty(\mathbf{D})$ such that

$$\omega_3 = a_1\omega_1 + a_2\omega_2,$$
$$\omega_1 \wedge \omega_2 = b(x, y)dx \wedge dy = b\Omega.$$

Smoothness of b is evident, and its non-vanishing follows from linear independence of ω_1 and ω_2. Also smoothness of a_1 and a_2 follows from Cramer's rule giving explicit formulae for the solution of a system of linear equations. Finally, since for $i = 1, 2$ and $x \in \mathbf{D}$ the pair of cotangent vectors $\omega_3(x)$ and $\omega_i(x)$ is also linearly independent, the functions a_1 and a_2 are everywhere non-zero.

Let $\varepsilon = \operatorname{sign}(a_1a_2b) \in \{-1, 1\}$ be the sign of the product a_1a_2b. Then the functions

$$f_1 = -\frac{a_1}{\sqrt{\varepsilon a_1a_2b}}, \qquad f_2 = -\frac{a_2}{\sqrt{\varepsilon a_1a_2b}}, \qquad f_3 = \frac{1}{\sqrt{\varepsilon a_1a_2b}},$$

satisfy (1.1). In particular,

$$f_i = -a_if_3, \qquad i = 1, 2.$$

If $g_i \in \mathcal{C}^\infty(\mathbf{D})$, $i = 1, 2, 3$, is another triple of functions satisfying

$$g_3 > 0, \qquad g_1\omega_1 + g_2\omega_2 + g_3\omega_3 = 0, \qquad g_1\omega_1 \wedge g_2\omega_2 = \Omega,$$

then $\omega_3 = -\frac{g_1}{g_3}\omega_1 - \frac{g_2}{g_3}\omega_2$ and $g_1g_2b = 1$. Hence $a_i = -\frac{g_i}{g_3}$, $i = 1, 2$, $b = \frac{1}{g_1g_2}$, and therefore $a_1a_2b = \frac{1}{g_3^2} > 0$. This implies that $g_3 = f_3$ and therefore $g_i = -a_ig_3 = f_i$, $i = 1, 2$. □

Thus not changing the foliations one can assume that

$$\omega_1 + \omega_2 + \omega_3 = 0, \qquad \omega_1 \wedge \omega_2 = \Omega. \tag{1.2}$$

A triple of differential 1-forms ω_i, $i = 1, 2, 3$, satisfying (1.2) will be called *normalized.*

2. Symplectic invariants

Definition 2.1. *We say that two planar* 3*-webs* W_3 *and* $\widetilde{W}_3$ *given in domains* $\mathbf{D}$ *and* $\widetilde{\mathbf{D}}$ *respectively are* ***symplectively equivalent*** *if there is a symplectomorphism* $\phi : \mathbf{D} \to \widetilde{\mathbf{D}}$*, such that* $\phi(W_3) = \widetilde{W}_3$*.*

Proposition 2.2. *Let* $W_3 = \langle \omega_1, \omega_2, \omega_3 \rangle$ *and* $\widetilde{W}_3 = \langle \widetilde{\omega}_1, \widetilde{\omega}_2, \widetilde{\omega}_3 \rangle$ *be two planar* 3*-webs in domains* $\mathbf{D}$ *and* $\widetilde{\mathbf{D}}$ *respectively given by normalized differential forms, see* (1.2)*. Then a diffeomorphism* $\phi : \mathbf{D} \to \widetilde{\mathbf{D}}$ *establishes a symplectic equivalence of* 3*-webs if and only if*

$$\phi^* \left(\widetilde{\omega}_i \right) = \varepsilon \omega_{\sigma(i)},$$

where $(\sigma, \varepsilon) \in \mathbb{A}_3 \times \mathbb{Z}_2$*, and* $\mathbb{A}_3 \subset \mathbb{S}_3$ *is the subgroup of even permutations and* $\mathbb{Z}_2 = \{1, -1\}$*.*

Proof. This follows from uniqueness of "normalizing" functions f_i from Lemma 1.1. The second condition in (1.2) requires that the numbering $i \to \omega_i$ is defined up to even permutations and the first condition in (1.2) requires that rescaling $\omega_i \to f\omega_i$ is possible if $f = 1$ or $f = -1$. □

Recall also that the first condition in (1.2) implies the existence of the Chern connection given by the differential 1-form γ such that

$$d\omega_i = \gamma \wedge \omega_i,$$

for all $i = 1, 2, 3$.

The Chern form γ satisfies the following condition

$$d(f\omega_i) = \big(\gamma + d\ln|f|\big) \wedge f\omega_i,$$

and its exterior differential $d\gamma \in \Omega^2(\mathbf{D})$ is an invariant of planar 3-webs with respect to the diffeomorphism group.

In our case the normalization (1.2) and the above proposition shows that the Chern form γ is itself a symplectic invariant of 3-webs.

Let us write down γ in following form

$$\gamma = x_1\omega_1 + x_2\omega_2 + x_3\omega_3,$$

where the functions $x_i \in \mathcal{C}^\infty(\mathbf{D})$ are barycentric coordinates of γ, i.e.

$$x_1 + x_2 + x_3 = 1.$$

Then we have

$$
\begin{aligned}
d\omega_1 &= (x_3 - x_2)\,\omega_1 \wedge \omega_2,\\
d\omega_2 &= (x_1 - x_3)\,\omega_1 \wedge \omega_2,\\
d\omega_3 &= (x_2 - x_1)\,\omega_1 \wedge \omega_2.
\end{aligned}
$$

Using the second normalization (1.2) condition we can rewrite these relations in the following form

$$
d\omega_i = \lambda_i \Omega, \quad i = 1, 2, 3,
$$
$$
\lambda_1 = x_3 - x_2, \qquad \lambda_2 = x_1 - x_3, \qquad \lambda_3 = x_2 - x_1,
$$

and

$$
x_1 = \tfrac{1}{3}(1 + \lambda_2 - \lambda_3), \quad x_2 = \tfrac{1}{3}(1 + \lambda_3 - \lambda_1), \quad x_3 = \tfrac{1}{3}(1 + \lambda_1 - \lambda_2).
$$

Theorem 2.3. *The functions*

$$
\begin{aligned}
J_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2,\\
J_2 &= \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2,\\
J_w &= (\lambda_2^2 - \lambda_1^2)(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2),\\
J_3 &= \lambda_1^2\lambda_2^2\lambda_3^2
\end{aligned}
$$

are symplectic invariants of 3*-webs.*

Proof. The action of the cyclic group $\mathbb{A}_3$ permute λ_i while the action of group $\mathbb{Z}_2$ change sign of λ_i. Therefore, the invariants are the functions of λ_i^2, that are $\mathbb{A}_3$ invariants. Hence they are generated by symmetric polynomials and the Vandermonde polynomial. Here J_1, J_2, J_3 correspond to the first elementary symmetric functions, and J_w is the Vandermonde polynomial. □

Remark 2.4. Condition $\lambda_1+\lambda_2+\lambda_3 = 0$ implies that the following syzygies hold:

$$
J_2 = \tfrac{1}{4}J_1^2, \qquad\qquad J_w^2 = \tfrac{1}{2}J_1^3 J_3 - 27J_3^2. \tag{2.1}
$$

Definition 2.5. *We say that a planar* 3*-web* W_3 *is* in general position *in the domain* **D** *if invariants* J_1 *and* J_3 *are independent, i.e.*

$$
dJ_1 \wedge dJ_3 \neq 0
$$

everywhere in **D**.

3. Coordinates

Assume that differential 1-forms ω_i have the following form in canonical coordinates (x, y), i.e. $\Omega = dx \wedge dy$, and

$$\omega_i = a_i(x,y)dx + b_i(x,y)dy, \quad i = 1, 2, 3.$$

Then

$$\lambda_i = b_{i,x} - a_{i,y},$$
$$\gamma = (a_1\lambda_2 - a_2\lambda_1)dx + (b_1\lambda_2 - b_2\lambda_1)dy$$

and the Blaschke curvature κ, where

$$d\gamma = \kappa\Omega,$$

equals

$$\kappa = (b_1\lambda_2 - b_2\lambda_1)_x - (a_1\lambda_2 - a_2\lambda_1)_y.$$

Invariants J_1 and J_3 are differential invariants of the first order:

$$J_1 = (b_{1,x} - a_{1,y})^2 + (b_{1,x} - a_{1,y})(b_{2,x} - a_{2,y}) + (b_{2,x} - a_{2,y})^2,$$
$$J_3 = (b_{1,x} - a_{1,y})^2 (b_{2,x} - a_{2,y})^2 (b_{1,x} + b_{2,x} - a_{1,y} - a_{2,y})^2.$$

4. Homogeneous 3-webs

We say that a 3-web is *homogenous* if invariants J_1, J_3 are constant. In this case syzygy relation (2.1) shows that all λ_i are constant too.

We consider in series the following four cases:

$$C_1 = \{\lambda_1 = \lambda_2 = 0\}, \qquad C_2 = \{\lambda_1 = 0, \lambda_2 \neq 0\},$$
$$C_3 = \{\lambda_1 \neq 0, \lambda_2 = 0\}, \qquad C_4 = \{\lambda_1 \neq 0, \lambda_2 \neq 0\}.$$

Case $\mathbf{C}_1$. Here we have

$$d\omega_1 = 0, \qquad d\omega_2 = 0,$$

and therefore

$$\omega_1 = df, \qquad \omega_2 = dg,$$

for some smooth functions in the domain $\mathbf{D}$.

Then the condition $\omega_1 \wedge \omega_2 = \Omega$ implies that the Poisson bracket $[f, g]$ equals 1 everywhere: $[f, g] = 1$. Therefore, there is a symplectic transformation such that $(f, g) \to (x, y)$ and

$$\omega_1 = dx, \qquad \omega_2 = dy, \qquad \omega_3 = d(-x - y).$$

Theorem 4.1. *Any planar* 3*-web with trivial invariants*

$$J_1 = J_2 = J_3 = J_w = 0$$

can be transformed by a symplectomorphism to the 3*-web formed by the family of* 3 *parallel straight lines*

$$x = \mathsf{const}_1, \qquad y = \mathsf{const}_2, \qquad x + y = \mathsf{const}_3.$$

Case C$_2$. Here $d\omega_1 = 0$ and therefore we could find canonical coordinates (x, y) such that $\omega_1 = dx$. Moreover, the condition $\omega_1 \wedge \omega_2 = \Omega$ implies that $b_2 = 1$.

We have also that $\lambda_2 = b_{2,x} - a_{2,y} = -a_{2,y}$, whence

$$a_2 = -\lambda_2 y + a(x),$$
$$\omega_2 = (dy - \lambda_2\, y\, dx) + a(x)dx.$$

Notice that the following symplectic transformations $(x, y) \to \big(x, y+\psi(x)\big)$ transform the differential 1-form ω_2 to the form

$$(dy - \lambda_2\, y\, dx) + a(x)dx + d\psi - \lambda_2\psi dx.$$

Therefore, the transformation associated with function ψ satisfying the following differential equation

$$\psi' - \lambda_2\psi + a(x) = 0,$$

transforms the pair (ω_1, ω_2) to the pair $(dx, dy - \lambda_2 y dx)$.

Theorem 4.2. *Any planar* 3*-web with invariants*

$$J_1 = J_3 = J_w = 0, \qquad J_2 = 2\alpha^2$$

can be transformed by a symplectomorphism to the 3*-web formed by the family of curves*

$$x = \mathsf{const}_1, \qquad y = \mathsf{const}_2 \exp(\alpha x), \qquad y = \mathsf{const}_3 \exp(\alpha x) + 1.$$

Case C$_3$. Here $d\omega_2 = 0$ and therefore we could take such canonical coordinates (x, y) that $\omega_2 = dy$. Moreover, the condition $\omega_1 \wedge \omega_2 = \Omega$ gives us $a_1 = 1$.

We have also that $\lambda_1 = b_{1,x} - a_{1,y} = b_{1,x}$, and therefore

$$b_1 = \lambda_1 x + b(y),$$
$$\omega_1 = (dx + \lambda_1\, x\, dy) + b(y)dy.$$

As above, the symplectic transformations $(x, y) \to (x+\phi(y), y)$ transform the differential 1-form ω_1 to the form

$$(dx + \lambda_1\, x\, dy) + b(y)dy + d\phi + \lambda_1\phi dy.$$

Therefore this transformation, with function ϕ satisfying the following differential equation

$$\phi' + \lambda_1 \phi + b(y) = 0,$$

transforms the pair (ω_1, ω_2) to $(dx + \lambda_1\, x\, dy, dy)$, and the symplectic transformation $(x, y) \to (y, -x)$ send it back to case C_2.

Case C$_4$. Here $d\omega_1 = \lambda_1 \Omega$, $d\omega_2 = \lambda_2 \Omega$. Therefore,

$$d(\lambda_2\omega_1 - \lambda_1\omega_2) = 0, \qquad \lambda_2\omega_1 - \lambda_1\omega_2 = dx$$

in some canonical coordinates (x, y).

Thus we have the following equations for coefficients (a_1, a_2, b_1, b_2) of differential forms ω_1, ω_2:

$$b_{1,x} - a_{1,y} = \lambda_1, \qquad b_{2,x} - a_{2,y} = \lambda_2,$$
$$\lambda_2 a_1 - \lambda_1 a_2 = 1, \qquad \lambda_2 b_1 - \lambda_1 b_2 = 0,$$
$$a_1 b_2 - a_2 b_1 = 1.$$

The last three equations imply

$$b_1 = \lambda_1, b_2 = \lambda_2, \qquad a_2 = \tfrac{\lambda_2}{\lambda_1} a_1 - \tfrac{1}{\lambda_1}.$$

Therefore, we get from the first two equations that

$$a_1 = -\lambda_1 y + \alpha(x), \qquad a_2 = -\lambda_2 y + \beta(x),$$

where

$$\beta = \tfrac{\lambda_2}{\lambda_1}\alpha - \tfrac{1}{\lambda_1}.$$

Therefore, the pair (ω_1, ω_2) reduces to the form

$$\omega_1 = \lambda_1(dy - y\, dx) + \alpha dx,$$
$$\omega_2 = \lambda_2(dy - y\, dx) + \beta dx.$$

As above, by using the symplectic transformation $(x, y) \to (x, y + \phi(x))$, where function ϕ satisfies the equation $\lambda_1(\phi' - \phi) + \alpha = 0$, we transform these forms to the following

$$\omega_1 = \lambda_1(dy - y\, dx),$$
$$\omega_2 = \lambda_2(dy - y\, dx) - \tfrac{1}{\lambda_1} dx.$$

Theorem 4.3. *Any planar 3-web with constant invariants J_1, J_2, J_3, J_w could be transformed by a symplectomorphism to the 3-web formed by the family of curves*

$$y = \mathsf{const}_1 \exp(x),$$
$$y = \mathsf{const}_2 \exp(x) + \tfrac{1}{\lambda_1\lambda_2},$$
$$y = \mathsf{const}_3 \exp(x) + \tfrac{1}{\lambda_1(\lambda_1+\lambda_2)},$$

if $\lambda_1 + \lambda_2 \neq 0$*, and*

$$y = \mathsf{const}_1 \exp(x),$$
$$y = \mathsf{const}_2 \exp(x) + \tfrac{1}{\lambda_1 \lambda_2},$$
$$x = \mathsf{const}_3,$$

if $\lambda_1 + \lambda_2 = 0$.

5. General case

In addition to the above classification of homogeneous webs we consider also the singular case, when $J_3 = 0$.

Assume that $\lambda_1 = 0$. Then $d\omega_1 = 0$ so we can choose coordinates (x, y) so that $\omega_1 = dx$. Hence, the normalization conditions imply that

$$a_1 = 1, \qquad b_1 = 0, \qquad b_2 = 1, \qquad b_3 = -1,$$
$$a_2 + a_3 + 1 = 0, \qquad a_{2,y} = 0,$$

and

$$\omega_1 = dx, \qquad \omega_2 = a(x)dx + dy, \qquad \omega_2 = -(a(x) + 1)dx - dy.$$

Hence, applying, as above, the symplectic transformation

$$(x, y) \to \big(x, y - \int_0^x a(s)ds\big),$$

we get the following forms

$$\omega_1 = dx, \qquad \omega_2 = dy, \qquad \omega_3 = -dx - dy.$$

Theorem 5.1. *A planar* 3*-web could be transformed by a symplectomorphism to the* 3*-web formed by the family of* 3 *parallel straight lines*

$$x = \mathsf{const}_1, \qquad y = \mathsf{const}_2, \qquad x + y = \mathsf{const}_3$$

if and only if the value of invariant J_3 *on this web equals zero.*

This theorem shows that for 3-webs in general position, i.e. such that $dJ_1 \wedge dJ_3 \neq 0$, we have $\lambda_i \neq 0$, for all i. Assume now that in domain $\mathbf{D}$ we have $J_3 \neq 0$. Then we eliminate the action of group $\mathbb{Z}_2$ on triples $(\omega_1, \omega_2, \omega_3)$:

$$(\omega_1, \omega_2, \omega_3) \to (\pm\omega_1, \pm\omega_2, \pm\omega_3),$$

by requirement

$$\lambda_1 \lambda_2 \lambda_3 > 0.$$

The normalized triples $(\omega_1, \omega_2, \omega_3)$ that satisfy this requirement we will call *strictly normalized.*

Then strictly normalized triples $(\omega_1, \omega_2, \omega_3)$ and $(\widetilde{\omega}_1, \widetilde{\omega}_2, \widetilde{\omega}_3)$ define the same 3-web if and only if $\widetilde{\omega}_i = \omega_{\sigma(i)}$, for some even permutation $\sigma \in \mathbb{A}_3$.

Let's consider the following invariants of the $\mathbb{A}_3$ action:

$$j_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,$$
$$j_w = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1),$$
$$j_3 = \lambda_1\lambda_2\lambda_3.$$

Then, as above, we get the following syzygy:

$$j_w^2 = \tfrac{1}{2} j_3 j_2^2 - 27 j_3^2,$$

and the following proposition.

Proposition 5.2. *Two strictly normalized triples*

$$(\omega_1, \omega_2, \omega_3), \qquad (\widetilde{\omega}_1, \widetilde{\omega}_2, \widetilde{\omega}_3)$$

define the same 3*-web if and only if their invariants* j_2, j_w, j_3 *coincide.*

Definition 5.3. *We say that* 3*-web* $W_3 = \langle \omega_1, \omega_2, \omega_3 \rangle$*, defined by the strict normalized triple* $(\omega_1, \omega_2, \omega_3)$*, is* ***in general position*** *whenever*

$$dj_2 \wedge dj_3 \neq 0.$$

Let now 3-web $W_3 = \langle \omega_1, \omega_2, \omega_3 \rangle$ defined by the strict normalized triple $(\omega_1, \omega_2, \omega_3)$ in the domain $\mathbf{D}$ be in general position and let

$$\Phi_\omega : \mathbf{D} \to \mathbf{D}_\omega \subset \mathbb{R}^2(X, Y),$$
$$\Phi_\omega : (x, y) \to (X = j_2(\omega), Y = j_3(\omega))$$

be the diffeomorphism of domain $\mathbf{D}$ on a domain $\mathbf{D}_\omega$ defined by values $j_2(\omega)$, $j_3(\omega)$ of invariants j_2, j_3 on the 3-web $\langle \omega_1, \omega_2, \omega_3 \rangle$.

Let $W_\omega^I = \Phi_\omega(W_3)$ be the image of the 3-web W_3 under this diffeomorphism. We call it *invariantization* of 3-web W_3.

Summarize we get the following.

Theorem 5.4. *Suppose* 3*-webs*

$$W_3 = \langle \omega_1, \omega_2, \omega_3 \rangle, \qquad \widetilde{W}_3 = \langle \widetilde{\omega}_1, \widetilde{\omega}_2, \widetilde{\omega}_3 \rangle$$

are given in simply connected domains $\mathbf{D}$ *and* $\widetilde{\mathbf{D}}$*, where they are in general position and* $J_3(\omega) > 0, J_3(\widetilde{\omega}) > 0$*. Then these webs are equivalent with respect to group of symplectomorphisms if and only if* $j_w(\omega) j_w(\widetilde{\omega}) \geqslant 0$*, and their invariantizations coincide.*

References

[1] M. A. Akivis. Differential geometry of webs. In *Problems in geometry, Vol. 15*, Itogi Nauki i Tekhniki, pages 187–213. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1983.

[2] Wilhelm Blaschke. *Einführung in die Geometrie der Waben*. Birkhäuser Verlag, Basel-Stuttgart, 1955.
[3] Wilhelm Blaschke, Gerrit Bol. *Geometrie der Gewebe. Topologische Fragen der Differentialgeometrie*. J. W. Edwards, Ann Arbor, Michigan, 1944.
[4] Shiing Shen Chern. Web geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(1):1–8, 1982, doi: 10.1090/S0273-0979-1982-14955-2.
[5] Boris Kruglikov, Valentin Lychagin. *Global Lie-Tresse theorem*, volume 22. 2016, doi: 10.1007/s00029-015-0220-z.

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Konovenko Nadiia
ONTU, Odesa, Ukraine
Email: ngkonovenko@gmail.com
ORCID: 0000-0002-8631-0688