

# On homotopy nilpotency of some suspended spaces

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**Abstract.** A homological criterium from [14] is applied to investigate the homotopy nilpotency of some suspended spaces. We investigate the homotopy nilpotency of the wedge sum and smash products of Moore spaces  $M(A, n)$  with  $n \geq 1$ . The homotopy nilpotency of homological spheres are studied as well.

**Анотація.** В роботі використовується гомологічний критерій з [14] для дослідження гомотопічної нільпотентності надбудов деякого класу просторів. Досліджується гомотопічна нільпотентність букетів та смеш-добутків просторів Мура  $M(A, n)$ ,  $n \geq 1$ . Також вивчається гомотопічна нільпотентність гомологічних сфер.

## INTRODUCTION

In the group theory, if we consider only nilpotent groups, the nilpotency class is the one which measures a distance from commutativity. Already G. Whitehead [26] had the insight that the (J.H.C.) Whitehead products satisfy identities which reflect commutator identities for groups. Berstein and Ganea [3] adapted the nilpotency to  $H$ -spaces as follows. Let  $X$  be an  $H$ -space,  $\varphi_{X,1} = \text{id}_X$ , and  $\varphi_{X,2} : X^2 \rightarrow X$  the commutator map. Put  $\varphi_{X,n+1} = \varphi_{X,1} \circ (\text{id}_X \times \varphi_{X,n})$  for  $(n+1)$ -fold commutator map of  $X$  with  $n \geq 2$ . An  $H$ -space  $X$  is called homotopy nilpotent of class  $n$  if  $\varphi_{X,n+1} \simeq *$ , is nullhomotopic but  $\varphi_{X,n}$  is not, [3]. In this case, we write  $\text{nil } X = n$ .

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*Ключові слова:* абелева група, декартовий добуток, гомологічна сфера, клас нільпотентності, відображення  $n$ -кратного комутування,  $H$ -простір, простір петель, простір Мура, локалізація, добуток Самельсона, смеш-добуток, надбудова, букет просторів, добуток Уайтхеда

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Then, Bernstein and Ganea [3] introduced a concept of the homotopy nilpotency of a pointed space by means of its loop space. In particular, the  $m$ -iterated Samelson products vanish in the loop space  $\Omega(X)$ , or equivalently, the  $m$ -iterated Whitehead products vanish in  $X$  provided  $m > \text{nil}\Omega(X)$ . We refer to [27, Chapter X] for details on Samelson and Whitehead products.

The homotopy nilpotency classes  $\text{nil} X$  of associative  $H$ -spaces  $X$  has been extensively studied as well as their homotopy commutativity. Work of Hopkins [17] drew renewed attention to such problems by relating this classical nilpotency notion with the nilpotence theorem of Devinatz, Hopkins, and Smith [6]. Although many results on the homotopy nilpotency have been obtained, the homotopy nilpotency classes have been determined in very few cases.

It is well-known that for the loop space  $\Omega(\mathbb{S}^m)$  of the  $m$ -sphere  $\mathbb{S}^m$  we have  $\text{nil}\Omega(\mathbb{S}^m) = 1$  if and only if  $m = 1, 3, 7$  and

$$\text{nil}\Omega(\mathbb{S}^m) = \begin{cases} 2 & \text{for odd } m \text{ and } m \neq 1, 3, 7 \text{ or } m = 2, \\ 3 & \text{for even } m \geq 4. \end{cases}$$

Write  $\mathbb{K}P^m$  for the projective  $m$ -space for  $\mathbb{K} = \mathbb{R}, \mathbb{C}$ , the field of reals or complex numbers and  $\mathbb{H}$  for the skew  $\mathbb{R}$ -algebra of quaternions. Then, the homotopy nilpotency of  $\Omega(\mathbb{K}P^m)$  has been first studied by Ganea [10], Snaith [24] and then their  $p$ -localization  $\Omega((\mathbb{K}P^m)_{(p)})$  by Meier [19]. The homotopy nilpotency of the loop spaces of Grassmann and Stiefel manifolds, and their  $p$ -localization have been extensively studied in [13].

Let  $\mathbb{S}_{(p)}^{2m-1}$  be the  $p$ -localization of the sphere  $\mathbb{S}^{2m-1}$  at a prime  $p$ . The main result of the paper [12] is the explicit determination of the homotopy nilpotence class of a wide range of homotopy associative multiplications on localized spheres  $\mathbb{S}_{(p)}^{2m-1}$  for  $p > 3$ . Furthermore, the paper [11] reviews known and states some new results on the homotopy nilpotency and coinpotency of spaces.

The article [14] grew out of our desire to develop techniques in the study of the homotopy nilpotency classes of Moore spaces  $M(A, n)$  for  $n \geq 1$ . In this partially survey paper [14, Proposition 3.2] is applied to investigate the homotopy nilpotency of the wedge sum and smash products of Moore spaces  $M(A, n)$  with  $n \geq 1$  and the homotopy nilpotency of homological spheres as well. To have kept all results together, we decided to include the main part of [14] here.

In Section 1 we set stages for developments to come. This introductory section is devoted to a general discussion and establishes notations on the homotopy nilpotency of  $H$ -spaces used in the rest of the paper. Given a space  $X$ , we consider the iterated Samelson product  $s_k : X^{\wedge k} \rightarrow \Omega\Sigma(X)$  to

show in Proposition 1.11 (see [14, Proposition 3.2]) that the space  $\Omega\Sigma(X)$  is not homotopy nilpotent provided the homology  $\tilde{H}_*(X, \mathbb{F})$  has at least two primitive generators, where  $\mathbb{F}$  is a field.

Section 2 first takes up the systematic study of the homotopy nilpotency of Moore spaces  $M(A, n)$  for  $n \geq 1$  already presented in [14]. Then, we generalize [14, Theorem 3.8] and state the main result

**Theorem 2.5.** *Let  $m \geq 1$ ,  $n_1, \dots, n_m \geq 2$  and  $M(A_k, n_k)$  be Moore spaces of type  $(A_k, n_k)$  for  $k = 1, \dots, m$ . Then:*

- (1)  $\text{nil}\Omega((M(A_1, n_1) \times \dots \times M(A_m, n_m))) < \infty$  if and only if  $A_k$  are torsion-free groups with rank  $r(A_k) = 1$  for  $k = 1, \dots, m$ ;
- (2)  $\text{nil}\Omega((M(A_1, n_1) \vee \dots \vee M(A_m, n_m))) < \infty$  if and only if  $m = 1$  and  $A_1$  is a torsion-free group with rank  $r(A_1) = 1$ .

Unfortunately, Moore spaces of type  $(A, 1)$  are not determined uniquely (up to homotopy) by an abelian group  $A$ . We present constructions and analyse the homotopy nilpotency of some such spaces and their wedge sums.

Next, we study the homotopy nilpotency of the smash products of Moore spaces and state

**Proposition 2.8.** *If  $m, n \geq 1$  then*

$$\text{nil}\Omega(M(A, m) \wedge M(B, n)) < \infty$$

*if and only if the group  $A \otimes B$  is torsion free with the rank  $r(A \otimes B) \leq 1$  and  $\text{Tor}(A, B) = 0$ .*

At the end, the homotopy nilpotency of homology spheres and their suspensions are taken into account as well.

## 1. PREREQUISITES

All spaces and maps in this note are assumed to be connected and based with the homotopy type of  $CW$ -complexes unless we assume otherwise. We also do not distinguish notationally between a continuous map and its homotopy class. We write  $\Omega(X)$  (resp.  $\Sigma(X)$ ) for the loop (resp. suspension) space on a space  $X$  and  $[X, Y]$  for the set of homotopy classes of maps  $X \rightarrow Y$ .

Given a space  $X$ , we use the customary notations  $X \vee X$  and  $X \wedge X$  for the *wedge* and the *smash square* of  $X$ , respectively. Recall that an  $H$ -space is a pair  $(X, \mu)$ , where  $X$  is a space and  $\mu : X \times X \rightarrow X$  is a map such that the diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{\mu} & X \\ \uparrow & \nearrow \Delta & \\ X \vee X & & \end{array}$$

commutes up to homotopy, where  $\nabla : X \vee X \rightarrow X$  is the folding map. We call  $\mu$  a *multiplication* or an *H-structure* for  $X$ . Two examples of *H-spaces* come in mind: topological groups and the loop spaces  $\Omega(X)$ . In the sequel, we identify an *H-space*  $(X, \mu)$  with the space  $X$ .

An *H-space*  $X$  is called a *group-like space* if  $X$  satisfies all the axioms of groups up to homotopy. Recall that a homotopy associative *H-CW-complex* always has a homotopy inverse. More precisely, according to [28, 1.3.2. Corollary] (see also [1, Proposition 8.4.4]), we have

**Proposition 1.1.** *If  $X$  is a homotopy associative H-CW-complex then  $X$  is a group-like space.*

From now on, we assume that any *H-space*  $X$  is group-like.

Given spaces  $X_1, \dots, X_n$ , we use the customary notations  $X_1 \times \dots \times X_n$  for their Cartesian product and  $T_m(X_1, \dots, X_n)$  for the subspace of  $X_1 \times \dots \times X_n$  consisting of those points with at least  $m$  coordinates at base points with  $m = 0, 1, \dots, n$ . Then,  $T_0(X_1, \dots, X_n) = X_1 \times \dots \times X_n$ ,  $T_1(X_1, \dots, X_n)$  is the so called the *fat wedge* of spaces  $X_1, \dots, X_n$  and  $T_{n-1}(X_1, \dots, X_n) = X_1 \vee \dots \vee X_n$ , the wedge sum of spaces  $X_1, \dots, X_n$ . We write

$$j_m(X_1, \dots, X_n) : T_m(X_1, \dots, X_n) \rightarrow X_1 \times \dots \times X_n$$

for the inclusion map with  $m = 0, 1, \dots, n$  and

$$X_1 \wedge \dots \wedge X_n = X_1 \times \dots \times X_n / T_1(X_1, \dots, X_n)$$

for the *smash product* of spaces  $X_1, \dots, X_n$ .

Let  $f_m : (X_m, \star_m) \rightarrow (Y_m, \star_m)$  be continuous maps of pointed topological spaces for  $m = 1, \dots, n$ . The map

$$f_1 \times \dots \times f_n : (X_1 \times \dots \times X_n, (\star_1, \dots, \star_n)) \rightarrow (Y_1 \times \dots \times Y_n, (\star_1, \dots, \star_n))$$

sends each point  $(x_1, \dots, x_n) \in X_1 \times \dots \times X_n$  to  $(f_1(x_1), \dots, f_n(x_n))$  and restricts to maps

$$T_m(f_1, \dots, f_n) : T_m(X_1, \dots, X_n) \rightarrow T_m(Y_1, \dots, Y_n)$$

with  $m = 0, 1, \dots, n$ . If  $X_m = X$  and  $f_m = f$  for  $m = 1, \dots, n$  then we write

$$\begin{aligned} X^n &= X_1 \times \dots \times X_n, & X^{\wedge n} &= X_1 \wedge \dots \wedge X_n, \\ f^n &= f_1 \times \dots \times f_n, & f^{\wedge n} &= f_1 \wedge \dots \wedge f_n. \end{aligned}$$

The identity map of a space  $X$  involved is consistently denoted by  $\iota_X$ .

Given an *H-group*  $X$ , the functor  $[-, X]$  takes its values in the category of groups. One may then ask when those functors take their values in

various subcategories of groups. For example,  $X$  is homotopy commutative if and only if  $[Y, X]$  is abelian for all  $Y$ .

Given an  $H$ -space  $X$ , we write  $\varphi_{X,1} = \iota_X$ ,  $\varphi_{X,2} : X^2 \rightarrow X$  for the basic commutator map and  $\varphi_{X,n+1} = \varphi_{X,2} \circ (\varphi_{X,n} \times \iota_X)$  for  $n \geq 2$ .

**The nilpotency class.** The *nilpotency class*  $\text{nil}(X, \mu)$  of an  $H$ -space  $(X, \mu)$  is the least integer  $n \geq 0$  for which the map  $\varphi_{X,n+1} \simeq *$  is nullhomotopic and we call the homotopy associative  $H$ -space  $X$  homotopy nilpotent. If no such integer exists, we put  $\text{nil}(X, \mu) = \infty$ . In the sequel, we simply write  $\text{nil} X$  for the nilpotency class of an  $H$ -space  $X$ . Thus,  $\text{nil} X = 0$  if and only if  $X$  is contractible and, as is easily seen,  $\text{nil} X \leq 1$  if and only if  $X$  is homotopy commutative.

It is obvious that for homotopy nilpotent group-like spaces  $X_1, \dots, X_m$ , we have

$$\text{nil}(X_1 \times \dots \times X_m) = \max\{\text{nil} X_1, \dots, \text{nil} X_m\}. \tag{1.1}$$

The set  $\pi_0(X)$  of all path-components of an  $H$ -space  $X$  is known to be a group. The following result is easy to prove

**Lemma 1.2.** *If  $X$  is an  $H$ -space and the path component of the base-point  $\star \in X$  is contractible then  $\text{nil} \pi_0(X) = \text{nil} X$ .*

The definition of the nilpotency classes may be extended to maps. The nilpotency class  $\text{nil} f$  of an  $H$ -map  $f : X_1 \rightarrow X_2$  is the least integer  $n \geq 0$  for which the map  $f \circ \varphi_{X,n+1} : X_1^{n+1} \rightarrow X_2$  is nullhomotopic. If no such integer exists, we put  $\text{nil} f = \infty$ .

In the sequel we need the following

**Lemma 1.3.** *If  $X$  is an  $H$ -space then the composite map*

$$T_1(X, \dots, X) \xrightarrow{j_1(X, \dots, X)} X^n \xrightarrow{\varphi_{X,n}} X$$

*is nullhomotopic.*

Since the space  $X^{\wedge n}$ , the  $n$ -th smash power of  $X$ , is the homotopy cofiber of the map

$$j_1(X, \dots, X) : T_1(X, \dots, X) \rightarrow X^n,$$

the result above implies an existence of a map  $\bar{\varphi}_{X,n} : X^{\wedge n} \rightarrow X$  for  $n \geq 1$  with  $\bar{\varphi}_{1,X} = \varphi_{X,1}$ .

It is well known that the quotient map  $X^n \rightarrow X^{\wedge n}$  has a right homotopy inverse after suspending for  $n \geq 1$ , and the fact that  $X$  is an  $H$ -space means that the suspension map  $[Y, X] \rightarrow [\Sigma Y, \Sigma X]$  is a monomorphism for any space  $Y$ . Thus, we may state

**Proposition 1.4.** *Let  $X$  be an  $H$ -space. Then  $\varphi_{X,n} \simeq *$  if and only if  $\bar{\varphi}_{X,n} \simeq *$  for  $n \geq 1$ .*

Hence [3, 2.7. Theorem] and Proposition 1.4 lead to

**Theorem 1.5.** *If  $X$  is an  $H$ -space then*

$$\text{nil } X = \sup_m \text{nil } [X^m, X] = \sup_m \text{nil } [X^{\wedge m}, X] = \sup_Y \text{nil } [Y, X],$$

where  $m$  ranges over all integers and  $Y$  over all topological spaces.

Furthermore, in view of [28, Lemma 2.6.1], we may state

**Corollary 1.6.** *A connected  $H$ -space  $X$  is homotopy nilpotent if and only if the functor  $[-, X]$  on the category of all spaces is nilpotent group valued.*

**Proof.** Certainly, the homotopy nilpotency of a connected associative  $H$ -space  $X$  implies that the functor  $[-, X]$  on the category of all pointed spaces is nilpotent group valued.

Now, suppose that the functor  $[-, X]$  is nilpotent groups valued and  $\text{nil } [\prod_1^\infty X, X] < n$ . Then, for the projection map  $\prod_1^\infty X \rightarrow X^n$  on the first  $n$  factors, the composite map

$$\prod_1^\infty X \rightarrow X^n \xrightarrow{\varphi_{X,n}} X$$

is nullhomotopic. Since, the projection  $\prod_1^\infty X \rightarrow X^n$  has a retraction, we deduce that the map  $\varphi_{X,n} : X^n \rightarrow X$  is also nullhomotopic and the proof is complete.  $\square$

Further, notice that for a map  $f : X \rightarrow Y$  of  $H$ -spaces, we have the commutative (up to homotopy) diagram

$$\begin{array}{ccc} X^{\wedge n} & \xrightarrow{\bar{\varphi}_{X,n}} & X \\ f^{\wedge n} \downarrow & & \downarrow f \\ Y^{\wedge n} & \xrightarrow{\bar{\varphi}_{Y,n}} & Y \end{array}$$

with  $n \geq 1$ . This yields

**Remark 1.7.** If  $X'$  is an  $H$ -subspace of an  $H$ -space  $X$  and  $r : X \rightarrow X'$  its a homotopy  $H$ -retract then one can easily derive from the above that

$$\text{nil } X' \leq \text{nil } X.$$

**Homotopy nilpotency of spaces.** With any based path-connected space  $X$ , we associate the integer  $\text{nil } \Omega(X)$  called the *nilpotency class* of  $X$ . Evidently,  $\text{nil } \pi_1(X) \leq \text{nil } \Omega(X)$ . We give an extension of this result involving Whitehead products, generally denoted by  $[\alpha_1, \alpha_2] \in \pi_{m_1+m_2-1}(X)$  if  $\alpha_i \in \pi_{m_i}(X)$  for  $m_i \geq 1$  with  $i = 1, 2$ . Define  $(n+1)$ -fold Whitehead products  $[\alpha_1, \dots, \alpha_{n+1}]$  as  $[[\alpha_1, \dots, \alpha_n], \alpha_{n+1}]$  if  $\alpha_i \in \pi_{m_i}(X)$  for  $m_i \geq 1$  with  $i = 1, \dots, n+1$  agreeing that, for  $n = 0$ ,  $[\alpha] = \alpha$ .

Recall that W-length  $X$ , the *Whitehead length* of a space  $X$  is the least integer  $n \geq 0$  such that  $[\alpha_1, \dots, \alpha_{n+1}] = 0$  for all  $\alpha_i \in \pi_{m_i}(X)$ ,  $m_i \geq 1$ ; if no such integer exists, we put W-length  $X = \infty$ .

Then, according to [3, 4.6. Theorem], we have

**Theorem 1.8.** W-length  $X \leq \text{nil } \Omega(X)$ .

**Example 1.9.** (1) It is well-known that

$$\text{W-length } \mathbb{S}^n = \text{nil } \Omega(\mathbb{S}^n) = \begin{cases} 3, & \text{for } n \text{ even with } n \neq 2, \\ 2, & \text{for } n \text{ odd with } n \neq 1, 3, 7 \text{ or } n = 2, \\ 1, & \text{for } n = 1, 3, 7. \end{cases}$$

(2) For the wedge  $\mathbb{S}^m \vee \mathbb{S}^n$  of two spheres with  $m, n \geq 2$ , there is an iterated nontrivial Whitehead product of any length. Therefore, by Theorem 1.8, we conclude that

$$\text{nil } \Omega(\mathbb{S}^m \vee \mathbb{S}^n) = \infty. \quad (1.2)$$

The concept of a nilpotent space is due to Dror [8]. Recall that a pointed path-connected space  $X$  is said to be *nilpotent* if its fundamental group  $\pi_1(X)$  acts nilpotently on the higher homotopy groups  $\pi_n(X)$  for  $n \geq 1$ . But, the action of  $\pi_1(X)$  on  $\pi_n(X)$  for  $n \geq 1$  may be written in terms of Whitehead products. Therefore, the nilpotency of a space  $X$  is a lower bound of its homotopy nilpotency  $\text{nil } \Omega(X)$ . Hence, by Theorem 1.8, the space  $X$  is nilpotent if  $\Omega(X)$  is homotopy nilpotent. But, by Example 1.9(2), not every space  $\Omega(X)$  is homotopy nilpotent if  $X$  is nilpotent or even simply connected.

Dror [8] has also published a far-reaching generalization of a classical theorem of J. H. C. Whitehead useful in the next section.

**Theorem 1.10.** *If  $f : X \rightarrow Y$  is a map of connected, pointed, CW-complexes which induces an isomorphism on integral homology, and if  $X$  and  $Y$  are nilpotent spaces, then  $f$  is a homotopy equivalence.*

To formulate the next result needed in the next section, let us fix first some notations and recall a definition. Given a pointed space  $X$ , write

$i_1, i_2 : X \rightarrow X \times X$  for the canonical embedding maps and  $\Delta : X \rightarrow X \times X$  for the diagonal map. If  $H_m(X, A)$  is the  $m$ -th homology group of  $X$  with coefficient in an abelian group  $A$  then an element  $\alpha \in H_m(X, A)$  is said to be *primitive* if  $\Delta_*(\alpha) = i_{1*}(\alpha) + i_{2*}(\alpha)$  for the induced homomorphisms

$$i_{1*}, i_{2*}, \Delta_* : H_m(X, A) \rightarrow H_m(X \times X, A).$$

Now, we show that the space  $\Omega\Sigma(X)$  is not homotopy nilpotent provided the homology  $H_*(X, \mathbb{F})$  has at least two primitive generators, where  $\mathbb{F}$  is a field.

**Proposition 1.11.** *If  $\tilde{H}_*(X, \mathbb{F})$  has at least two primitive generators, where  $\mathbb{F}$  is a field then  $\Omega\Sigma(X)$  is not homotopy nilpotent.*

**Proof.** To see this, take homology  $H_*(X, \mathbb{F})$  with  $\mathbb{F}$ -coefficients and recall that the Bott-Samelson theorem, [4], states that

$$H_*(\Omega\Sigma X, \mathbb{F}) \approx T(\tilde{H}_*(X, \mathbb{F})),$$

the tensor algebra on  $\tilde{H}_*(X, \mathbb{F})$ . This may be rewritten as  $UL\langle \tilde{H}_*(X, \mathbb{F}) \rangle$ , where  $L\langle \tilde{H}_*(X, \mathbb{F}) \rangle$  is the free Lie algebra generated by  $\tilde{H}_*(X, \mathbb{F})$  and  $UL\langle \tilde{H}_*(X, \mathbb{F}) \rangle$  is its universal enveloping algebra. Further, the suspension  $E : X \rightarrow \Omega\Sigma X$  induces the inclusion of the generating set in homology. Now, consider the iterated Samelson product

$$s_k : X^{\wedge k} \longrightarrow \Omega\Sigma(X),$$

where  $s_1 = E, s_{k+1} = \langle E, s_k \rangle$ , the Samelson product of  $s_k$  and  $E$  for  $n \geq 2$ . For any  $x \in \tilde{H}_*(X)$  we have  $E_*(x) = x$ .

Recall that the effect of the Samelson product map on homology classes of loop spaces is presented e.g., in [27, Chapter X, Section 6]. If  $x_1, \dots, x_k \in \tilde{H}_*(X, \mathbb{F})$  are primitive then the class

$$x_1 \otimes \dots \otimes x_k \in \tilde{H}_*(X, \mathbb{F})^{\otimes k} \approx \tilde{H}_*(X^{\wedge k}, \mathbb{F})$$

is sent by  $(s_k)_*$  to the iterated bracket  $[x_1, [x_2, \dots [x_{k-1}, x_k]] \dots]$ . In particular, if  $x, y \in \tilde{H}_*(X, \mathbb{F})$  are distinct primitive generators then the class

$$[x, [x, \dots [x, y]] \dots] \in UL\tilde{H}_*(X, \mathbb{F}) \approx \tilde{H}_*(\Omega\Sigma(X), \mathbb{F})$$

is in the image of  $(s_k)_*$ . Hence,  $s_k$  cannot be nullhomotopic.

Notice that for the map  $s_k : X^{\wedge k} \longrightarrow \Omega\Sigma(X)$  defined above, there is the factorization

$$\begin{array}{ccc} X^{\wedge k} & \xrightarrow{s_k} & \Omega\Sigma(X) \\ & \searrow & \nearrow \varphi_{\Omega\Sigma(X), k} \\ & & (\Omega\Sigma(X))^{\wedge k}. \end{array}$$

Consequently, the map  $\bar{\varphi}_{\Omega\Sigma X,k}$  is not nullhomotopic provided the map  $s_k$  is so and the proof is complete.  $\square$

Notice that, in view of (1.1) and

$$\tilde{H}_*(X_1 \vee \cdots \vee X_m, \mathbb{F}) = \tilde{H}_*(X_1, \mathbb{F}) \oplus \cdots \oplus \tilde{H}_*(X_m, \mathbb{F})$$

for a field  $\mathbb{F}$ , Proposition 1.11 yields to the following

**Corollary 1.12.** *If  $\tilde{H}_*(X_k, \mathbb{F})$  has at least two primitive generators for some  $k$  with  $k = 1, \dots, m$  then*

$$\begin{aligned} \text{nil}(\Omega\Sigma(X_1) \times \cdots \times \Omega\Sigma(X_m)) &= \infty, \\ \text{nil}(\Omega(\Sigma(X_1) \vee \cdots \vee \Sigma(X_m))) &= \infty. \end{aligned}$$

## 2. MOORE SPACES

We take up the systematic study of the homotopy nilpotency of Moore spaces  $M(A, n)$  for  $n \geq 1$ .

In Eckmann-Hilton duality, Moore  $CW$ -complexes play the role of dual objects of Eilenberg-MacLane  $CW$ -complexes. Let  $A$  be an abelian group and  $n$  any integer  $\geq 1$ . A space  $X$  (if one such exists) satisfying  $\pi_j(X) = 0$  for  $j < n$ ,  $\pi_n(X) \approx A$  and  $H_i(X) = 0$  for  $i > n$  is known as a *Moore space* of type  $(A, n)$ , or simply an  $M(A, n)$  space. By [21], it is known that a  $CW$ -complex Moore space  $M(A, n)$  with  $n \geq 2$  exists and, in view of [15, Example 4.34], the homotopy type of a Moore space  $M(A, n)$  is uniquely determined by  $A$  and  $n \geq 2$ . This implies that every Moore space  $M(A, n)$  with  $n \geq 3$ , is the suspension  $\Sigma M(A, n - 1)$ . Furthermore, in [2, Section 2], it was shown that also  $M(A, 2)$  is a suspension  $\Sigma L(A)$  for some  $CW$ -complex  $L(A)$ .

By means of [25, Proposition 1.1], there exists an  $M(A, 1)$  space if and only if the homology  $H_2(A, \mathbb{Z}) = 0$  for the ring  $\mathbb{Z}$  of integers. Recall that, by [20, Theorem 3],  $H_2(A, \mathbb{Z}) = 0$  if and only if  $A \otimes A$  coincides with its subgroup generated by the diagonal,  $\{a \otimes a; a \in A\}$ .

It is well-known that any abelian group homomorphism yields a map

$$\tilde{\varphi} : M(A_1, n) \rightarrow M(A_2, n)$$

with  $H_n(\tilde{\varphi}) = \varphi$  provided  $n \geq 2$ . We shown such a result for  $n = 1$  as well.

**Proposition 2.1.** *Let  $X_1, X_2$  be Moore spaces of types  $(A_1, 1), (A_2, 1)$  constructed in [25] and  $\varphi : A_1 \rightarrow A_2$  an abelian group homomorphism. Then, there is a map  $\tilde{\varphi} : X_1 \rightarrow X_2$  with  $H_1(\tilde{\varphi}) = \varphi$ . If the spaces  $X_1, X_2$  are nilpotent then  $\tilde{\varphi} : X_1 \rightarrow X_2$  is a homotopy equivalence.*

*In particular, if  $X_1, X_2$  are Moore spaces of type  $(A, 1)$  constructed in [25] then there is a map  $\text{id}_A : X_1 \rightarrow X_2$  with  $H_1(\text{id}_A) = \text{id}_A$ .*

**Proof.** For any space  $X$ , let  $S(X)$  denote the subgroup of  $H_2(X, \mathbb{Z})$  consisting of the spherical homology classes. Next, given an abelian group  $A$  with  $H_2(A, \mathbb{Z})$ , consider the Eilenberg-MacLane  $CW$ -complex  $K(A, 1)$ . Then, by [25],

$$H_2(K(A, 1)^{(2)}, \mathbb{Z}) = S(K(A, 1)^{(2)})$$

for the 2-skeleton  $K(A, 1)^{(2)}$  of  $K(A, 1)$ . Since  $H_2(K(A, 1)^{(2)}, \mathbb{Z})$  is a free abelian group with a basis  $\{a_\mu\}_{\mu \in I}$ , there exist a map  $f_\mu : \mathbb{S}^2 \rightarrow K(A, 1)^{(2)}$  such that  $f_{\mu*}(\gamma) = a_\mu$  for

$$(f_\mu)_* : H_2(\mathbb{S}^2, \mathbb{Z}) \rightarrow H_2(K(A, 1)^{(2)}, \mathbb{Z})$$

and a generator  $\gamma \in H_2(\mathbb{S}^2, \mathbb{Z})$ . The  $CW$ -complex  $M(A, 1)$  is obtained by attaching 3-cells to  $K(A, 1)^{(2)}$  by means of the maps  $f_\mu : \mathbb{S}^2 \rightarrow K(A, 1)^{(2)}$  for  $\mu \in I$ . Consequently, we have the pushout diagram

$$\begin{array}{ccc} \bigvee_{\mu \in I} \mathbb{S}^2 & \longrightarrow & K(A, 1)^{(2)} \\ \downarrow & & \downarrow \\ \bigvee_{\mu \in I} \mathbb{D}^3 & \dashrightarrow & X \end{array}$$

with  $X$  a Moore space of type  $(A, 1)$ .

Now, given a group homomorphism  $\varphi : A_1 \rightarrow A_2$ , the induced cellular map  $\bar{\varphi} : K(A_1, 1) \rightarrow K(A_2, 1)$  restricts to

$$\bar{\varphi}^{(2)} : K(A_1, 1)^{(2)} \rightarrow K(A_2, 1)^{(2)}.$$

We show an existence of the induced map  $\tilde{\varphi} : X_1 \rightarrow X_2$  of corresponding Moore spaces  $X_1, X_2$  of types  $(A_1, 1), (A_2, 1)$ , respectively. Let  $\{a_\mu\}_{\mu \in I_1}$  be a basis of  $H_2(K(A_1, 1)^{(2)}, \mathbb{Z})$ . Then the map  $f_\mu : \mathbb{S}^2 \rightarrow K(A_1, 1)^{(2)}$  determined by  $a_\mu$  implies  $\bar{\varphi}^{(2)} f_\mu : \mathbb{S}^2 \rightarrow K(A_2, 1)^{(2)}$  for  $\mu \in I_1$ . Next,

$$(\bar{\varphi}^{(2)} f_\mu)_* : H_2(\mathbb{S}^2, \mathbb{Z}) \rightarrow H_2(K(A_2, 1)^{(2)}, \mathbb{Z})$$

yields an element

$$b = (\bar{\varphi}^{(2)} f_\mu)_*(\gamma) \in H_2(K(A_2, 1)^{(2)}, \mathbb{Z}).$$

Since  $H_2(K(A_2, 1)^{(2)}, \mathbb{Z})$  is a free abelian group with a basis  $\{b_\lambda\}_{\lambda \in I_2}$ , we get an isomorphism

$$H_2(K(A_2, 1)^{(2)}, \mathbb{Z}) \approx \pi_2\left(\bigvee_{\lambda \in J} \mathbb{S}^2\right).$$

If  $g_\mu : \mathbb{S}^2 \rightarrow \bigvee_{\lambda \in I_2} \mathbb{S}^2$  is a map corresponding to

$$b = (\bar{\varphi}^{(2)} f_\mu)_*(\gamma) \in \pi_2\left(\bigvee_{\lambda \in I_2} \mathbb{S}^2\right)$$

then we get a map

$$\bigvee_{\mu \in I_1} g_\mu : \bigvee_{\mu \in I_1} \mathbb{S}^2 \longrightarrow \bigvee_{\lambda \in I_2} \mathbb{S}^2$$

and the associated map of cones

$$\bigvee_{\mu \in I_1} C(g_\mu) : \bigvee_{\mu \in I_1} \mathbb{D}^3 \longrightarrow \bigvee_{\lambda \in I_2} \mathbb{D}^3.$$

Consequently, those maps lead the required map  $\tilde{\varphi} : X_1 \rightarrow X_2$  with  $H_1(\tilde{\varphi}) = \varphi$ . If the spaces  $X_1, X_2$  are nilpotent then Theorem 1.10 implies that  $\tilde{\varphi} : X_1 \rightarrow X_2$  is a homotopy equivalence.

If  $X_1, X_2$  are Moore spaces of type  $(A, 1)$  constructed in [25] then, by the above, the identity  $\text{id}_A : A \rightarrow A$  yields a map  $\widetilde{\text{id}}_A : X_1 \rightarrow X_2$  with  $H_1(\widetilde{\text{id}}_A) = \text{id}_A$  and the proof is complete.  $\square$

However, the homotopy type of a CW-complex Moore space  $M(A, 1)$  is not uniquely determined by  $A$ . Hatcher [15, Example 4.35] constructed the space  $X = (\mathbb{S}^1 \vee \mathbb{S}^n) \cup e^{n+1}$  such that the inclusion  $\mathbb{S}^1 \hookrightarrow X$  induces an isomorphism on all homology groups and on  $\pi_k$  for  $k < n$ , but not on  $\pi_n$ . More precisely, from [15, Example 4.27] we have

$$\pi_n(\mathbb{S}^1 \vee \mathbb{S}^n) \approx \mathbb{Z}[t, t^{-1}]/(2t - 1).$$

Then,  $X$  is obtained from  $\mathbb{S}^1 \vee \mathbb{S}^n$  by attaching a cell  $e^{n+1}$  via a map  $\mathbb{S}^n \rightarrow \mathbb{S}^1 \vee \mathbb{S}^n$  corresponding to  $2t - 1 \in \mathbb{Z}[t, t^{-1}]$ .

**Moore spaces of type  $(A, n)$  with  $n \geq 2$ .** To examine the homotopy nilpotency of  $M(A, n)$  with  $\geq 2$ , we notice that Proposition 1.11 yields

**Corollary 2.2.** *If  $n \geq 2$ , then*

$$\text{nil } \Omega(M(\mathbb{Z}_p^m, n)) = \infty$$

for  $m = 1, 2, \dots, \infty$  and  $n \geq 2$ .

**Proof.** Let  $X_n(A) = M(A, n)$  with  $n \geq 2$  or  $X_1(A) = L(A)$ . Then, by the Universal Coefficient Theorem, we have

$$\tilde{H}_k(X_n(\mathbb{Z}_p^m), \mathbb{F}_p) \approx \begin{cases} \mathbb{Z}_p^m \otimes \mathbb{F}_p \approx \mathbb{F}_p, & \text{for } k = n, \\ \text{Tor}(\mathbb{Z}_p^m, \mathbb{F}_p) \approx \mathbb{F}_p, & \text{for } k = n + 1, \\ 0, & \text{for } k \neq n, n + 1, \end{cases}$$

for  $m, n \geq 1$ .

Since the Moore space  $M(\mathbb{Z}_p^m, n) \simeq \Sigma X_{n-1}(\mathbb{Z}_p^m)$  for  $n \geq 2$ , we can consider the iterated Samelson product

$$s_n : (X_{n-1}(\mathbb{Z}_p^m))^{\wedge k} \rightarrow \Omega M(\mathbb{Z}_p^m, n) \simeq \Omega \Sigma(X_{n-1}(\mathbb{Z}_p^m)).$$

Thus, Proposition 1.11 implies that

$$\text{nil}\Omega(M(\mathbb{Z}_{p^m}, n)) = \infty \tag{2.1}$$

for  $m \geq 1$  and  $n \geq 2$ .

Since  $\mathbb{Z}_{p^\infty} \otimes \mathbb{F}_p = 0$ , we have

$$\tilde{H}_k(X_n(\mathbb{Z}_{p^\infty}), \mathbb{F}_p) \approx \begin{cases} \text{Tor}(\mathbb{Z}_{p^\infty}, \mathbb{F}_p) \approx \mathbb{F}_p, & \text{for } k = n + 1, \\ 0, & \text{for } k \neq n + 1, \end{cases}$$

for  $n \geq 1$ . Thus, the argument above collapses.

Therefore, we proceed as follows. Given  $n \geq 2$  and a prime  $p$ , consider the mapping telescope  $T$  determined by the sequence of maps

$$\mathbb{S}^n \xrightarrow{p} \mathbb{S}^n \xrightarrow{p} \mathbb{S}^n \xrightarrow{p} \dots$$

Recall that  $T$  is the union of the mapping cylinders  $M_k$  with the copies of  $\mathbb{S}^n$  in  $M_k$  and  $M_{k-1}$  identified for all  $k$ . Thus,  $T$  is the quotient space of the disjoint union

$$\bigsqcup_{k=1}^{\infty} \mathbb{S}^n \times [k, k + 1]$$

in which each point

$$(x_k, k + 1) \in \mathbb{S}^n \times [k, k + 1]$$

is identified with

$$(p(x_k), k + 1) \in \mathbb{S}^1 \times [k + 1, k + 2].$$

In the mapping telescope  $T$ , let  $T_m$  be the union of the first  $m$  mapping cylinders. This deformation retracts onto  $\mathbb{S}^n$  by deformation retracting each mapping cylinder onto its right end in turn. Since the maps  $p : \mathbb{S}^n \rightarrow \mathbb{S}^n$  are cellular, each mapping cylinder is a  $CW$ -complex and the telescope  $T$  is the increasing union of the subcomplexes  $T_m \simeq \mathbb{S}^n$ .

If we attach a cell  $e^{n+1}$  to the first  $\mathbb{S}^n$  in  $T$  via the identity map of  $\mathbb{S}^n$ , we obtain a space  $X$  which is the increasing union of its subspaces  $X_m = T_m \cup e^{n+1}$  being  $M(\mathbb{Z}_{p^m}, n)$ 's. Since,

$$H_n(X, \mathbb{Z}) \approx \text{colim}_m H_n(X_m, \mathbb{Z}) = \text{colim}_m \mathbb{Z}_{p^m} = \mathbb{Z}_{p^\infty}$$

and  $H_k(X, \mathbb{Z}) = 0$  for  $k \neq n$ , we derive that  $X$  is the Moore space of type  $(\mathbb{Z}_{p^\infty}, n)$ .

Furthermore,  $\Omega(X) = \text{colim}_m \Omega(X_m)$  implies a homotopy equivalence

$$\Omega(M(\mathbb{Z}_{p^\infty}, n)) \simeq \text{colim}_m \Omega(M(\mathbb{Z}_{p^m}, n)).$$

Thus, the non-trivial maps

$$\tilde{\varphi}_{\Omega(M(\mathbb{Z}_{p^m}, n)), k} : \Omega(M(\mathbb{Z}_{p^m}, n))^{\wedge k} \longrightarrow \Omega(M(\mathbb{Z}_{p^m}, n))$$

for  $k, m \geq 1$  determined by (2.1) yield the non-trivial the maps

$$\tilde{\varphi}_{\Omega(M(\mathbb{Z}_{p^\infty}, n)), k} : \Omega(M(\mathbb{Z}_{p^\infty}, n))^{\wedge k} \longrightarrow \Omega(M(\mathbb{Z}_{p^\infty}, n))$$

for  $k \geq 1$  and  $n \geq 2$ . Consequently,

$$\text{nil}\Omega(M(\mathbb{Z}_{p^\infty}, n)) = \infty \tag{2.2}$$

for  $n \geq 2$  and this concludes the proof. □

Now, recall that due to [9, Corollary 27.4], an abelian group  $A$  with elements of finite order contains a direct summand  $\mathbb{Z}_{p^m}$  for some prime  $p$  and  $m = 1, 2, \dots$  or  $\infty$ . Then, such an abelian group  $A \approx \mathbb{Z}_{p^m} \oplus B$  for some abelian group  $B$  and  $m = 1, \dots, \infty$ . Hence,

$$M(A, n) = M(\mathbb{Z}_{p^m}, n) \vee M(B, n),$$

and therefore (2.1), (2.2), and Remark 1.7 imply that

$$\text{nil}\Omega(M(A, n)) = \infty$$

for  $n \geq 2$ .

Given an abelian group  $A$ , we have  $A \otimes \mathbb{Q} = \bigoplus_1^{r(A)} \mathbb{Q}^+$  for the rank  $r(A)$  of  $A$  and the additive group  $\mathbb{Q}$  of the field  $\mathbb{Q}$  of rationals. Therefore,

$$\tilde{H}_k(X_n(A \otimes \mathbb{Q}), \mathbb{Q}) \approx \begin{cases} \bigoplus_1^{r(A)} \mathbb{Q}, & \text{for } k = n, \\ 0, & \text{otherwise,} \end{cases}$$

for  $n \geq 1$ .

Next, given a nilpotent space  $X$  and a set of primes  $I$ , write  $X_{(I)}$  for the  $I$ -localization of  $X$ . Then, by [19, Proposition 3.5], we have

**Proposition 2.3.** *Let  $X$  be a nilpotent space. If  $\text{nil}\Omega(X) < n$  then  $\text{nil}\Omega(X_{(I)}) < n$  for every set of primes  $I$ .*

Since the Moore space  $M(A, n) \simeq \Sigma X_{n-1}(A)$  for  $n \geq 2$ , we can consider the iterated Samelson product

$$s_n : (X_{n-1}(A))^{\wedge k} \longrightarrow \Omega M(A, n) \simeq \Omega \Sigma(X_{n-1}(A)).$$

Furthermore, if  $r(A) \geq 2$  then Proposition 1.11 yields

$$\text{nil}\Omega(M(A \otimes \mathbb{Q}, n)) = \infty.$$

Since  $M(A \otimes \mathbb{Q}, n) = M(A, n)_{(0)}$ , the rationalization of  $M(A, n)$ , Proposition 2.3 leads to  $\text{nil}\Omega(M(A, n)) = \infty$ . Thus, in view of (1.1), Proposition 1.11 and Corollary 2.2, we may state

**Corollary 2.4.** *Let  $A_1, \dots, A_m$  be abelian groups and  $n_1, \dots, n_m \geq 2$ .*

- (1) If  $A_k$  has an element of finite order or  $r(A_k) \geq 2$  for some  $k = 1, \dots, m$  then

$$\text{nil } \Omega((M(A_1, n_1) \times \cdots \times (M(A_m, n_m))) = \infty$$

for  $m \geq 1$ .

- (2) If  $A_k$  are non-trivial for  $k = 1, \dots, m$  then

$$\text{nil } \Omega((M(A_1, n_1) \vee \cdots \vee M(A_m, n_m)) = \infty$$

for  $m \geq 2$ .

If  $r(A) = 1$  and  $A$  is a torsion-free abelian group then by [9, Chapter IV, Section 24], we know that  $A$  is a subgroup of  $\mathbb{Q}$ . Notice that

$$M(\mathbb{Q}, n) = S_{(0)}^n = K(\mathbb{Q}, n),$$

the Eilenberg-MacLane of type  $(\mathbb{Q}, n)$  provided  $n$  is odd. Therefore,  $M(\mathbb{Q}, n)$  is a homotopy commutative and associative  $H$ -space and

$$\text{nil } M(\mathbb{Q}, n) = \text{nil } \Omega(M(\mathbb{Q}, n)) = 1.$$

Given any subgroup  $A < \mathbb{Q}$ , we have a sequence  $\mathbb{Z} \xrightarrow{n_0} \mathbb{Z} \xrightarrow{n_1} \mathbb{Z} \xrightarrow{n_2} \cdots$  and  $A = \text{colim}_{n_k} \mathbb{Z}$ . Next, for  $n \geq 2$ , the mapping telescope  $T$  of the associated sequence of maps

$$\mathbb{S}^n \xrightarrow{n_0} \mathbb{S}^n \xrightarrow{n_1} \mathbb{S}^n \xrightarrow{n_2} \cdots$$

is the union of the mapping cylinders  $M_{n_k}$  with the copies of  $\mathbb{S}^n$  in  $M_{n_k}$  and  $M_{n_{k-1}}$  identified for all  $k$ . In the mapping telescope  $T$ , let  $T_m$  be the union of the first  $m$  mapping cylinders. This deformation retracts onto  $\mathbb{S}^n$  by deformation retracting each mapping cylinder onto its right end in turn. Since the maps  $n_k : \mathbb{S}^n \rightarrow \mathbb{S}^n$  are cellular, each mapping cylinder is a  $CW$ -complex and the telescope  $T$  is the increasing union of the subcomplexes  $T_m \simeq \mathbb{S}^n = M(\mathbb{Z}, n)$ . Next, by  $\tilde{H}_n(T, \mathbb{Z}) \approx A = \text{colim}_m \mathbb{Z}$  and  $\tilde{H}_k(T, \mathbb{Z}) = 0$  for  $k \neq n$ , we derive that  $T = M(A, n) = \text{colim}_m T_m$ . Then,

$$\Omega(M(A, n)) = \text{colim}_m \Omega(T_m), \quad \text{nil } \Omega(T_m) = \text{nil } \Omega(\mathbb{S}^n) \leq 3$$

imply that  $\text{nil } \Omega(M(A, n)) \leq 3$  provided  $n \geq 2$  and we get the main result

**Theorem 2.5.** *Let  $m \geq 1, n_1, \dots, n_m \geq 2$  and  $M(A_k, n_k)$  be Moore spaces of type  $(A_k, n_k)$  for  $k = 1, \dots, m$ . Then*

- (1)  $\text{nil } \Omega((M(A_1, n_1) \times \cdots \times (M(A_m, n_m))) < \infty$  if and only if  $A_k$  are torsion-free groups with rank  $r(A_k) = 1$  for  $k = 1, \dots, m$ ;
- (2)  $\text{nil } \Omega((M(A_1, n_1) \vee \cdots \vee M(A_m, n_m)) < \infty$  if and only if  $m = 1$  and  $A_1$  is a torsion-free group with rank  $r(A_1) = 1$ .

**Moore spaces of type  $(A, 1)$ .** We present constructions and analyse homotopy nilpotency of some Moore spaces of type  $(A, 1)$ .

(1) The space  $X = (\mathbb{S}^1 \vee \mathbb{S}^n) \cup e^{n+1}$  constructed by Hatcher [15, Example 4.35] is a Moore space of type  $(\mathbb{Z}, 1)$  for the infinite cyclic group  $\mathbb{Z}\langle t \rangle$ . Since

$$\pi_1(X) \approx \mathbb{Z}\langle t \rangle, \quad \pi_n(\mathbb{S}^1 \vee \mathbb{S}^n) \approx \mathbb{Z}[t, t^{-1}]/(2t - 1),$$

we get that  $(2t - 1)\alpha = 0$  implies  $2t\alpha = \alpha$  for  $\alpha \in \mathbb{Z}[t, t^{-1}]/(2t - 1)$ . Hence, the action of  $\pi_1(X)$  on  $\pi_n(X)$  is non-trivial.

Note that the map  $\mathbb{Z}[t, t^{-1}] \rightarrow \mathbb{Q}$  given by  $t \mapsto 1/2$  yields a ring isomorphism

$$\mathbb{Z}[t, t^{-1}]/(2t - 1) \xrightarrow{\cong} \mathbb{Z}[1/2]$$

for the subring  $\mathbb{Z}[1/2] \subseteq \mathbb{Q}$  consisting of rationals with denominator a power of 2. Then,  $\pi_1(X) \approx \mathbb{Z}\langle t \rangle$  acts on  $\pi_n(X) \approx \mathbb{Z}[1/2]$  by  $t\alpha = (1/2)\alpha$  for  $\alpha \in \mathbb{Z}[1/2]$ . Consequently, the Whitehead product  $[t, \alpha] = (-1/2)\alpha$  and the  $(n + 1)$ -fold Whitehead product  $[\alpha, t, t \dots, t] = (-1/2)^n \alpha$  is non-trivial provided  $\alpha \neq 0$ . Therefore in view of Theorem 1.8,

$$\text{nil } \Omega(X) = \infty.$$

(2) The space  $\mathbb{R}P_m^2 = \mathbb{S}^1 \cup_m e^2$  is a Moore space of type  $(\mathbb{Z}_m, 1)$  for the cyclic group  $\mathbb{Z}_m$  of order  $m$ . Then,  $\pi_1(\mathbb{R}P_m^2) = \mathbb{Z}_m\langle t \rangle$ , where  $t$  is represented by the canonical map  $i : \mathbb{S}^1 \rightarrow \mathbb{R}P_m^2$ . Next, by [23], the group  $\pi_2(\mathbb{R}P_m^2)$  can be identified with the ideal of the group ring  $\mathbb{Z}[\mathbb{Z}_m]$  generated by  $\alpha = 1 - t$ , so that as an abelian group  $\pi_2(\mathbb{R}P_m^2)$  is free of rank  $m - 1$  and, as  $\pi_1$ -module,  $\pi_2(\mathbb{R}P_m^2)$  has a single generator  $\alpha$ , subject solely to the relation  $(1 + t + \dots + t^{m-1})\alpha = 0$ . Then, the Whitehead product

$$[\alpha, t] = t\alpha - \alpha = 2\alpha - t^2\alpha - \dots - t^{m-1}\alpha.$$

Thus, we derive that the  $(n + 1)$ -fold Whitehead product  $[\alpha, t, t \dots, t]$  is non-trivial for  $n \geq 1$ . Consequently, in view of Theorem 1.8, we derive that

$$\text{nil } \Omega(\mathbb{R}P_m^2) = \infty$$

for  $m \geq 2$ .

(3) If  $r(A) = 1$  and  $A$  is a torsion-free abelian group then  $A < \mathbb{Q}$ , we have a sequence  $\mathbb{Z} \xrightarrow{n_0} \mathbb{Z}^n \xrightarrow{n_1} \mathbb{Z} \xrightarrow{n_2} \dots$  and  $A = \text{colim}_{k_i} \mathbb{Z}$ . Next, the mapping telescope  $T$  of the associated sequence of maps

$$\mathbb{S}^1 \xrightarrow{n_0} \mathbb{S}^1 \xrightarrow{n_1} \mathbb{S}^1 \xrightarrow{n_2} \dots$$

is the union of the mapping cylinders  $M_{n_k}$  with the copies of  $\mathbb{S}^1$  in  $M_{n_k}$  and  $M_{n_{k-1}}$  identified for all  $k$ . In the mapping telescope  $T$ , let  $T_m$  be the union of the first  $m$  mapping cylinders. This deformation retracts onto  $\mathbb{S}^1$  by deformation retracting each mapping cylinder onto its right end in turn.

Since the maps  $n_k : \mathbb{S}^1 \rightarrow \mathbb{S}^n$  are cellular, each mapping cylinder is a  $CW$ -complex and the telescope  $T$  is the increasing union of the subcomplexes  $T_m \simeq \mathbb{S}^1$ .

Next, from

$$\tilde{H}_1(T, \mathbb{Z}) \approx \operatorname{colim}_m H_1(T_m) = \operatorname{colim}_{k_i} \mathbb{Z} = A, \quad \tilde{H}_k(T, \mathbb{Z}) = 0, \quad k \neq 1,$$

we derive that  $T = \operatorname{colim}_m T_m$  is a Moore space of type  $(A, 1)$ . Furthermore,  $\pi_k(T) = \operatorname{colim}_m \pi_k(T_m)$  implies that  $\pi_1(T) = \operatorname{colim}_m \mathbb{Z} = A$  and  $\pi_k(T) = 0$  for  $k \neq 1$ . Consequently,  $T = \operatorname{colim}_m T_m$ , as the Eilenberg-MacLane space  $K(A, 1)$ , is a homotopy commutative and an associative  $H$ -space. Finally, we get that

$$\operatorname{nil} T = 1.$$

(4) At the end of this subsection, given a prime  $p$ , consider the telescope  $T$  determined by the sequence of maps

$$\mathbb{S}^1 \xrightarrow{p} \mathbb{S}^1 \xrightarrow{p} \mathbb{S}^1 \xrightarrow{p} \dots$$

If we attach a cell  $e^2$  to the first  $\mathbb{S}^1$  in  $T$  via the identity map of  $\mathbb{S}^1$ , we obtain a space  $X$  which is a Moore space of type  $(\mathbb{Z}_{p^\infty}, 1)$  since  $X$  is the increasing union of its subspaces  $X_m = T_m \cup e^2$ , which are  $\mathbb{R}P_{p^m}^2$ 's. Then, (2) leads to

$$\operatorname{nil} \Omega(X) = \infty.$$

To conclude, notice that by Seifert-Van Kampen Theorem

$$\pi_1(M(A_1, 1) \vee M(A_2, 1)) = A_1 * A_2,$$

the free product of  $A_1$  and  $A_2$ . Furthermore, if  $n \geq 2$  then

$$\begin{aligned} \pi_1(M(A_1, 1) \vee M(A_2, n)) &= A_1, \\ \pi_n(M(A_1, 1) \vee M(A_2, n)) &\approx \mathbb{Z}[A_1] \otimes A_2 \end{aligned}$$

for the group ring  $\mathbb{Z}[A_1]$ . Therefore, the spaces  $M(A_1, 1) \vee M(A_2, n)$  for  $n \geq 1$  are non-nilpotent. Then, Remark 1.7 and Proposition 1.11 yield

**Corollary 2.6.** *If  $m \geq 2$ ,  $n_1, \dots, n_m \geq 1$  and  $A_k$  are non-trivial Abelian group for  $k = 1, \dots, m$  then  $\operatorname{nil} \Omega((M(A_1, n_1) \vee \dots \vee M(A_m, n_m))) = \infty$ .*

**Smash product of Moore spaces.** Recall that the *smash product* of pointed spaces  $X, Y$  is the quotient space

$$X \wedge Y = (X \times Y) / (X \vee Y)$$

of the product space  $X \times Y$  by the wedge sum  $X \vee Y$ . It is well-known that the connectivity

$$\operatorname{conn}(X \wedge Y) \geq \operatorname{conn}(X) + \operatorname{conn}(Y) + 1.$$

Since  $\text{conn}(M(A, n)) = n - 1$  for  $n \geq 1$ , we deduce that

$$\text{conn}(M(A, m) \wedge M(B, n)) \geq m + n - 1 \geq 1$$

for  $m, n \geq 1$ . Furthermore, by the Künneth Theorem, we have

$$\tilde{H}_k(M(A, m) \wedge M(B, n), \mathbb{Z}) = \begin{cases} A \otimes B, & \text{for } k = m + n, \\ \text{Tor}(A, B), & \text{for } k = m + n + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that the rationalization

$$(M(A, m) \wedge M(B, n))_{(0)} = M(A \otimes B \otimes \mathbb{Q}, m + n) = M\left(\bigoplus_1^{r(A)r(B)} \mathbb{Q}, m + n\right).$$

Consequently, Propositions 1.11 and 2.3 yield that

$$\text{nil } \Omega(M(A, m) \wedge M(B, n)) = \infty$$

provided  $\max\{r(A), r(B)\} \geq 2$ .

To analyse the case  $\max\{r(A), r(B)\} \leq 1$  and  $\max\{m, n\} \geq 2$ , we need

**Lemma 2.7.** *For any abelian groups  $A$  and  $B$ , the group  $\text{Tor}(A, B)$  is torsion.*

**Proof.** Certainly, we can assume that  $A$  and  $B$  are torsion groups. We then form an exact sequence

$$0 \rightarrow \ker \rightarrow \bigoplus_{0 \neq b \in B} \mathbb{Z}_{|b|} \rightarrow B \rightarrow 0.$$

where  $|b|$  is the order of  $b \in B$  and the map to  $B$  is given by sending a generator of the factor corresponding to  $b$  to the element  $b$ . If we write  $C$  for the group in the middle, we obtain from the exact sequence for  $\text{Tor}$  the short exact sequence

$$\begin{aligned} 0 \rightarrow \text{coker}(\text{Tor}(A, \ker) \rightarrow \text{Tor}(A, C)) \rightarrow \\ \rightarrow \text{Tor}(A, B) \rightarrow \ker(A \otimes \ker \rightarrow A \otimes C) \rightarrow 0. \end{aligned}$$

We have that

$$\text{Tor}(A, C) = \bigoplus_{0 \neq b \in B} \text{Tor}(A, \mathbb{Z}_{|b|}) = \bigoplus_{0 \neq b \in B} \ker(A \xrightarrow{|b|} A)$$

and  $A \otimes \ker$  are torsion groups because  $A$  is a torsion group. Hence the first and third term of the sequence above are torsion. The result now follows from the general result that if in an exact sequence  $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$  the groups  $G'$  and  $G''$  are torsion, then so is  $G$  and the proof is complete.  $\square$

Since  $\max\{m, n\} \geq 2$ , the smash product  $M(A, m) \wedge M(B, n)$  is a suspension. For  $r(A) = 0$  or  $r(B) = 0$ , the groups  $A \otimes B$  and  $\text{Tor}(A, B)$  (Lemma 2.7) are torsion. If  $A \otimes B \neq 0$  or  $\text{Tor}(A, B) \neq 0$  then Proposition 1.11 implies that

$$\text{nil}\Omega(M(A, m) \wedge M(B, n)) = \infty.$$

If  $A \otimes B = 0$  and  $\text{Tor}(A, B) = 0$  then  $M(A, m) \wedge M(B, n)$  is an acyclic space. Since the space  $M(A, m) \wedge M(B, n)$  has the homotopy type of a *CW*-complex with the trivial fundamental group, the Whitehead and Hurewicz Theorems yield the its contractibility and so

$$\text{nil}\Omega(M(A, m) \wedge M(B, n)) = 0.$$

Next,  $r(A) = r(B) = 1$  implies  $r(A \otimes B) = 1$ . If  $A \otimes B$  is a mixed group or  $\text{Tor}(A, B) \neq 0$  then Proposition 1.11 yields

$$\text{nil}\Omega(M(A, m) \wedge M(B, n)) = \infty.$$

If  $A \otimes B$  is torsion-free and  $\text{Tor}(A, B) = 0$  then

$$\text{nil}\Omega(M(A, m) \wedge M(B, n)) < \infty.$$

Now, let  $m = n = 1$  and write

$$T(A) = \bigoplus_{p \in I_A} T_p(A), \quad T(B) = \bigoplus_{p \in I_B} T_p(B)$$

for primary decompositions of the subgroups of  $T_p(A) < A$  and  $T_p(B) < B$ , respectively. But, if  $2 \notin I_A \cap I_B$  then

$$\pi_3(M(\text{Tor}(A, B), 2)) = (\text{Tor}(A, B)) \otimes \mathbb{Z}_2 = 0.$$

Consequently, by the Universal Coefficient Theorem for Homotopy Groups ([1, Theorem 2.5.9] and [16, Chpter 5]) there is a short exact sequence:

$$\begin{aligned} 0 \rightarrow \text{Ext}(A \otimes B, \pi_3(M(\text{Tor}(A, B), 2))) \rightarrow \\ \rightarrow [M(A \otimes B, 2), M(\text{Tor}(A, B), 2)] \rightarrow \\ \rightarrow \text{Hom}(A \otimes B, \text{Tor}(A, B)) \rightarrow 0, \end{aligned}$$

It follows that

$$M(A \otimes B, 2) \rightarrow M(\text{Tor}(A, B), 2)$$

is the only homology trivial map. Thus, from the homology decomposition (see [16, Chapter 8] for details) of  $M(A, 1) \wedge M(B, 1)$ , we get that

$$M(A, 1) \wedge M(B, 1) = M(A \otimes B, 2) \vee M(\text{Tor}(A, B), 3)$$

is a suspension and so Proposition 1.11 yields

$$\text{nil}\Omega(M(A, 1) \wedge M(B, 1)) < \infty$$

if and only if the group  $A \otimes B$  is torsion free with  $r(A \otimes B) \leq 1$  and  $\text{Tor}(A, B) = 0$ .

To examine the case  $2 \in I_A \cap I_B$ , we begin from the following situations.

(I) If  $k \geq 2$  then in view of [22, Proposition 6.2.2], we have

$$M(\mathbb{Z}_{2^k}, 1) \wedge M(\mathbb{Z}_{2^k}, 1) = M(\mathbb{Z}_{2^k}, 2) \vee M(\mathbb{Z}_{2^k}, 3).$$

Hence, Proposition 1.11 leads to

$$\text{nil } \Omega(M(\mathbb{Z}_{2^k}, 1) \wedge M(\mathbb{Z}_{2^k}, 1)) = \infty.$$

(II) If  $k = 1$  then, by the short exact sequence [1, Theorem 2.5.9], there is the short exact sequence

$$0 \rightarrow \text{Ext}(\mathbb{Z}_2, \pi_3(M(\mathbb{Z}_2, 2))) \rightarrow [M(\mathbb{Z}_2, 2), M(\mathbb{Z}_2, 2)] \rightarrow \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) \rightarrow 0$$

with

$$\text{Ext}(\mathbb{Z}_2, \pi_3(M(\mathbb{Z}_2, 2))) \approx \mathbb{Z}_2,$$

$$[M(\mathbb{Z}_2, 2), M(\mathbb{Z}_2, 2)] = [\Sigma \mathbb{R}P^2, \Sigma \mathbb{R}P^2] \approx \mathbb{Z}_4.$$

But,  $M(\mathbb{Z}_2, 1) = \mathbb{R}P^2$  and  $[\mathbb{R}P^2, \mathbb{R}P^2] \approx \mathbb{Z}_2$ . Then, the suspension map  $\Sigma\alpha : M(\mathbb{Z}_2, 2) \rightarrow M(\mathbb{Z}_2, 2)$  is homology trivial for the only non-trivial map  $\alpha : \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ . Thus,

$$\mathbb{R}P^2 \wedge \mathbb{R}P^2 \simeq \Sigma(C\mathbb{R}P^2 \cup_{\alpha} \mathbb{R}P^2)$$

and Proposition 1.11 yields that

$$\text{nil } \Omega(\mathbb{R}P^2 \wedge \mathbb{R}P^2) = \infty.$$

From now on, we restrict to Moore spaces of type  $(A, 1)$  constructed in [25]. If  $2 \in I_A \cap I_B$  then, in view of [9, Corollary 27.4],  $A = \mathbb{Z}_{2^k} \oplus A'$  and  $B = \mathbb{Z}_{2^l} \oplus B'$  with some  $k, l = 1, \dots, \infty$ . Hence, by Proposition 2.1, we get a homotopy retract

$$M(\mathbb{Z}_{2^k}, 1) \wedge M(\mathbb{Z}_{2^l}, 1) \rightarrow M(A, 1) \wedge M(B, 1) \rightarrow M(\mathbb{Z}_{2^k}, 1) \wedge M(\mathbb{Z}_{2^l}, 1).$$

Next, notice that the canonical inclusions

$$\mathbb{Z}_{2^{\min\{k,l\}}} \hookrightarrow \mathbb{Z}_{2^k}, \quad \mathbb{Z}_{2^{\min\{k,l\}}} \hookrightarrow \mathbb{Z}_{2^l}$$

yield a homology isomorphism

$$M(\mathbb{Z}_{2^{\min\{k,l\}}, 1) \wedge M(\mathbb{Z}_{2^{\min\{k,l\}}, 1) \longrightarrow M(\mathbb{Z}_{2^k}, 1) \wedge M(\mathbb{Z}_{2^l}, 1)$$

of 1-connected spaces. Thus, Whitehead Theorem leads to a homotopy equivalence

$$M(\mathbb{Z}_{2^k}, 1) \wedge M(\mathbb{Z}_{2^l}, 1) \simeq M(\mathbb{Z}_{2^{\min\{k,l\}}, 1) \wedge M(\mathbb{Z}_{2^{\min\{k,l\}}, 1).$$

Now, we analyse the following cases.

(1) If  $k = \infty$  or  $l = \infty$  then

$$M(\mathbb{Z}_{2^k}, 1) \wedge M(\mathbb{Z}_{2^l}, 1) = M(\mathbb{Z}_{2^\infty}, 3).$$

Since,  $\text{nil } \Omega(M(\mathbb{Z}_{2^\infty}, 3)) = \infty$ , Remark 1.7 implies

$$\text{nil } \Omega(M(A, 1) \wedge M(B, 1)) = \infty.$$

(2) If  $2 \leq \min\{k, l\} < \infty$  then

$$M(\mathbb{Z}_{2^k}, 1) \wedge M(\mathbb{Z}_{2^l}, 1) \simeq M(\mathbb{Z}_{2^{\min\{k, l\}}}, 1) \wedge M(\mathbb{Z}_{2^{\min\{k, l\}}}, 1),$$

Remark 1.7 and (I) yield

$$\text{nil } \Omega(M(A, 1) \wedge M(B, 1)) = \infty.$$

(3) If  $\min\{k, l\} = 1$  then a homotopy equivalence

$$M(\mathbb{Z}_{2^k}, 1) \wedge M(\mathbb{Z}_{2^l}, 1) \simeq M(\mathbb{Z}_2, 1) \wedge M(\mathbb{Z}_2, 1),$$

Remark 1.7 and (II) lead to

$$\text{nil } \Omega(M(A, 1) \wedge M(B, 1)) = \infty.$$

Summarizing the discussion above, we get

**Proposition 2.8.** *If  $m, n \geq 1$  then*

$$\text{nil } \Omega(M(A, m) \wedge M(B, n)) < \infty$$

*if and only if the group  $A \otimes B$  is torsion free with the rank  $r(A \otimes B) \leq 1$  and  $\text{Tor}(A, B) = 0$ .*

**Homology spheres.** Recall that a map  $f: X \rightarrow Y$  is called a *homology isomorphism* if it induces isomorphisms on all integral homology groups. Then, according to [7] a space  $\mathcal{S}^n$  is called a *homology  $n$ -sphere* if there is an isomorphism of homologies  $H_k(\mathcal{S}^n) \approx H_k(\mathbb{S}^n)$  for all  $k \geq 0$ . Given a homology  $n$ -sphere  $\mathcal{S}^n$  with  $n \geq 1$ , the fundamental group  $\pi_1(\mathcal{S}^n)$  is perfect for  $n \geq 2$  and the abelianization  $\pi_1(\mathcal{S}^1)^{\text{ab}} \approx \mathbb{Z}$ , the infinite cyclic group. Thus, the result of Bousfield-Kan [5, Chapter VII, Proposition 3.2, p. 206] implies that there is a homology isomorphism  $\mathfrak{h}_n: \mathcal{S}^n \rightarrow \mathbb{S}^n$  for  $n \geq 2$ . It is not difficult to show that there is also a homology isomorphism  $\mathfrak{h}_1: \mathcal{S}^1 \rightarrow \mathbb{S}^1$ .

**Remark 2.9.** Let  $\mathcal{S}^n$  be a homology  $n$ -sphere with  $n \geq 1$  and  $\mathfrak{h}_n: \mathcal{S}^n \rightarrow \mathbb{S}^n$  a homology isomorphism.

(1) Since the fundamental group  $\pi_1(\mathcal{S}^n)$  is perfect for  $n \geq 2$ , we get that  $\text{nil } \pi_1(\mathcal{S}^n) = \infty$  provided  $\pi_1(\mathcal{S}^n) \neq E$ . Thus,  $\text{nil } \pi_1(\mathcal{S}^n) \leq \text{nil } \Omega(\mathcal{S}^n)$  implies that  $\text{nil } \Omega(\mathcal{S}^n) = \infty$  provided  $\pi_1(\mathcal{S}^n) \neq E$ .

(2) Since a homology  $n$ -sphere  $\mathcal{S}^n$  is path-connected, the homology isomorphism  $\mathfrak{h}_n: \mathcal{S}^n \rightarrow \mathbb{S}^n$  leads to a weak homotopy equivalence

$$\Sigma(\mathfrak{h}_n): \Sigma(\mathcal{S}^n) \rightarrow \Sigma(\mathbb{S}^n) = \mathbb{S}^{n+1}, \quad n \geq 1.$$

In particular,  $\Sigma(\mathcal{S}^n)$  has the homotopy type of the sphere  $\mathbb{S}^{n+1}$  provided  $\mathcal{S}^n$  is a  $CW$ -complex. This implies that

$$\text{nil } \Omega(\Sigma(\mathcal{S}^n)) = \text{nil } \Omega(\mathbb{S}^{n+1}) \leq 3, \quad n \geq 1.$$

(3) If  $X, Y$  are  $CW$ -complexes then the connectivity of the smash product satisfies the following inequality:

$$\text{conn}(X \wedge Y) \geq \text{conn}(X) + \text{conn}(Y) + 1$$

(if both  $X$  and  $Y$  are connected).

Thus,  $\mathfrak{h}_m: \mathcal{S}^m \rightarrow \mathbb{S}^m$  and  $\mathfrak{h}_n: \mathcal{S}^n \rightarrow \mathbb{S}^n$  induce a homotopy equivalence

$$\mathfrak{h}_m \wedge \mathfrak{h}_n: \mathcal{S}^m \wedge \mathcal{S}^n \rightarrow \mathbb{S}^m \wedge \mathbb{S}^n = \mathbb{S}^{m+n}$$

provided that  $\mathcal{S}^m$  and  $\mathcal{S}^n$  are connected  $CW$ -complexes.

(4) Since the fundamental group of a co- $H$ -space is free, a co- $H$ -structure on  $\mathcal{S}^n$  implies that  $\mathcal{S}^n$  is 1-connected for  $n \geq 2$  and  $\pi_1(\mathcal{S}^1) \approx \mathbb{Z}$ . Consequently,  $\mathfrak{h}_n: \mathcal{S}^n \rightarrow \mathbb{S}^n$  is a weak homotopy equivalence for  $n \geq 1$ , provided  $\mathcal{S}^n$  has a co- $H$ -structure.

Let  $M^n$  be a *smooth homology  $n$ -sphere*, i.e., a smooth  $n$ -dimensional manifold such that  $H_*(M^n) \approx H_*(\mathbb{S}^n)$ . Then, by Kervire [18], the fundamental group  $\pi = \pi_1(M^n)$  of  $M^n$  satisfies the following three conditions:

- (1)  $\pi$  as a finite presentation,
- (2)  $H_1(\pi) = 0$ ,
- (3)  $H_2(\pi) = 0$ ,

where  $H_k(\pi)$  denotes the  $k$ -th homology group of  $\pi$  with coefficients in the trivial  $\mathbb{Z}[\pi]$ -module  $\mathbb{Z}$ . Concretely, this means that the abelianization of  $\pi$  is trivial (it has no non-trivial abelian quotients) and that its Schur multiplier is trivial (it has no non-trivial perfect central extensions).

Furthermore, by [18, Theorem 2], we have

**Theorem 2.10.** *Let  $M^3$  be a 3-dimensional manifold such that  $\pi_1(M^3)$  is finite and  $H_*(M^3) \approx H_*(\mathbb{S}^3)$ . Then, either  $\pi_1(M^3) = E$  or else,  $\pi_1(M^3)$  is isomorphic to the binary icosahedral group with presentation*

$$\langle x, y : x^2 = y^3 = (xy)^5 \rangle.$$

The Poincarè homology sphere (also known as Poincarè dodecahedral space) is a particular example of a smooth homology sphere, first constructed by Henri Poincaré. Being a spherical 3-manifold, it is the only

homology 3-sphere (besides the 3-sphere itself) with a finite fundamental group.

Alternatively, the Poincaré homology sphere can be constructed as the quotient space  $SO(3)/I$ , where  $SO(3)$  is the special orthogonal group in dimension 3 and  $I$  is the icosahedral group (i.e., the rotational symmetry group of the regular icosahedron and dodecahedron, isomorphic to the alternating group  $A_5$ ).

One can also pass instead to the universal cover of  $SO(3)$  which can be realized as the group of unit quaternions and is homeomorphic to the 3-sphere  $\mathbb{S}^3$ . In this case, the Poincaré homology sphere is isomorphic to  $\mathbb{S}^3/\tilde{I}$ , where  $\tilde{I}$  is the binary icosahedral group, the perfect double cover of  $I$  embedded in  $\mathbb{S}^3$ .

Consequently,  $\text{nil } \Omega(\mathbb{S}^3/\tilde{I}) = \infty$ , whereas  $\text{nil } \Omega(\Sigma(\mathbb{S}^3/\tilde{I})) = \text{nil } \Omega(\mathbb{S}^4) = 3$ .

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