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When is the space of semi-additive functionals an absolute (neighbourhood) retract?

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Abstract. In the present paper proved that if for a given compact Hausdorff space X the hyperspace $\exp X$ is a contractible compact space then the space $OS_f(X)$ is also a contractible compact space. As a consequence it is established that the space $OS_f(X)$ of semi-additive functionals is absolute (neighbourhood) retract if and only if the hyperspace $\exp X$ is so.

Анотація. В роботі доведено, що якщо для даного компакту X гіперпростір $\exp X$ є стягуваним компактом, то простір $OS_f(X)$ також є стягуваним компактом. В якості наслідку отримано, що простір $OS_f(X)$ напівадитивних функціоналів є абсолютним (околовим) ретрактом тоді і лише тоді, коли таким є гіперпростір $\exp X$.

1. Introduction

Let X be a compact Hausdorff space. By $C(X)$ we denote the Banach algebra of all continuous functions $\varphi\colon X \to \mathbb{R}$ with pointwise operations and the sup-norm, *i.e.* the norm given by

$$\|\varphi\| = \sup\{|\varphi(X)| : x \in X\},$$

e.g. [1]. For each $c \in \mathbb{R}$ we denote by c_X the constant function defined by the formula $c_X(x) = c$, $x \in X$. Let $\varphi, \psi \in C(X)$. Then the inequality $\varphi \leqslant \psi$ means that $\varphi(x) \leqslant \psi(x)$ for all $x \in X$.

Definition 1.1. *A functional $\mu\colon C(X) \to \mathbb{R}$ is called:*

1) *weakly additive if $\mu(\varphi + c_X) = \mu(\varphi) + c$ for all $c \in \mathbb{R}$ and $\varphi \in C(X)$;*
2) *order-preserving if for functions φ, $\psi \in C(X)$ the inequality $\varphi \leqslant \psi$ implies $\mu(\varphi) \leqslant \mu(\psi)$;*

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3) **normed** *if* $\mu(1_X) = 1$;
4) **positively homogeneous** *if* $\mu(t\varphi) = t\mu(\varphi)$ *for all* $\varphi \in C(X)$, $t \in \mathbb{R}_+$, *where* $\mathbb{R}_+ = [0, +\infty)$;
5) **semi-additive** *if* $\mu(\varphi + \psi) \leqslant \mu(\varphi) + \mu(\psi)$ *for all* φ, $\psi \in C(X)$.

For a compact Hausdorff space X let $OS(X)$ be the set of all functionals $\nu\colon C(X) \to \mathbb{R}$ satisfying the above five conditions. For brevity such functionals will be called *semi-additive*.

The set $OS(X)$ is equipped with the pointwise convergence topology. The base of neighborhoods of a functional $\mu \in OS(X)$ is formed by sets of the form

$$\langle \mu; \varphi_1, \ldots, \varphi_n; \varepsilon \rangle = \{\nu \in OS(X) : |\mu(\varphi_i) - \nu(\varphi_i)| < \varepsilon,\ i = 1, \ldots, n\},$$

where $\varphi_i \in C(X)$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, $\varepsilon > 0$.

It is clear that if μ, $\nu \in OS(X)$, then $\alpha\mu + \beta\nu \in OS(X)$, where α, $\beta \geqslant 0$, $\alpha + \beta = 1$. Moreover, the following statement is true: *for any compact Hausdorff space* X*, the space* $OS(X)$ *is a convex subspace of* $\mathbb{R}^{C(X)}$. The case of Tychonoff spaces were considered in [10] for bigger space of functionals. In [7] the first author investigated order-preserving functionals with finite support.

Let X and Y be compact Hausdorff spaces and $f\colon X \to Y$ be a continuous map. For each $\varphi \in C(Y)$ define the map $OS(f)\colon OS(X) \to OS(Y)$ by the following formula:

$$OS(f)(\mu)(\varphi) = \mu(\varphi \circ f), \qquad \mu \in OS(X),$$

The construction OS determines a functor acting in the category $\mathfrak{Comp}$ of compact Hausdorff spaces and their continuous maps.

Moreover, the functor $OS\colon \mathfrak{Comp} \to \mathfrak{Comp}$ is normal [4]. The notion of normality of functors one can find in [6].

Let X be a compact Hausdorff space, F a normal functor, and $x \in F(X)$. The *degree of a point* x is the smallest natural number n such that x belongs to the image $F(f)F(K)$ for some map $f\colon K \to X$ of an n-point space K, [6, Definition 16]. If such a finite n does not exist, then the degree of x is considered as *infinite*. The *degree of the functor* F is the maximum of the degrees of all possible points $x \in F(X)$ for all possible compact sets X and is denoted by $\deg F$.

The *support* of a point $x \in F(X)$ is a closed subset $\operatorname{supp} x \subset X$ such that the relations $\operatorname{supp} x \subset A$ and $x \in F(A)$ are equivalent for any closed $A \subset X$, [6, Definition 18]. For a normal functor F, the support exists for any $x \in F(X)$ and is determined by the relation

$$\operatorname{supp} x = \cap\{A \subset X : cl(A) = A,\ x \in F(A)\}, \tag{1.1}$$

where $cl(A)$ is the closure of the set A.

Obviously, for every compact Hausdorff space X, the space $P(X)$ of probability measures on X (*i.e.* the space of all linear, nonnegative, normed functionals on $C(X)$) is a subspace of $OS(X)$.

Let A be a non-empty subset of $P(X)$ and $\varphi \in C(X)$. Then $|\mu(\varphi)| < \|\varphi\|$ for any $\mu \in A$, and therefore the number set $\{\mu(\varphi) : \mu \in A\}$ is bounded. Therefore, for every $\varphi \in C(X)$, there exists the following upper bound

$$\nu_A(\varphi) = \sup\{\mu(\varphi) : \mu \in A\}. \tag{1.2}$$

Let A be a non-empty subset of $P(X)$. Then we have [4]:

a) the functional $\nu_A \colon C(X) \to \mathbb{R}$ belongs to $OS(X)$;
b) $\nu_A = \nu_{co(A)}$ where $co(A)$ is the convex hull of A;
c) $\nu_A = \nu_{cl(A)}$ where $cl(A)$ is the closure of A;
d) $\nu_A = \nu_{co(cl(A))}$.

The following statement shows that the formula (1.2) gives a general form of functionals from $OS(X)$: *for every $\nu \in OS(X)$ there exists a non-empty convex compact set A in $P(X)$ such that $\nu = \nu_A$, and for each $\varphi \in C(X)$ there exists $\mu \in A$ such that $\nu_A(\varphi) = \mu(\varphi)$*, [4].

For a compact Hausdorff space X, we denote by $\exp X$ the hyperspace of the space X, *i.e.* the space of all non-empty closed subsets of X provided with the Vietoris topology. It is well known (see, for example [6]) that for every compact Hausdorff space X its hyperspace $\exp X$ is also a compact Hausdorff space. By $\exp_\omega X$ we denote a set consisting of all non-empty finite subsets of X. Note that $\exp_\omega X$ is everywhere dense in $\exp X$.

For each $F \in \exp X$, we define the functional $\mu_F \colon C(X) \to \mathbb{R}$ as follows

$$\mu_F(\varphi) = \max_{x \in F} \varphi(x), \quad \varphi \in C(X). \tag{1.3}$$

It is clear that μ_F is a weakly additive, order-preserving, normed, positively-homogeneous and semi-additive functional. Moreover, the correspondence $F \mapsto \mu_F$ is one-to-one, [4]. Therefore, we can identify the set F with the functional μ_F, and thus $\exp X \subset OS(X)$.

Let K be a convex closed subset of a locally convex space E. By $cc(K)$ we denote the set consisting of all convex closed nonempty subsets of K and on $cc(K)$ consider the induced from $\exp K$ topology. For a convex subset $A \subset K$ by ext$A$ we denote the set of all extremal points of A.

For a compact Hausdorff space X the spaces $OS(X)$ and $cc(P(X))$ are homeomorphic [4]. Namely, a homeomorphism $\tau \colon cc(P(X)) \to OS(X)$ may be given by the rule

$$\tau(A) = \nu_A, \quad A \in cc(P(X)).$$

A functional $\nu \in OS(X)$ is said to be *concentrated on a closed subset of* $Z \subset X$ if $\mu \in OS(Z)$. The least (with respect to inclusion) closed set $Z \subset X$

for which $\mu \in OS(Z)$ is called the *support* of the functional $\mu \in OS(X)$ and is denoted by $\operatorname{supp}\mu$. Evidently, this notion is a partial case of the general notion of support, see (1.1), and we have

$$\operatorname{supp}\mu = \cap\{Z \subset X : cl(Z) = Z,\, \mu \in OS(Z)\}.$$

For a compact Hausdorff space X denote by $OS_n(X)$ the set of all functionals $\mu \in OS(X)$ for which $|\operatorname{supp}\mu| \leqslant n$ where n is a positive integer. Regard $OS_n(X)$ as a subspace of $OS(X)$ and put

$$OS_\omega(X) = \bigcup_{i=1}^{\infty} OS_n(X).$$

A functional $\mu \in OS_\omega(X)$ is called a *semi-additive functional with finite support*. Note that $OS_\omega(X)$ is everywhere dense in $OS(X)$. For every semi-additive functional μ with finite support there exist, see [4], a unique convex closed set $A \in cc(P(X))$, such that

$$\mu = \nu_A, \tag{1.4}$$

In the present paper for a given compact Hausdorff space X a subspace of $OS(X)$ is highlighted, which is a contractible compact space if the hyperspace $\exp X$ is so. Then it is obtained that this subspace is an absolute (neighbourhood) retract if and only if the hyperspace $\exp X$ is an absolute (neighbourhood) retract.

2. Constructions

Let μ be a semi-additive functional with finite support, and $A \in cc(P(X))$ a set for which (1.4) holds. Then it is clear that each element $\xi \in A$ is a probability measure with finite support and

$$\operatorname{supp}\mu = \bigcup_{\xi\in A} \operatorname{supp}\xi. \tag{2.1}$$

Let $|\operatorname{supp}\mu| \leqslant k$. Then $|\operatorname{supp}\xi| \leqslant k$ for each $\xi \in A$, and

$$\operatorname{supp}\xi = \{x_{\xi 1}, \ldots, x_{\xi\, n_\xi}\}.$$

Therefore

$$\xi = \sum_{i=1}^{n_\xi} \alpha_{\xi\, i}\delta_{x_{\xi\, i}},$$

where $\sum\limits_{i=1}^{n_\xi} \alpha_{\xi\, i} = 1$, $\alpha_{\xi\, i} \geqslant 0$, $i = 1, \ldots, n_\xi$. Hence, the formula (1.2) can be rewritten as follows:

$$\mu(\varphi) = \sup\left\{\sum_{i=1}^{n_\xi} \alpha_{\xi\, i}\delta_{x_{\xi\, i}}(\varphi) : \xi \in A\right\}. \tag{1.2$'$}$$

The following set was proposed by E. V. Shchepin (for more detailed information, see [5, 8]).

$$P_f(X) = \Big\{\mu \in P_\omega(X) : \text{ if } \mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}, \quad \text{then there exists } i_0 \in \{1, \ldots, n\} \text{ such that } \alpha_{i_0} \geqslant 1 - \frac{1}{n+1}\Big\}. \tag{2.2}$$

Note that the case of idempotent probability measures was investigated in [11]. In [12] the space of idempotent measures were studied as an object of the category theory, and in [9] the following results were obtained in this direction.

For a compact Hausdorff space X, define the following sets

$$OS_{f\omega}(X) = \big\{\nu_A \in OS_\omega(X) : \operatorname{ext} A \subset P_f(X)\big\}, \tag{2.3}$$

$$OS_f(X) = \big\{\nu_A \in OS(X) : \operatorname{ext} A \subset P_f(X)\big\}. \tag{2.4}$$

It is easy to see that the following statement hold.

Lemma 2.1. *For each compact Hausdorff space X one has that*

$$OS_{f\omega}(X) = OS_f(X) \cap OS_\omega(X).$$

Moreover, $OS_{f\omega}(X)$ is everywhere dense in $OS_f(X)$.

Lemma 2.2. *For every compact Hausdorff space X and each closed $F \subset X$, we have that $\mu_F \in OS_f(X)$. Hence one can assume that* $\exp X \subset OS_f(X)$. *Consequently,* $\exp_\omega X \subset OS_{f\omega}(X)$.

Proof. Let $F \in \exp X$. Consider the functional μ_F defined by the equality (1.3). Then

$$\mu_F = \nu_{\{\delta_x :\, x \in F\}}. \tag{2.5}$$

Hence, $\operatorname{ext} A \equiv \{\delta_x : x \in F\} \subset P_f(X)$ for $A = cl(co(\{\delta_x : x \in F\}))$, *i.e.* $\mu_F \in OS_f(X)$. □

Equality (2.5) for a finite set, say $\{x_1, \ldots, x_n\} \in \exp X$, can be rewritten as follows:

$$\mu_{\{x_1, \ldots, x_n\}} = \nu_{\{\delta_{x_1}, \ldots, \delta_{x_n}\}}.$$

Lemma 2.2 implies, in particular, that $OS_f(X) \neq \varnothing$. Indeed, for example, $\mu_X = \max\{\delta_x : x \in X\} \in OS_f(X)$. Note that for an infinite compact Hausdorff space X we have $\mu_X \notin OS_{f\omega}(X)$, although $\delta_x \in OS_{f\omega}(X)$ for all $x \in X$.

Proposition 2.3. *The topological space $OS_f(X)$ equipped with the pointwise convergence topology is a compact Hausdorff space.*

Proof. Note that $P_f(X)$ is a compact Hausdorff space [8]. Hence $P_f(X)$ is a closed subset of $P_\omega(X)$. Put

$$cc(P_\omega(X)) = \{A \cap P_\omega(X) : A \in cc(P(X))\}.$$

Then by the definition of the Vietoris topology any set of the form

$$\begin{aligned}\mathfrak{A}(X) =&\{A \in cc(P_\omega(X)) : \mathrm{ext} A \not\subset P_f(X)\} = \\ =&\{A \in cc(P_\omega(X)) : A \cap (P_\omega(X) \backslash P_f(X)) \neq \varnothing\}\end{aligned}$$

is open in $cc(P_\omega(X))$.

Consider now a net $\{\mu_\alpha\} \subset OS_f(X)$. By the construction there exists a net $\{A_\alpha\} \subset P_\omega(X)$ such that $\mu_\alpha = \nu_{A_\alpha}$ and $\mathrm{ext} A_\alpha \subset P_f(X)$. Assume that for its limit A_0 (with respect to the Vietoris topology) one has that

$$\mathrm{ext} A_0 \not\subset P_f(X).$$

Then $A_0 \in \mathfrak{A}(X)$, and by force of openness of $\mathfrak{A}(X)$ there exists an open neighbourhood $O(A_0)$ such that $O(A_0) \subset \mathfrak{A}$. Hence, A_0 can not be a limit of the net $\{A_\alpha\}$. The obtained contradiction implies that $A_0 \subset P_f(X)$, and consequently, the limit $\mu_0 = \nu_{A_0}$ of the net $\{\mu_\alpha\}$ is in $OS_f(X)$. □

It is well known (see, for example [6]) that for every compact Hausdorff space X its hyperspace $\exp X$ is also a compact Hausdorff space. Then Lemma 2.2 and Proposition 2.3 yield the following statement:

Corollary 2.4. *For every compact Hausdorff space X, its hyperspace $\exp X$ is homeomorphic to some closed subset of the compact Hausdorff space $OS_f(X)$.*

Now Lemma 2.1 implies the following

Corollary 2.5. *One may assume that $\exp_\omega X \subset OS_{f\omega}(X)$ for any compact Hausdorff space X.*

For given compact Hausdorff spaces X and Y and a continuous map $f : X \to Y$. Then it is easy to see that $OS(f)(OS_f(X)) \subset OS_f(Y)$. So one may define the following the restriction map:

$$OS_f(f) = OS(f)|_{OS_f(X)}.$$

On the other hand $OS_f(f)$ is a continuous map as a restriction of the continuous map $OS(f)$. So, we get a continuous map

$$OS_f(f) : OS_f(X) \to OS_f(Y), \quad \varphi \in C(Y).$$

Thus, the construction OS_f transforms compact Hausdorff spaces to compact Hausdorff spaces, and continuous maps of compact Hausdorff spaces to continuous maps of compact Hausdorff spaces.

Proposition 2.6. *The construction OS_f is a covariant functor in the category $\mathfrak{Comp}$ of compact Hausdorff spaces and their continuous maps.*

Moreover, repeating the procedure performed in [4] we will obtain the following result.

Theorem 2.7. *Functor $OS_f\colon \mathfrak{Comp} \to \mathfrak{Comp}$ is a normal functor.*

3. Contractibility of the space of semi-additive functionals

In this section we establish that if for a given compact Hausdorff space X its hyperspace $\exp X$ is a contractible compact space, then $OS_f(X)$ is also a contractible compact space.

A subset Y of the topological space X is called a *retract* of the space X, if there exists a map $r\colon X \to Y$ (called a *retraction* of X onto Y) whose restriction $r|_Y\colon Y \to Y$ is the identity map $\mathrm{id}_Y\colon Y \to Y$ (*i.e.* $r(y) = y$ for all $y \in Y$), [3, p. 14]. If $r\colon X \to Y$ is a retraction and there exists a homotopy $h\colon X \times [0;1] \to Y$ such that $h(x,0) = x$ and $h(x,1) = r(x)$ for all $x \in X$, then r is called a *deformation retraction* and Y, a *deformation retract* of the space X. A deformation retraction $r\colon X \to F$ is a *strong deformation retraction* if, given any homotopy $h\colon X \times [0;1] \to Y$, we have $h(x,t) = x$ for all $x \in F$, $t \in [0;1]$. A space Y is called an *absolute retract* (notation: $Y \in AR$) if, for each homeomorphism h taking Y to a closed subset hY of an arbitrary space X, the set hY is a retract of X. A space Y is called an *absolute neighborhood retract* (notation: $Y \in ANR$) if, for any homeomorphism h mapping Y to a closed subset hY of an arbitrary space X, hY has a neighborhood U (in X) such that hY is a retract of U.

Let X and Y be two compact subspaces of spaces M and N, respectively, such that $M, N \in AR$. Following [3, p. 17] we say that a sequence of maps $f_k\colon M \to N$, $k = 1, 2, \ldots$, is a *fundamental sequence* from X to Y if, for each neighborhood V of Y (in N) the compact space X has a neighborhood U (in M), such that

$$f_k|_U \simeq f_{k+1}|_U \text{ in } V \text{ for almost all } k = 1, 2, \ldots.$$

Here "for almost all" means "for all but finitely many". The relation $f_k|_U \simeq f_{k+1}|_U$ means that there exists a homotopy $\varphi_k\colon U \times [0;1] \to V$ such that $\varphi_k(x,\, 0) = f_k(x)$, $\varphi_k(x,\, 1) = f_{k+1}(x)$ for all $x \in U$. We denote this fundamental sequence by $\{f_k;\, X,\, Y\}_{M,\, N}$ or, briefly, by $\mathbf{f}$, and write $\mathbf{f}\colon X \to Y$ in M and N. A fundamental sequence $\mathbf{f} = \{f_k;\, X,\, Y\}_{M,N}$ is said to be *generated by a map* $f\colon X \to Y$, if $f_k(x) = f(x)$ for all $x \in X$, $k = 1, 2, \ldots$

Let $\mathbf{f} = \{f_k;\, X,\, Y\}_{M,\, N}$ and $\mathbf{g} = \{g_k;\, X,\, Y\}_{M,\, N}$ be two fundamental sequences. They say [3, p. 27] that $\mathbf{f}$ and $\mathbf{g}$ are homotopic (symbolically $\mathbf{f} \simeq \mathbf{g}$) if for every closed subset A in X there exists a closed subset B in Y that for each neighbourhood V of B in N there exists a neighbourhood U of A in M such that

$$f_k|_U \simeq g_k|_U \text{ in } V \text{ for almost all } k = 1, 2, \ldots.$$

Let X and Y be closed subsets containing in AR-spaces M and N, respectively. They say [3, p. 29] that spaces X and Y are *fundamentally equivalent* (with respect to M, N) if there exist two fundamental sequences $\mathbf{f}\colon X \to Y$ and $\mathbf{g}\colon Y \to X$ for which $\mathbf{gf} \simeq \mathbf{id}_{X,M}$ and $\mathbf{fg} \simeq \mathbf{id}_{Y,\, N}$. Here

$$\mathbf{gf} = \{g_k \circ f_k;\, X,\, X\}_{M,\, M}, \qquad \mathbf{id}_{X,\, M} = \{\mathrm{id}_X;\, X,\, X\}_{M,\, M},$$
$$\mathbf{fg} = \{f_k \circ g_k;\, Y,\, Y\}_{N,\, N}, \qquad \mathbf{id}_{Y,\, N} = \{\mathrm{id}_Y;\, Y,\, Y\}_{N,\, N}.$$

The relation of fundamental equivalence is an equivalence relation. Therefore, the class of all spaces splits into pairwise disjoint classes of spaces, which are called *shapes*, [3, p. 31]. Two spaces belong to the same shape if and only if they are fundamentally equivalent. The shape containing a space X is called the *shape of the space* X and denoted by $Sh(X)$. The notion of shape is topological, *i.e.* two homeomorphic spaces have the same shape. It is known that for two neighborhood retracts A and B, $Sh(A) = Sh(B)$ if and only if A and B are homotopy equivalent.

Take any functional $\nu_A \in OS_f(X)$. Note that each probability measure ξ with finite support, say $\{x_1, \ldots, x_n\}$, is represented as an affine combination $\xi = \sum\limits_{i=1}^{n} \alpha_i \delta_{x_i}$ of the Dirac measures δ_{x_i}, $i = 1, \ldots, n$, uniquely (see, for example, [5]).

Let $\mathrm{ext} A = \{\xi_s : s \in S\}$. By definition $\mathrm{ext} A \subset P_f(X)$. It is clear that the support of each $\xi_s \in \mathrm{ext} A$ is finite, suppose $|\mathrm{supp}\, \xi_s| = n_s$. Let $\xi_s = \sum\limits_{i=1}^{n_s} \alpha_{s,i} \delta_{x_{s,i}}$, $s \in S$. Here for every $s \in S$ we have

$$\alpha_{s,i} \geqslant 0,\ i \in \{1, \ldots, n_s\}, \qquad \sum_{i=1}^{n_s} \alpha_{s,i} = 1. \tag{3.1}$$

Remark 3.1. By the construction, see (2.2), for each ξ_s there exists $i(s) \in \{1, \ldots, n_s\}$ such that $\alpha_{s,i(s)} \geqslant 1 - \frac{1}{n_s+1}$. This inequality and the relations in (3.1) provide the uniqueness such $i(s)$.

To the functional ν_A we associate the set $F_A := \{x_{s,i(s)} : s \in S\}$. The resulting correspondence $OS_f(X) \to \exp X$ is denoted by

$$r_e^O := r_{\exp X}^{OS_f(X)}. \tag{3.2}$$

The map $r_e^O\colon OS_f(X) \to \exp X$ is well defined, see Remark 3.1.

In particular, we have that for each $F \in \exp X$

$$r_e^O(\nu_{\{\delta_x\colon x\in F\}}) = F. \tag{3.3}$$

Lemma 3.2. *For any compact Hausdorff space X the map r_e^O is a retraction.*

Proof. Equality (3.3) immediately implies that under the map r_e^O the points of the space $\exp X$ remain fixed.

To show that the map r_e^O is continuous, take a functional $\nu_{A_0} \in OS_f(X)$ and a net $\{\nu_{A_\alpha}\} \subset OS_f(X)$ converging to ν_{A_0}. Then the net $\{F_{A_\alpha}\} \subset \exp X$ and the set $F_{A_0} \in \exp X$ appear as the values of the map r_e^O at $\{\nu_{A_\alpha}\}$ and ν_{A_0} respectively. Using (2.5) and Remark 3.1 one has that $\mu_{F_{A_\alpha}} = \nu_{A_\alpha}$. Suppose the net $\{F_{A_\alpha}\}$ does not converge to F_{A_0}. Then the net $\{\mu_{F_{A_\alpha}}\}$ also does not converge to $\mu_{F_{A_0}}$. The obtained contradiction ends the proof. □

So, r_e^O is a retraction, and the set $\exp X$ is a retract of the set $OS_f(X)$. Let us establish a stronger statement. While the proof of the following result we identify a set $F \in \exp X$ with the functional $\mu_F \in OS_f(X)$.

Theorem 3.3. *For an arbitrary compact Hausdorff space X its hyperspace $\exp X$ is a strong deformation retract of $OS_f(X)$.*

Proof. Consider the map $h\colon OS_f(X) \times [0;1] \to OS_f(X)$, defined by the formula

$$h(\mu, t) = h_t(\mu) = (1-t)\cdot\mu + t\cdot r_e^O(\mu),$$

for $(\mu, t) \in OS_f(X) \times [0;1]$. It is easy to check that the map h is well defined. Moreover, $h_0 = \mathrm{id}_{OS_f(X)}$ and $h_1 = r_e^O$, that is, h is a homotopy connecting the maps $\mathrm{id}_{OS_f(X)}$ and r_e^O. Further, we have

$$h(\mu_F, t) = (1-t)\cdot\mu_F + t\cdot r_e^O(\mu_F) = \mu_F,$$

i.e. $h_t(\mu_F) = \mu_F$ for all $F \in \exp X$ and $t \in [0;1]$. Thus, $\exp X$ is a strong deformation retract of the compact set $OS_f(X)$. □

From the theorem and statement [3, (5.4), p. 32] we obtain

Corollary 3.4. *For an arbitrary compact Hausdorff space X,*

$$Sh(\exp X) = Sh(OS_f(X)).$$

Recall, [2, p. 29], that a set $A \subset X$ is said to be *contractible* on X *to a set* $B \subset X$ if the embedding $i_A \colon A \to X$ is homotopic to a map $f \colon A \to X$ such that $f(A) \subset B$. If, moreover, B consists of one point, then A *is contractible on* X.

A space X is said to be *locally contractible at a point* $x_0 \in X$ if any neighborhood U of the point x_0 contains a neighborhood U_0 contractible on U to a point, [2, p. 31]. A space X is said to be *locally contractible* if it is locally contractible at each point.

Theorem 3.5. *If for a compact Hausdorff space X the set* $\exp X$ *is contractible, then the space $OS_f(X)$ is also contractible.*

Proof. We will show more: each homotopy $h\colon \exp X \times [0;1] \to \exp Y$ connecting maps $h_0, h_1 \colon \exp X \to \exp Y$ has an extension

$$\tilde{\tilde{h}}\colon OS_f(X) \times [0;1] \to OS_f(Y)$$

which is a homotopy connecting maps $\tilde{\tilde{h}}_0$, $\tilde{\tilde{h}}_1 \colon OS_f(X) \to OS_f(Y)$, here $\tilde{\tilde{h}}_0$ and $\tilde{\tilde{h}}_1$ are extensions of h_0 and h_1 respectively.

Let $g\colon \exp_\omega X \to \exp_\omega Y$ be a map. Note that by (1.3) it has a representation

$$g'(\mu_F)(\varphi) = \mu_{g(F)}(\varphi) = \max\{\delta_y(\varphi) : y \in g(\{x\}),\, x \in F\}, \qquad \varphi \in C(Y).$$

Further, for brevity, put $g(x) = g(\{x\})$.

Now by force of $(1.2')$ an extension $\tilde{g}\colon OS_{f\omega}(X) \to OS_{f\omega}(Y)$ of g may be defined by naturally way using the following equality

$$\tilde{g}\left(\sup\left\{\sum_{i=1}^{n_s} \alpha_{si}\delta_{x_{si}} : s \in S\right\}\right) = \sup\left\{\sum_{i=1}^{n_s} \sum_{y \in g(x_{si})} \frac{\alpha_{si}}{|g(x_{si})|}\delta_y : s \in S\right\}.$$

Consider a homotopy $h\colon \exp_\omega X \times [0;1] \to \exp_\omega Y$ connecting homotopic maps $h_0, h_1 \colon \exp_\omega X \to \exp_\omega Y$, *i.e.* $h(F, 0) = h_0(F)$, $h(F, 1) = h_1(F)$, $F \in \exp_\omega X$.

We define the map $\tilde{h}\colon OS_{f\omega}(X) \times [0;1] \to OS_{f\omega}(Y)$ by the equality

$$\tilde{h}\left(\sup\left\{\sum_{i=1}^{n_s} \alpha_{si}\delta_{x_{si}} : s \in S\right\}, t\right) =$$

$$= \sup\left\{\sum_{i=1}^{n_s} \sum_{y \in h(x_{si}, t)} \frac{\alpha_{si}}{|h(x_{si}, t)|}\delta_y : s \in S\right\}.$$

Then we have

$$\begin{aligned}
\widetilde{h}\left(\sup\left\{\sum_{i=1}^{n_s}\alpha_{si}\delta_{x_{si}} : s\in S\right\}, 0\right) = \\
&= \sup\left\{\sum_{i=1}^{n_s}\sum_{y\in h(x_{si},0)}\frac{\alpha_{si}}{|h(x_{si}, 0)|}\delta_y : s\in S\right\} = \\
&= \sup\left\{\sum_{i=1}^{n_s}\sum_{y\in h_0(x_{si})}\frac{\alpha_{si}}{|h_0(x_{si})|}\delta_y : s\in S\right\} = \\
&= \widetilde{h}_0\left(\sup\left\{\sum_{i=1}^{n_s}\alpha_{si}\delta_{x_{si}} : s\in S\right\}\right),
\end{aligned}$$

$$\begin{aligned}
\widetilde{h}\left(\sup\left\{\sum_{i=1}^{n_s}\alpha_{si}\delta_{x_{si}} : s\in S\right\}, 1\right) = \\
&= \sup\left\{\sum_{i=1}^{n_s}\sum_{y\in h(x_{si}, 1)}\frac{\alpha_{si}}{|h(x_{si}, 1)|}\delta_y : s\in S\right\} = \\
&= \sup\left\{\sum_{i=1}^{n_s}\sum_{y\in h_1(x_{si})}\frac{\alpha_{si}}{|h_1(x_{si})|}\delta_y : s\in S\right\} = \\
&= \widetilde{h}_1\left(\sup\left\{\sum_{i=1}^{n_s}\alpha_{si}\delta_{x_{si}} : s\in S\right\}\right),
\end{aligned}$$

i.e. $\widetilde{h}(\mu, 0) = \widetilde{h}_0(\mu)$ and $\widetilde{h}(\mu, 1) = \widetilde{h}_1(\mu)$ for every $\mu \in OS_{f\omega}(X)$. Lemma 2.1 implies the uniqueness of extensions $\widetilde{\widetilde{h}}\colon OS_f(X)\times[0, 1]\to OS_f(X)$ and $\widetilde{\widetilde{h}}_0, \widetilde{\widetilde{h}}_1\colon OS_f(X)\to OS_f(X)$ such that

$$\widetilde{\widetilde{h}}|_{OS_{f\omega}(X)\times[0,1]} = \widetilde{h}, \qquad \widetilde{\widetilde{h}}_0|_{OS_{f\omega}(X)} = \widetilde{h}_0, \qquad \widetilde{\widetilde{h}}_1|_{OS_{f\omega}(X)} = \widetilde{h}_1.$$

In other words, $\widetilde{\widetilde{h}}$ is a homotopy connecting the maps $\widetilde{\widetilde{h}}_0$ and $\widetilde{\widetilde{h}}_1$. □

Theorem 3.6. *$OS_f(X)\in A(N)R \Leftrightarrow \exp X\in A(N)R$.*

Proof. Theorem 3.3 implies the implication

$$OS_f(X)\in A(N)R \;\Rightarrow\; \exp X\in A(N)R.$$

Let us establish the converse implication. Let $\exp X$ be a neighbourhood retract of some compact Hausdorff space, say, Y. Then one may identify

$$\exp X \equiv \{\{F\} : F \in \exp X\} \subset \exp Y. \tag{3.4}$$

If $\mathcal{U}$ be an open subset of Y such that $\mathcal{U} \supset h(\exp X)$, where $h\colon \exp X \to \mathcal{U}$ is an embedding, then $\langle \mathcal{U} \rangle = \{G \in \exp Y : G \subset \mathcal{U}\}$ is open in $\exp Y$, and according to (3.4) we have $\exp X \subset \langle \mathcal{U} \rangle$.

Now take any retraction $r\colon \langle \mathcal{U} \rangle \to \exp X$ and consider a set

$$\langle \mathcal{U};\, \tfrac{1}{2} \rangle = \{\mu \in P(Y) : \text{ there is } F \in \langle \mathcal{U} \rangle \text{ such that } \mu(F) > \tfrac{1}{2}\}$$

which is open in $P(Y)$ by definition of the pointwise convergence topology.

We will show that

$$P_f(X) \subset \langle \mathcal{U};\, \tfrac{1}{2} \rangle. \tag{3.5}$$

Let $\mu \in P_f(X)$. Then $\operatorname{supp} \mu \in \exp X$ and $r(\operatorname{supp} \mu) = \operatorname{supp} \mu \in \langle \mathcal{U} \rangle$ (since r is a retraction). We have $\mu(\operatorname{supp} \mu) = 1 > \frac{1}{2}$. Hence $\mu \in \langle \mathcal{U};\, \frac{1}{2} \rangle$.

Now we put

$$\langle \mathcal{U};\, \tfrac{1}{2} \rangle_f = \langle \mathcal{U};\, \tfrac{1}{2} \rangle \cap P_f(Y). \tag{3.6}$$

Since $\langle \mathcal{U};\, \frac{1}{2} \rangle_f$ is open in $P_f(Y)$, the set

$$\mathcal{U}_{OSf} := \{\nu_A \in OS(Y) : \operatorname{ext} A \subset \langle \mathcal{U};\, \tfrac{1}{2} \rangle_f\}$$

is open in $OS_f(Y) \subset OS(Y) \cong cc(P(Y))$ with respect to the Vietoris topology. Note that (2.4), (3.5), and (3.6) provide $OS_f(X) \subset \mathcal{U}_{OSf}$. In other words $\mathcal{U}_{OSf}$ is an open neighbourhood of $OS_f(X)$ in $OS_f(Y)$.

It remains to construct a retraction $r_{OS}\colon \mathcal{U}_{OSf} \to OS_f(X)$.

Let $\nu_A \in \mathcal{U}_{OSf} \subset OS_f(Y)$. Since $OS_{f\omega}(Y)$ is everywhere dense in $OS_f(Y)$ (see Lemma 2.1), for any sequence $\{\nu_{A_k}\} \subset OS_{f\omega}(Y)$ converging to ν_A we may assume that $\{\nu_{A_k}\} \subset \mathcal{U}_{OSf\omega} = \mathcal{U}_{OSf} \cap OS_{f\omega}(Y)$. Suppose

$$\nu_{A_k} = \sup \left\{ \sum_{i=1}^{n_{k\xi}} \alpha_{k\xi\, i} \delta_{x_{k\xi\, i}} : \xi \in A_k \right\},$$

where for all $\xi \in A_k$, $k = 1, 2, \ldots$ by construction we have

- $\sum\limits_{i=1}^{n_{k\xi}} \alpha_{k\xi\, i} = 1$, $\alpha_{k\xi\, i} \geqslant 0$, $i = 1, \ldots, n_{k\xi}$;
- there exists a unique $\alpha_{k\xi\, i_{k\xi}}$ such that $\alpha_{k\xi\, i_{k\xi}} \geqslant \frac{n_{k\xi}}{n_{k\xi}+1}$;
- $\operatorname{supp} \xi = \{x_{k\xi\, 1}, x_{k\xi\, 2}, \ldots, x_{k\xi\, n_{k\xi}}\} \in \langle \mathcal{U} \rangle_\omega = \langle \mathcal{U} \rangle \cap \exp_\omega Y$.

Then we put

$$F_{k\xi} = r(\{x_{k\xi\, i_{k\xi}}\}) \subset \exp X.$$

Take an arbitrary $y_\xi \in F_{k\xi}$ and consider a map

$$S_\omega(\nu_{A_k}) = \sup_{\xi\in A_k}\left\{\Big(\alpha_{k\xi\, i_{k\xi}} + \sum_{\{x_{k\xi\, i}\}\in\langle\mathcal{U}\rangle_\omega\setminus\exp_\omega X}\alpha_{k\xi\, i}\Big)\delta_{y_\xi} + \right.$$

$$\left. + \sum_{\{x_{k\xi\, i}\}\in\exp_\omega X}\alpha_{k\xi\, i}\delta_{x_{k\xi\, i}}\right\}. \quad (3.7)$$

It is clear that $\operatorname{supp} S_\omega(\xi) = m_{k\xi} \leqslant n_{k\xi} = \operatorname{supp}\xi$. Therefore

$$\alpha_{k\xi\, i_{k\xi}} + \sum_{\{x_{k\xi\, i}\}\in\langle\mathcal{U}\rangle_\omega\setminus\exp_\omega X}\alpha_{k\xi\, i} \geqslant \alpha_{k\xi\, i_{k\xi}} \geqslant \frac{n_{k\xi}}{n_{k\xi}+1} \geqslant \frac{m_{k\xi}}{m_{k\xi}+1},$$

for all $\xi \in A_k$, $k = 1, 2, \ldots$ Consequently, $S_\omega(\nu_{A_k}) \in OS_{f\omega}(X)$, and S_ω maps $\mathcal{U}_{OSf\omega}$ into $OS_{f\omega}(X)$. We have

$$\{S_\omega(\nu_{A_k})\} \subset OS_{f\omega}(X) \subset OS_f(X).$$

According to Proposition 2.3, $OS_f(X)$ is compact. Then $\{S_\omega(\nu_{A_k})\}$ has a limit in $OS_f(X)$. This limit we denote by $r_{OS}(\nu_A)$, where ν_A is the limit of $\{\nu_{A_k}\}$. Obviously, if $\nu_A \in \mathcal{U}_{OSf\omega}$ then $r_{OS}(\nu_A) = S_\omega(\nu_A)$. Therefore, we got a map $r_{OS}\colon \mathcal{U}_{OSf} \to OS_f(X)$ which is the unique extension of the map $S_\omega\colon \mathcal{U}_{OSf\omega} \to OS_{f\omega}(X)$. Note that at the same time we have proved that $r_{OS}\colon \mathcal{U}_{OSf} \to OS_f(X)$ is continuous.

To establish that r_{OS} is retract, take a $\nu_A \in OS_f(X)$, any sequence $\{\nu_{A_k}\} \subset OS_{f\omega}(X)$ converging to ν_A. Then

$$\operatorname{supp}\xi = \{x_{k\xi 1}, x_{k\xi 2}, \ldots, x_{k\xi n_{k\xi}}\} \in \exp_\omega X, \qquad \xi \in \nu_{A_k}.$$

In other words, $(\operatorname{supp}\xi) \cap \big(\langle\mathcal{U}\rangle_\omega\setminus\exp_\omega X\big) = \varnothing$. Since r is a retraction, we have $r(\{x_{k\xi\, n_{k\xi}}\}) = \{x_{k\xi\, n_{k\xi}}\}$. So, in this case formula (3.7) has the following form

$$S_\omega(\nu_{A_k}) = \sup_{\xi\in A_k}\left\{\sum_{\{x_{k\xi\, i}\}\in\exp_\omega X}\alpha_{k\xi\, i}\delta_{x_{k\xi\, i}}\right\} = \nu_{A_k}.$$

Hence, $r_{OS}(\nu_A) = \nu_A$. Thus, $r_{OS}\colon \mathcal{U}_{OSf} \to OS_f(X)$ is a continuous retraction. □

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