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Open finite-to-one functions on open topological graphs

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Abstract. The paper describes homotopy classes of open continuous functions on finite open topological graphs.

Анотація. В роботі описано класи гомотопії неперервних відкритих функцій на відкритих топологічних графах.

1. Introduction

By a *topological graph* we will mean a finite 1-dimensional CW-complex. Its 0-cells are called *vertices* and 1-cell are *edges*. We can always replace each vertex of degree 2 and a pair of edges incident to it with a single edge and assume that a topological graph has no vertices of degree 2.

Denote by Σ the class of all topological graphs. A topological space G is called an *open topological graph* if there is a topological graph $\bar{G}$ such that G is a complement to some set of vertices of degree 1 in $\bar{G}$. The corresponding topological graph $\bar{G}$ is called a *closure* of G. An edge of G incident to a vertex of degree 1 will be called *semifree*. We will also regard the real line $\mathbb{R}$ as a special open topological graph with 1 edge and without vertices.

Let $f : X \to Y$ be a continuous map between topological spaces. Then f is called *open* if the image of each open set is open. A map f is *open at a point* $x \in G$ if there exists an open base of topology B at x such that for every neighborhood U of x in B its image $f(U)$ is open. It is easy to see that f is open iff it is open at each point $x \in X$.

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Recall also that f is called *finite-to-one* whenever the preimage of each point is finite, [1].

2. FUNCTIONS ON OPEN TOPOLOGICAL GRAPHS

Lemma 2.1. *Let G be a topological graph without isolated vertices and $f : G \to \mathbb{R}$ a continuous function. Then f is an open map iff the following two conditions hold:*

(a) *f is strictly monotone on edges;*

(b) *for every vertex v the open edges incident to it split into two non-empty collections: $A = \{e_1, \dots, e_k\}$ and $B = \{e_{k+1}, \dots e_l\}$ such that $f(A) \subset (-\infty, f(v))$ and $f(B) \subset (f(v), +\infty)$.*

Proof. *Necessity.* (a) Suppose f is an open map. Notice that then f has no local extremes. Indeed, if $x \in G$ for example is a local maximum, then for a small neighborhood U of x we have that $f(U) \subset (-\infty, f(x)]$, and therefore $f(U)$ is not an open neighborhood of $f(x)$, which contradicts to the openness of f.

Assume that f is not monotone on an edge e, so there are points $x \neq y \in e$ such that $f(x) = f(y)$. Let $[x, y] \subset e$ be the closed segment between x and y, and (x, y) be its interior. Then $f([x, y])$ is a compact connected subset of $\mathbb{R}$, and therefore it is either a point or a closed segment. If $f([x, y])$ is a point, then $f\big((x, y)\big)$ is also a point and it not an open subset of $\mathbb{R}$, which contradicts to the openness of f. Otherwise, $f([x, y])$ is a segment of the form $[f(x), a]$ or $[a, f(x)]$ for some $a \in \mathbb{R}$, so there exists $z \in (x, y)$ with $f(z) = a$. But then z is a local extreme of the restriction of f to open set (x, y), which again contradicts to the openness of f. This proves (a).

(b) Notice that it follows from (a) that if e is an edge incident to some vertex v, then either $f(e) \subset (-\infty, f(v))$ or $f(e) \subset (f(v), +\infty)$.

Suppose condition (b) fails, so there exists a vertex such that for all edges $\{e_i\}$ incident to v we have that

- either $f(\cup_i e_i) \subset (-\infty, f(v))$
- or $f(\cup_i e_i) \subset (f(v), +\infty)$.

Then v is a local extreme of f, which contradicts to the opennes of f.

Sufficiency. Suppose (a) and (b) hold. Then (a) implies that f is open at points of edges, while (b) implies openness of f at vertices of G. □

Corollary 2.2. *If G is a finite topological graph, then every open function $f : G \to \mathbb{R}$ is finite-to-one.*

3. EXISTENCE OF FUNCTIONS FROM THE CLASS $\mathfrak{F}(G,\mathbb{R})$

Denote by Θ the class of all connected open topological graphs.

For G_1, $G_2 \in \Theta$ let $\mathfrak{F}(G_1, G_2)$ be the set of continuous open (and therefore finite-to-one) maps from G_1 to G_2. Such maps a used in particular for modelling of holomorphic maps of Riemann surfaces, e.g. [3].

The class $\mathfrak{F}(G,\mathbb{R})$ can be empty for some graphs in Θ. The simplest examples of a graph $G \in \Theta$ such that the class $\mathfrak{F}(G,\mathbb{R})$ is empty are the circle or a graph that has only a vertex and one semifree edge.

Let $G \in \Theta$ and let $\bar{G} \in \Sigma$ be a closure of G. Denote by $\mathfrak{M}(\bar{G}, \mathbb{R} \cup \{\pm\infty\})$ the class of functions on $\bar{G}$ which are strictly monotone on edges and have no local extremes at vertices of degree > 1.

By Lemma 2.1 for every function $\bar{f} \in \mathfrak{M}(\bar{G}, \mathbb{R} \cup \{\pm\infty\})$ its restriction to G is a function of the class $\mathfrak{F}(G,\mathbb{R})$. Conversely, if f belongs to $\mathfrak{F}(G,\mathbb{R})$, then f extends to a unique continuous function $\bar{f} \in \mathfrak{M}(\bar{G}, \mathbb{R} \cup \{\pm\infty\})$. In other words we have a canonical bijection:

$$\mathfrak{M}(\bar{G}, \mathbb{R} \cup \{\pm\infty\}) \to \mathfrak{F}(G,\mathbb{R}), \qquad \bar{f} \mapsto \bar{f}|_G = f.$$

Strictly monotone on edges functions on topological graphs were studied in connection with Kronrod-Reeb graphs of smooth functions on manifolds by V. V. Sharko in [2]. In that paper were given an existence criterion.

Definition 3.1. ([2, §3]) Let $\Gamma \in \Sigma$ be a connected topological graph. A *func-orientation* $\mathcal{O}$ of Γ is an orientation on its edges having the following properties:

(1) Γ has vertices of degree 1 with incoming and outcoming edges;
(2) each vertex of degree $\geqslant 2$ has an incoming and an outcoming edges;
(3) Γ has no oriented closed cycles[1].

Notice that *func*-orientation $\mathcal{O}$ of Γ uniquely determines a partial order P_0 on vertices of Γ: $x < y$ iff there exists an oriented sequence of edges that starts in x and ends in y.

Definition 3.2. ([2, §3]) A partial order P on the vertices of Γ is said to be *consistent with func-orientation* $\mathcal{O}$ if the relations that are true for P_0 are also true for P. In other words, the identity map of the set of vertices is a morphism $P_0 \to P$ of partially ordered sets.

Definition 3.3. ([2, §3]) A *func*-oriented graph is a triple $(\Gamma, \mathcal{O}, P)$, where $\Gamma \in \Sigma$, $\mathcal{O}$ is a *func*-orientation of Γ and P is a partial order on vertices of Γ which is consistent with the *func*-orientation $\mathcal{O}$.

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[1] A set of oriented edges of Γ that is homeomorphic to an oriented circle.

In particular, if $\Gamma \in \Sigma$ admits *func*-orientation $\mathcal{O}$, then $(\Gamma, \mathcal{O}, P_0)$ is a *func*-oriented graph.

Also notice that every function $\bar{f} \in \mathfrak{M}(\bar{G}, \mathbb{R} \cup \{\pm\infty\})$ determines a natural *func*-orientation $\mathcal{O}_f$ of edges towards increasing of f, and the consistent with it partial order P_f on vertices given by the rule: $v_1 < v_2$ if and only if $f(v_1) < f(v_2)$. In other words f determines a *func*-oriented graph $(\bar{G}, \mathcal{O}_f, P_f)$.

Theorem 3.4. ([2, Theorem 3.1]) *For each func-oriented graph* $(\bar{G}, \mathcal{O}, P)$ *there exists* $f \in \mathfrak{M}(\bar{G}, \mathbb{R} \cup \{\pm\infty\})$ *such that* $(\bar{G}, \mathcal{O}, P) = (\bar{G}, \mathcal{O}_f, P_f)$.

Since the assumption $f \in \mathfrak{F}(\bar{G}, \mathbb{R})$ is equivalent to $\bar{f} \in \mathfrak{M}(\bar{G}, \mathbb{R} \cup \{\pm\infty\})$, we obtain the following:

Corollary 3.5. *A graph* $G \in \Theta$ *admits a function* $f \in \mathfrak{F}(G, \mathbb{R})$ *iff* $\bar{G}$ *admits a func-orientation.*

4. HOMOTOPIES IN CLASS $\mathfrak{F}(G, \mathbb{R})$

Lemma 4.1. *Let* $G \in \Theta$ *and* $f, g \in \mathfrak{F}(G, \mathbb{R})$. *Then the func-orientations induced by* f *and* g *on* $\bar{G}$ *coincide if and only if the functions* f *and* g *are homotopic in the class* $\mathfrak{F}(G, \mathbb{R})$.

Proof. *Necessity.* Suppose f and g induce the same *func*-orientations on edges of $\bar{G}$ (and therefore of G). Then, for each vertex v of G its edges split into the same non-empty collections A_v and B_v such that

$$f(A_v) \subset (-\infty, f(v)), \qquad f(B_v) \subset (f(v), +\infty),$$
$$g(A_v) \subset (-\infty, g(v)), \qquad g(B_v) \subset (g(v), +\infty).$$

Consider the following homotopy $h^t = tf + (1-t)g$ between f and g in the class of continuous functions. This formula easily implies that each h^t is also monotone on each edge e, increases in the same direction as both $f|_e$ and $g|_e$, and defines the same splitting of edges incident to v so that

$$h^t(A_v) \subset (-\infty, h^t(v)), \qquad h^t(B_v) \subset (h^t(v), +\infty).$$

Then by Lemma 2.1 each h^t is open, whence that homotopy is in the class $\mathfrak{F}(G, \mathbb{R})$.

Sufficiency. Assume the functions f and g are homotopic in the class $\mathfrak{F}(G, \mathbb{R})$ by some homotopy h^t but there exists an edge $e \in G$ and distinct points $x \neq y \in e$ such that $f(x) < f(y)$ and $g(x) > g(y)$. Then by continuity of h there exists $t_0 \in (0, 1)$ such that $h^{t_0}(x) = h^{t_0}(y)$, whence h^{t_0} is not strictly monotone on edges, and thus does not belong to $\mathfrak{F}(G, \mathbb{R})$. This gives a contradiction.

This each h^t induces certain orientation $\mathcal{O}_t$ of edges of G. By continuity of the homotopy those orientation must coincide for all $t \in [0,1]$. But $f = h^0$ and $g = h^1$, whence $\mathcal{O}_f = \mathcal{O}_0 = \mathcal{O}_1 = \mathcal{O}_g$. $\square$

Corollary 4.2. *Let $G \in \Theta$. Then the homotopy classes of the set $\mathfrak{F}(G,\mathbb{R})$ in the class $\mathfrak{F}(G,\mathbb{R})$ are convex subsets of the linear space $C(G,\mathbb{R})$ of continuous functions on G.*

5. COUNTING HOMOTOPY CLASSES IN $\mathfrak{F}(G,\mathbb{R})$

Theorem 5.1. *Let $G \in \Theta$ be a tree (i.e. has no cycles) consisting of k vertices of degrees $c_1, \ldots, c_k$. Then the number of homotopy classes of $\mathfrak{F}(G,\mathbb{R})$ is*

$$2\prod_{i=1}^{k}(2^{c_i-1}-1).$$

Proof. Since G is a tree, it follows that $\bar{G}$ is also a tree, and therefore both such graphs admit *func*-orientations. Then by Corollary 3.5, the set $\mathfrak{F}(G,\mathbb{R})$ is non-empty. Moreover, due to Lemma 4.1, the number of path components of $\mathfrak{F}(G,\mathbb{R})$ is finite and is equal to the number of distinct orientations on edges of G such that each vertex has incoming and outcoming edges. Let us calculate this number.

Choose a semifree edge e with vertex v_1 of degree c_1. The map of e can either preserve or reverse its orientation. Fix any orientation of e, say assume that e is incoming for v. Then there are 2^{c_1-1} variants of orientation the remaining c_1-1 edges. All such variants are admissible except for the case when those c_1-1 edges are incoming for v as well as e. Thus we obtain $2^{c_1-1}-1$ orientations of edges incident to v assuming that one fixed edge e is incoming.

Fix any admissible orientation at v and consider the rest of the graph $G\backslash\{e_1 \cup c_1\}$. It consists of c_1-1 trees for which of them we have have fixed an orientation of one semifree edges incident to v_1. Then applying the same arguments we will get that the total number of admissible orientation of G with fixed orientation of e_1 is $\prod_{i=1}^{k}(2^{c_i-1}-1)$. Hence the total number of orientations G is $2\prod_{i=1}^{k}(2^{c_i-1}-1)$. $\square$

Theorem 5.2. *Let $G \in \Theta$ be a func-orientable graph consisting of k vertices of degrees $c_1, \ldots, c_k$ and has m cycles. Denote by $N_{\mathfrak{F}(G,\mathbb{R})}$ the number of homotopy classes of functions of $\mathfrak{F}(G,\mathbb{R})$. Then*

$$\sup_{G'} 2\Big(\prod_{i=1}^{k}(2^{c_i'-1}-1)\Big) \leqslant N_{\mathfrak{F}(G,\mathbb{R})} < 2\Big(\prod_{i=1}^{k}(2^{c_i-1}-1)\Big),$$

where G' runs over all spanning subtrees (that is subtrees obtained from G by removing m edges), and $\{c'_i\}$ are the degrees of vertices in G'.

Proof. Let G' be any tree obtained by removing m edges $e_1, \dots, e_m$ from G. According to Theorem 5.1, the tree G' has $2\prod_{i=1}^{k}(2^{c'_i-1}-1)$ homotopy classes of maps from $\mathfrak{F}(G', \mathbb{R})$ corresponding to distinct choices of orientations of its edges, where c'_i, $i = 1, \dots, k$, are degrees of vertices in the tree G'.

Evidently, every such an orientation of edges of G' extends to some orientation of edges of G, because the orientation for each removed edge e_i is uniquely defined by values of both vertices of e_i with a function that represents a homotopy class from $\mathfrak{F}(G', \mathbb{R})$. Hence each homotopy class for the tree G' is also a homotopy class for G which implies the inequality: $2(\prod_{i=1}^{k}(2^{c'_i-1}-1)) \leqslant N_{\mathfrak{F}(G,\mathbb{R})}$.

Note that this inequality is not always strict. For example, the equality is reached on a graph that has exactly two homotopy classes of open functions and consists of two vertices a and b, two edges incident to both a and b, a semifree edge in a and a semifree edge in b.

To prove the second inequality, instead of removing edges $e_1, \dots, e_m$ from G let us split each edge e_i into two semifree edges by removing a point from its interior. This will give a new tree G''. By construction each homotopy class for G is also a homotopy class for G''. This implies that $N_{\mathfrak{F}(G,\mathbb{R})} < 2(\prod_{i=1}^{k}(2^{c_i-1}-1))$. This inequality is in fact always strict, since there exist homotopy classes for G'' where the parts of split edges e_i get different orientations which is impossible for G. ☐

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