

Rectangular diagrams of surfaces: the basic moves

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Abstract. In earlier papers we introduced a representation of isotopy classes of compact surfaces embedded in the three-sphere \mathbb{S}^3 by so-called rectangular diagrams. The formalism proved useful for comparing Legendrian knots. The aim of this paper is to prove a Reidemeister type theorem for rectangular diagrams of surfaces.

Анотація. У попередніх роботах ми визначили представлення класів ізо-топії компактних поверхонь, вкладених в 3-сферу \mathbb{S}^3 , за допомогою так званих прямокутних діаграм. Цей формалізм виявився корисним для порівняння лежандрових вузлів. Метою даної роботи є доведення теореми типу Рейдемайстера для прямокутних діаграм поверхонь.

1. INTRODUCTION

We work in the piecewise smooth category. Unless otherwise specified, all homeomorphisms and isotopies are assumed to be piecewise smooth.

Throughout the paper, a *surface* $F \subset \mathbb{S}^3$ means a compact smooth surface with corners embedded in the three-sphere \mathbb{S}^3 . ‘With corners’ means that the boundary of F is a union of cusp-free piecewise smooth curves, and that F can be extended beyond the boundary to become a smooth submanifold of \mathbb{S}^3 with smooth boundary. Surfaces are not assumed to be orientable or to have a non-empty boundary.

If the tangent plane $T_p F$ to a surface $F \subset \mathbb{S}^3$ at a point $p \in F$ is said to be *preserved* by a self-homeomorphism ϕ of \mathbb{S}^3 this means that $\phi(p) = p$ and $\phi(F)$ has a well defined tangent plane at p coinciding with $T_p F$.

The work is supported by the Russian Science Foundation under grant 19-11-00151

Keywords: rectangular diagrams

Ключові слова: прямокутні діаграми

DOI: <http://dx.doi.org/10.15673/tmgc.v13i4.1855>

If $F_1, F_2 \subset \mathbb{S}^3$ are two surfaces, then by a *morphism from F_1 to F_2* we mean a connected component of the space of orientation preserving self-homeomorphisms ϕ of \mathbb{S}^3 such that $\phi(F_1) = F_2$. The morphism represented by a homeomorphism ϕ will be denoted by $[\phi]$.

In [3] we introduced rectangular diagrams of surfaces (see definitions below), and with every diagram Π we associated a surface $\hat{\Pi} \subset \mathbb{S}^3$. We also showed [3, Theorem 1] that every isotopy class of surfaces can be represented by a rectangular diagram of a surface.

In [1] we defined basic moves for rectangular diagrams of surfaces, which are transformations of rectangular diagrams such that the associated surface is transformed by an isotopy. So, every basic move $\Pi \mapsto \Pi'$ comes with a well defined morphism from $\hat{\Pi}$ to $\hat{\Pi}'$.

The main result of the present paper (which is announced in [1] in a slightly weaker form) is Theorem 2.9, which states that any morphism between surfaces represented by rectangular diagrams can be decomposed into basic moves, and if the given morphism preserves a sublink of the surface boundary, then so does each basic move in the decomposition.

The paper is organized as follows. In Section 2 we introduce rectangular diagrams and their moves, and formulate the main result of the paper. In Sections 3–5 we introduce auxiliary tools and prove intermediate results. Sections 6 and 7 contain the proof of the main result.

2. PRELIMINARIES. RECTANGULAR DIAGRAMS AND BASIC MOVES

We recall some definitions and notation from [1, 3].

For two distinct points x, y of the circle \mathbb{S}^1 we denote by $[x; y]$ a unique arc of \mathbb{S}^1 such that, with respect to the standard orientation of \mathbb{S}^1 , it has starting point x , and end point y .

By \mathbb{T}^2 we denote the two-torus $\mathbb{S}^1 \times \mathbb{S}^1$ and by θ and φ the angular coordinates on the first and the second \mathbb{S}^1 -factor, respectively.

We identify the three-sphere \mathbb{S}^3 with the join of two circles:

$$\mathbb{S}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times [0; 1] / \sim,$$

where \sim stands for the following equivalence relation:

$$(\theta, \varphi, 0) \sim (\theta', \varphi, 0), \quad (\theta, \varphi, 1) \sim (\theta, \varphi', 1) \quad \forall \theta, \theta', \varphi, \varphi' \in \mathbb{S}^1,$$

and use θ, φ, τ for the corresponding coordinate system.

The *torus projection* $\text{pr}_{\mathbb{T}^2}$ is defined as the following map from

$$\mathbb{S}^3 \setminus (\mathbb{S}_{\tau=0}^1 \cup \mathbb{S}_{\tau=1}^1)$$

to \mathbb{T}^2 :

$$\text{pr}_{\mathbb{T}^2}(\theta, \varphi, \tau) = (\theta, \varphi).$$

For a point $v \in \mathbb{T}^2$, we denote by \hat{v} the closed arc

$$\hat{v} = \overline{\text{pr}_{\mathbb{T}^2}^{-1}(v)}.$$

For a finite subset $X \subset \mathbb{T}^2$, we define \hat{X} to be the union

$$\hat{X} = \bigcup_{v \in X} \hat{v}.$$

For $\theta, \varphi \in \mathbb{S}^1$ we also denote by m_θ the *meridian* $\{\theta\} \times \mathbb{S}^1 \subset \mathbb{T}^2$, and by ℓ_φ the *longitude* $\mathbb{S}^1 \times \{\varphi\}$. By \hat{m}_θ and $\hat{\ell}_\varphi$ we denote the endpoints of the arc $\widehat{(\theta, \varphi)}$, lying on $\mathbb{S}_{\tau=1}^1$ and $\mathbb{S}_{\tau=0}^1$, respectively, that is,

$$\hat{m}_\theta = (\theta, *, 1)/\sim \in \mathbb{S}_{\tau=1}^1, \quad \hat{\ell}_\varphi = (*, \varphi, 0)/\sim \in \mathbb{S}_{\tau=0}^1.$$

By a *rectangle* we mean a subset $r \subset \mathbb{T}^2$ of the form $[\theta_1; \theta_2] \times [\varphi_1; \varphi_2]$. By $V(r)$ we denote the set of vertices of r : $V(r) = \{\theta_1, \theta_2\} \times \{\varphi_1, \varphi_2\}$. We also set

$$V_{/}(r) = \{(\theta_1, \varphi_2), (\theta_2, \varphi_1)\}, \quad V_{\setminus}(r) = \{(\theta_1, \varphi_1), (\theta_2, \varphi_2)\}.$$

Two rectangles r_1, r_2 are said to be *compatible* if their intersection satisfies one of the following:

- (1) $r_1 \cap r_2$ is empty;
- (2) $r_1 \cap r_2$ is a subset of vertices of r_1 (equivalently: of r_2);
- (3) $r_1 \cap r_2$ is a rectangle disjoint from the vertices of both rectangles r_1 and r_2 .

Definition 2.1. By a *rectangular diagram of a surface* we mean a collection $\Pi = \{r_1, \dots, r_k\}$ of pairwise compatible rectangles in \mathbb{T}^2 such that every meridian $\{\theta\} \times \mathbb{S}^1$ and every longitude $\mathbb{S}^1 \times \{\varphi\}$ of the torus contains at most two free vertices, where by a *free vertex* we mean a point that is a vertex of exactly one rectangle in Π .

The set of all free vertices of Π is called *the boundary of Π* and denoted by $\partial\Pi$.

All elements of the union $\bigcup_{r \in \Pi} V(r)$ are called *vertices* of Π . The elements of the union $\bigcup_{r \in \Pi} V_{/}(r)$ (respectively, of $\bigcup_{r \in \Pi} V_{\setminus}(r)$) are called */-vertices* (respectively, *\-vertices*) of Π , and said to be *of type /* (respectively, *of type *).

All meridians and longitudes of \mathbb{T}^2 passing through vertices of Π are called *occupied levels* of Π .

With every rectangle $r \subset \mathbb{T}^2$ one can associate a surface $\hat{r} \subset \mathbb{S}^3$ homeomorphic to a two-disc so that the following holds for the map $r \mapsto \hat{r}$:

- (1) for any rectangle $r \subset \mathbb{T}^2$, the torus projection $\text{pr}_{\mathbb{T}^2}$ takes the interior of \hat{r} to the interior of r homeomorphically;

- (2) $\partial\hat{r} = \widehat{V}(r)$;
- (3) whenever r_1 and r_2 are compatible rectangles, the interiors of \hat{r}_1 and \hat{r}_2 are disjoint;
- (4) whenever r_1 and r_2 are compatible and share a vertex, the union $\hat{r}_1 \cup \hat{r}_2$ is a surface (that is to say, there are no singularities at $\hat{r}_1 \cap \hat{r}_2$).

In [3], the discs \hat{r} satisfying these properties are defined explicitly, but any other choice will work as well. The choice is supposed to be fixed from now on.

Definition 2.2. Let Π be a rectangular diagram of a surface. *The surface $\hat{\Pi}$ associated with Π* is defined as

$$\hat{\Pi} = \bigcup_{r \in \Pi} \hat{r}.$$

One can see that the boundary $\partial\Pi$ of a rectangular diagram of a surface is a rectangular diagram of a link in the sense of [2, 3], and we have

$$\partial\hat{\Pi} = \partial\Pi.$$

By a *basic move* of rectangular diagrams of surfaces we mean any of the transformations defined below in this section. These include: (half-) wrinkle creation and reduction moves, (de)stabilization moves, exchange moves, and flypes.

‘Transformations’ means pairs (Π, Π') of diagrams endowed with a morphism $\hat{\Pi} \rightarrow \hat{\Pi}'$. In each definition of a basic move $\Pi \mapsto \Pi'$ defined below, the description of the pair (Π, Π') naturally suggests what the morphism $\hat{\Pi} \rightarrow \hat{\Pi}'$ should be. So, we omit the description of this morphism in Definitions 2.3–2.7 and provide the necessary hints afterwards.

Definition 2.3. Let Π be a rectangular diagram of a surface, and

$$v_1 = (\theta_0, \varphi_1) \quad \text{and} \quad v_2 = (\theta_0, \varphi_2)$$

be a \setminus -vertex and a \swarrow -vertex of Π , respectively, lying on the same meridian m_{θ_0} . Choose an $\varepsilon > 0$ so that no meridian in

$$[\theta_0 - 2\varepsilon; \theta_0 + 2\varepsilon] \times \mathbb{S}^1 \subset \mathbb{T}^2$$

other than m_{θ_0} is an occupied level of Π . Also choose an orientation-preserving self-homeomorphism ψ of the interval $[\theta_0 - 2\varepsilon; \theta_0 + 2\varepsilon]$.

Let Π' be the rectangular diagram of a surface obtained from Π by making the following modifications:

- (1) every rectangle of the form

$$[\theta_0; \theta_1] \times [\varphi'; \varphi''], \quad (\text{resp. } [\theta_1; \theta_0] \times [\varphi'; \varphi''])$$

with $[\varphi'; \varphi''] \subset [\varphi_1; \varphi_2]$ is replaced by

$$[\psi(\theta_0 + \varepsilon); \theta_1] \times [\varphi'; \varphi''], \quad (\text{resp. } [\theta_1; \psi(\theta_0 + \varepsilon)] \times [\varphi'; \varphi'']).$$

(2) every rectangle of the form

$$[\theta_0; \theta_1] \times [\varphi'; \varphi''], \quad (\text{resp. } [\theta_1; \theta_0] \times [\varphi'; \varphi''])$$

with $[\varphi'; \varphi''] \subset [\varphi_2; \varphi_1]$ is replaced by

$$[\psi(\theta_0 - \varepsilon); \theta_1] \times [\varphi'; \varphi''], \quad (\text{respectively, by } [\theta_1; \psi(\theta_0 - \varepsilon)] \times [\varphi'; \varphi'']),$$

(3) the following two new rectangles are added:

$$[\psi(\theta_0 - \varepsilon); \psi(\theta_0)] \times [\varphi_1; \varphi_2] \quad \text{and} \quad [\psi(\theta_0); \psi(\theta_0 + \varepsilon)] \times [\varphi_2; \varphi_1].$$

Then we say that the passage from Π to Π' is a *vertical wrinkle creation move*. The inverse operation is referred to as a *vertical wrinkle reduction move*.

Horizontal wrinkle creation and reduction moves are defined similarly with the roles of θ and φ exchanged.

A vertical wrinkle move is illustrated in Figure 2.1. The left pair of pictures shows how the rectangular diagram changes, and the right pair of pictures shows the change in the corresponding tiling of $\hat{\Pi}$.

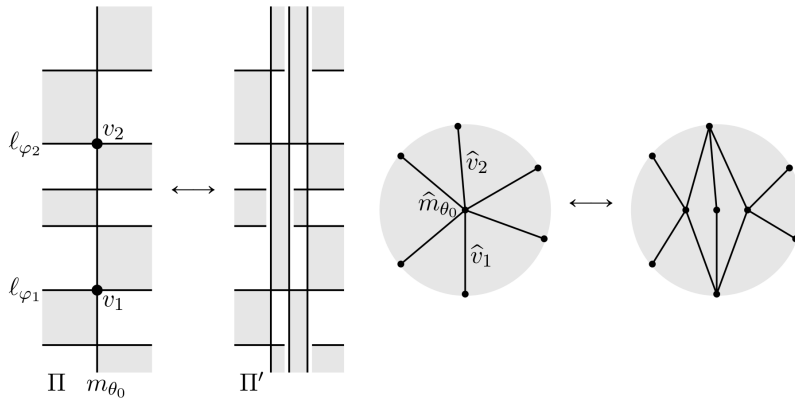


FIGURE 2.1. A vertical wrinkle move

Definition 2.4. Let Π , v_1 , v_2 , and ψ be as in Definition 2.3 and suppose additionally that we have $v_1, v_2 \in \partial\Pi$. Let Π' be obtained from Π as described in Definition 2.3 with the following one distinction:

- if Π has no rectangle of the form $[\theta_0; \theta_1] \times [\varphi'; \varphi'']$ or $[\theta_1; \theta_0] \times [\varphi'; \varphi'']$ with $[\varphi'; \varphi''] \subset [\varphi_1; \varphi_2]$, we do not add the rectangle

$$[\psi(\theta_0); \psi(\theta_0 + \varepsilon)] \times [\varphi_2; \varphi_1];$$

- if Π has no rectangle of the form $[\theta_0; \theta_1] \times [\varphi'; \varphi'']$ or $[\theta_1; \theta_0] \times [\varphi'; \varphi'']$ with $[\varphi'; \varphi''] \subset [\varphi_2; \varphi_1]$, we do not add the rectangle

$$[\psi(\theta_0 - \varepsilon); \psi(\theta_0)] \times [\varphi_1; \varphi_2].$$

One of these two cases must occur.

Then we say that the passage from Π to Π' is a *vertical half-wrinkle creation move*, and the inverse operation is a *vertical half-wrinkle reduction move*.

Horizontal half-wrinkle moves are defined similarly with the roles of θ and φ exchanged.

A vertical half-wrinkle move is illustrated in Figure 2.2. In the case

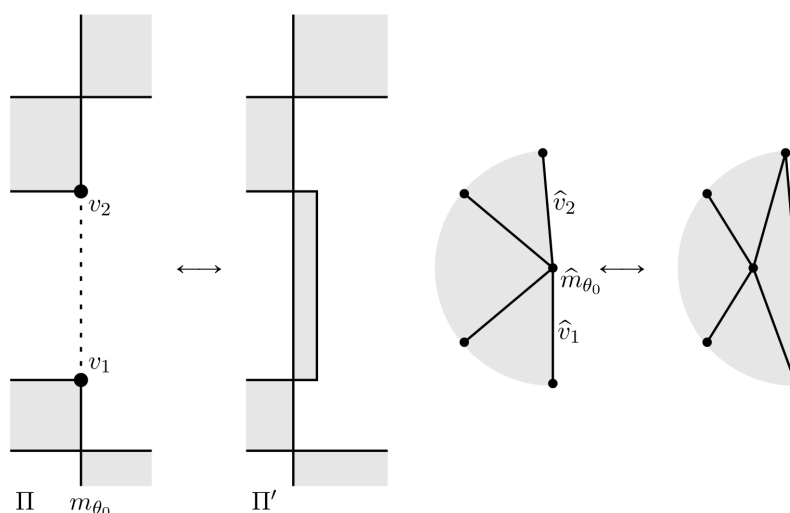


FIGURE 2.2. A vertical half-wrinkle move

$v_1, v_2 \in \partial\Pi$ the respective wrinkle creation move can be decomposed into two half-wrinkle creation moves, which justifies the term ‘half-wrinkle’.

Definition 2.5. Let Π , v_1 , v_2 , and ψ be as in Definition 2.3 except that $v_1 \in m_{\theta_0}$ is not a vertex of Π and, moreover, v_1 does not belong to any rectangle and any occupied longitude of Π . Let Π' be obtained from Π by exactly the same modification as the one described in Definition 2.3, and let T be a transformation of rectangular diagrams induced by any of the following symmetries or any composition of them (including the identity transformation):

$$(\theta, \varphi) \mapsto (\varphi, \theta), \quad (\theta, \varphi) \mapsto (-\theta, \varphi).$$

Then the passage from $T(\Pi)$ to $T(\Pi')$ is called a *stabilization move* and the inverse one a *destabilization move*.

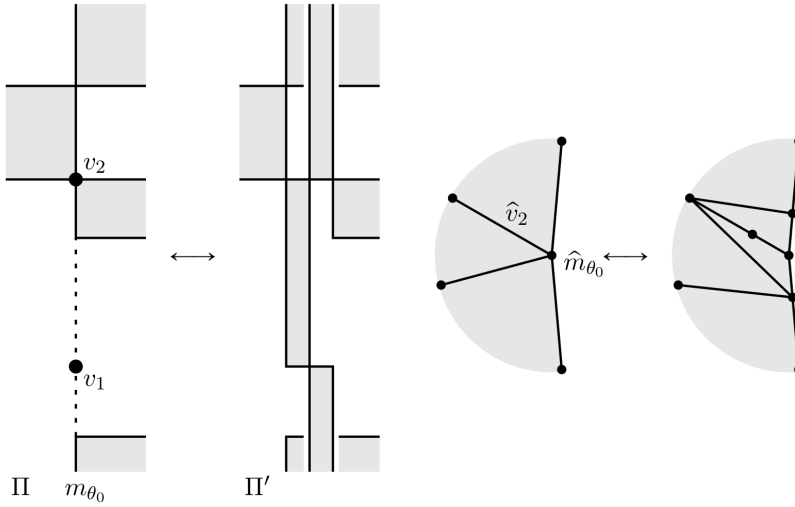


FIGURE 2.3. Stabilization/destabilization moves

An example of a (de)stabilization move is shown in Figure 2.3.

Note also that if $\Pi \mapsto \Pi'$ is a stabilization of a rectangular diagram of a surface, then $\partial\Pi \mapsto \partial\Pi'$ is a stabilization of a rectangular diagram of a link (in the generalized sense of [2]).

Definition 2.6. Let Π be a rectangular diagram of a surface, and let $\theta_1, \theta_2, \theta_3, \varphi_1, \varphi_2 \in \mathbb{S}^1$ be such that:

- (1) we have $\theta_2 \in (\theta_1; \theta_3)$;
- (2) the rectangles $r_1 = [\theta_1; \theta_2] \times [\varphi_1; \varphi_2]$ and $r_2 = [\theta_2; \theta_3] \times [\varphi_2; \varphi_1]$ contain no vertices of Π ;
- (3) the vertices of r_1 and r_2 are disjoint from the rectangles of Π .

Let $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a map that is identical on $[\theta_3; \theta_1]$, and exchanges the intervals $(\theta_1; \theta_2)$ and $(\theta_2; \theta_3)$ as follows:

$$f(\theta) = \begin{cases} \theta - \theta_2 + \theta_3 & \text{if } \theta \in (\theta_1; \theta_2], \\ \theta - \theta_2 + \theta_1 & \text{if } \theta \in (\theta_2; \theta_3). \end{cases}$$

Choose a self-homeomorphism ψ of \mathbb{S}^1 identical on $[\theta_3; \theta_1]$, and let

$$\Pi' = \{[\psi(f(\theta')); \psi(f(\theta''))] \times [\varphi'; \varphi''] \mid [\theta'; \theta''] \times [\varphi'; \varphi''] \in \Pi\}.$$

Then we say that the passage from Π to Π' , or the other way, is a *vertical exchange move*.

A *horizontal exchange move* is defined similarly with the roles of θ and φ exchanged.

An example of a vertical exchange move is shown in Figure 2.4.

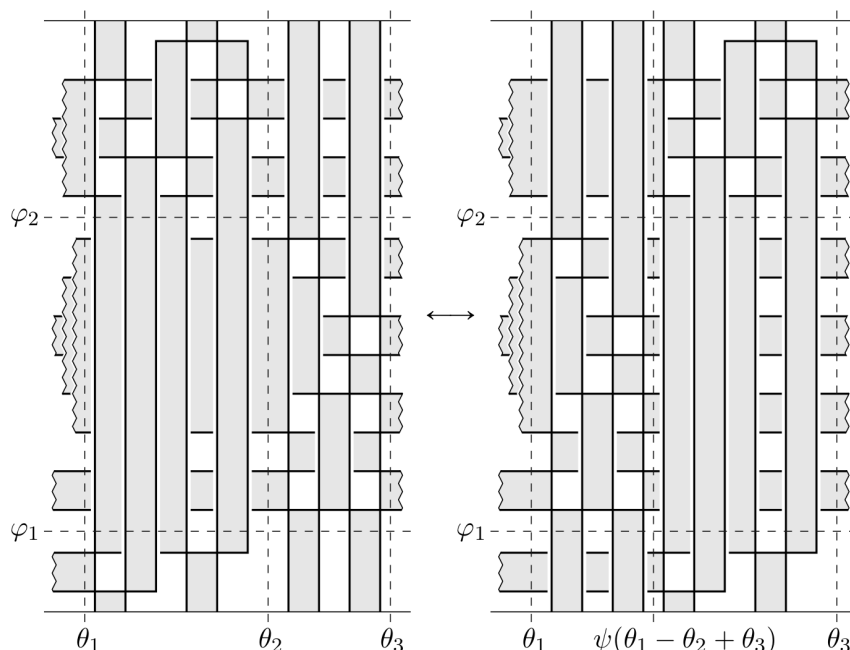


FIGURE 2.4. An exchange move

Definition 2.7. Let Π be a rectangular diagram of a surface, and let $\theta_1, \theta_2, \theta_3, \varphi_1, \varphi_2, \varphi_3 \in \mathbb{S}^1$ be such that:

- (1) $\theta_2 \in (\theta_1; \theta_3)$, $\varphi_2 \in (\varphi_1; \varphi_3)$;
- (2) points $v_1, v_2, v_3, v_4, v_5 \in \mathbb{T}^2$ having coordinates (θ_1, φ_3) , (θ_2, φ_3) , (θ_3, φ_3) , (θ_3, φ_2) , (θ_3, φ_1) , respectively, are vertices of Π , and, moreover, v_1, v_3, v_5 are \swarrow -vertices, and v_2, v_4 are \searrow -vertices;
- (3) none of v_2, v_3 , and v_4 belong to $\partial\Pi$;
- (4) there are no more vertices of Π in $[\theta_1; \theta_3] \times [\varphi_1; \varphi_3]$.

These assumptions imply that Π contains, among others, four rectangles of the following form:

$$\begin{aligned} r_1 &= [\theta_1; \theta_2] \times [\varphi'; \varphi_3], & r_2 &= [\theta_2; \theta_3] \times [\varphi_3; \varphi''], \\ r_3 &= [\theta_3; \theta''] \times [\varphi_2; \varphi_3], & r_4 &= [\theta'; \theta_3] \times [\varphi_1; \varphi_2] \end{aligned}$$

with some $\theta', \theta'' \in (\theta_3; \theta_1)$, $\varphi', \varphi'' \in (\varphi_3; \varphi_1)$.

Let Π' be obtained from Π by replacing these four rectangles with the following ones:

$$\begin{aligned} r'_1 &= [\theta_1; \theta_2] \times [\varphi'; \varphi_1], & r'_2 &= [\theta_2; \theta_3] \times [\varphi_1; \varphi''], \\ r'_3 &= [\theta_1; \theta''] \times [\varphi_2; \varphi_3], & r'_4 &= [\theta'; \theta_1] \times [\varphi_1; \varphi_2]; \end{aligned}$$

see Figure 2.5.

Then we say that the passage from Π to Π' , or the other way, is a *flype*. Note that the other rectangles of Π and Π' , not shown in Figure 2.5, are

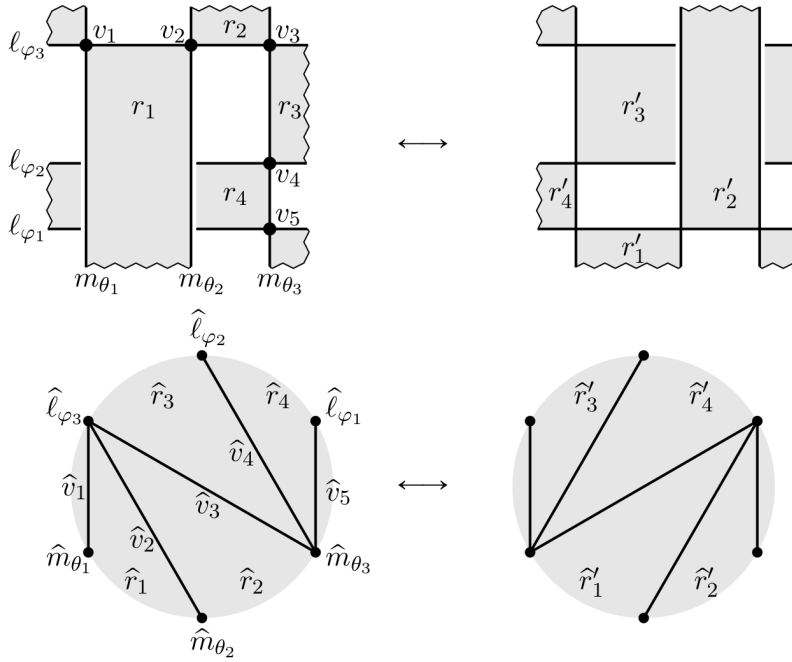


FIGURE 2.5. A flype

allowed to pass through $[\theta_1; \theta_3] \times [\varphi_1; \varphi_3]$, the region where the modification occurs.

If $\Pi \mapsto \Pi'$ is a flype and T is the transformation of rectangular diagrams induced by the map $(\theta, \varphi) \mapsto (-\theta, \varphi)$, then $T(\Pi) \mapsto T(\Pi')$ is also called a flype.

To complete Definitions 2.3-2.7 we need to define the respective morphisms $\hat{\Pi} \rightarrow \hat{\Pi}'$ in each case. In [1] we described an isotopy from $\hat{\Pi}$ to $\hat{\Pi}'$, which can be continued to the whole of \mathbb{S}^3 thus producing the required morphism. Here we outline the general principles of the construction.

First, let $\Pi \mapsto \Pi'$ be any basic move other than an exchange move. Then there are open discs or open half-discs $d \subset \hat{\Pi}$ and $d' \subset \hat{\Pi}'$ such that

$$\bar{d} = \bigcup_{r \in \Pi \setminus \Pi'} \hat{r}, \quad \bar{d}' = \bigcup_{r \in \Pi' \setminus \Pi} \hat{r}'$$

(by an open *half-disc* in a surface F we mean an open subset $d \subset F$ such that d is homeomorphic to $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1, y \geq 0\}$, and $\bar{d} \setminus d$ is not a single point). There is also an open 3-ball $B \subset \mathbb{S}^3$ such that $B \cap \hat{\Pi} = d$ and $B \cap \hat{\Pi}' = d'$. The morphism $\Pi \mapsto \Pi'$ is represented by any homeomorphism $(\mathbb{S}^3, \hat{\Pi}) \rightarrow (\mathbb{S}^3, \hat{\Pi}')$ which is identical outside B .

Now let $\Pi \mapsto \Pi'$ be a vertical exchange move. We use the notation from Definition 2.6. Pick an $\varepsilon > 0$ so small that no occupied meridian of Π is

contained in

$$((\theta_1; \theta_1 + \varepsilon] \cup [\theta_2 - \varepsilon; \theta_2 + \varepsilon] \cup [\theta_3 - \varepsilon; \theta_3)) \times \mathbb{S}^1.$$

Choose self-homeomorphisms ψ_1, ψ_2 of \mathbb{S}^1 such that:

- (1) $\psi_1(\theta) = \psi_2(\theta) = \theta$ if $\theta \in [\theta_3; \theta_1]$;
- (2) $\psi_1(\theta) = \psi(f(\theta))$ if $\theta \in [\theta_1 + \varepsilon; \theta_2 - \varepsilon]$;
- (3) $\psi_2(\theta) = \psi(f(\theta))$ if $\theta \in [\theta_2 + \varepsilon; \theta_3 - \varepsilon]$.

Here and in the sequel we use *the open book decomposition* of \mathbb{S}^3 with the *binding* $\mathbb{S}_{\tau=0}^1$ and *pages* of the form

$$\mathcal{P}_{\theta_0} = \{\theta_0\} * \mathbb{S}_{\tau=0}^1, \quad \theta_0 \in \mathbb{S}^1.$$

In other words, for any $\theta_0 \in \mathbb{S}^1$, the page \mathcal{P}_{θ_0} is defined by the equation $\theta = \theta_0$.

The union $\hat{r}_1 \cup \hat{r}_2$ is a 2-disc. There is a small open neighborhood U of this disc such that U is disjoint from $\hat{\Pi}$, $\mathbb{S}^3 \setminus U$ is homeomorphic to a closed 3-ball, and, for any $\theta_0 \in [\theta_1; \theta_3]$, the intersection of \bar{U} with the page \mathcal{P}_{θ_0} is a regular neighborhood of $(\hat{r}_1 \cup \hat{r}_2) \cap \mathcal{P}_{\theta_0}$ in \mathcal{P}_{θ_0} .

For any $\theta \in [\theta_1; \theta_3]$, the complement $\mathcal{P}_{\theta} \setminus U$ is a union of two closed 2-discs, which we denote by \mathcal{P}'_{θ} and \mathcal{P}''_{θ} using the following rule:

$$\mathcal{P}'_{\theta} \cap \mathbb{S}_{\tau=0}^1 \subset [\varphi_2; \varphi_1], \quad \mathcal{P}''_{\theta} \cap \mathbb{S}_{\tau=0}^1 \subset [\varphi_1; \varphi_2].$$

There is then an embedding $\sigma : \mathbb{S}^3 \setminus U \rightarrow \mathbb{S}^3$ such that:

- (1) $\sigma(\hat{\Pi}) = \hat{\Pi}'$;
- (2) σ is identical on $\mathbb{S}_{\tau=0}^1 \setminus U$;
- (3) if $\theta \in [\theta_3; \theta_1]$, then $\sigma(\mathcal{P}_{\theta} \setminus U) \subset \mathcal{P}_{\theta}$;
- (4) if $\theta \in [\theta_1; \theta_3]$, then $\sigma(\mathcal{P}'_{\theta}) \subset \mathcal{P}_{\psi_1(\theta)}$ and $\sigma(\mathcal{P}''_{\theta}) \subset \mathcal{P}_{\psi_2(\theta)}$.

This embedding can be extended to a homeomorphism $\mathbb{S}^3 \rightarrow \mathbb{S}^3$ that represents the morphism $\hat{\Pi} \rightarrow \hat{\Pi}'$ assigned to the exchange move $\Pi \mapsto \Pi'$.

Definition 2.8. Let Π be a rectangular diagram of a surface, and let X be a subset of $\partial\Pi$. A basic move $\Pi \mapsto \Pi'$ is said to be *fixed* on X if $X \subset \partial\Pi'$ and the types of any $v \in X$ as a vertex of Π and as a vertex of Π' (which are ‘/’ or ‘\’) coincide.

Theorem 2.9. *Let Π and Π' be rectangular diagrams of surfaces, and let R be a rectangular diagram of a link such that $L = \hat{R}$ is a (possibly empty) sublink of $\partial\hat{\Pi}$. Let also ϕ be a self-homeomorphism of \mathbb{S}^3 that takes $\hat{\Pi}$ to $\hat{\Pi}'$. Then the following two conditions are equivalent.*

- (i) *There exists an isotopy from $\text{id}|_{\mathbb{S}^3}$ to ϕ fixed on L and such that the tangent plane to the surface at any point $p \in L$ is preserved during the isotopy.*
- (ii) *There exists a sequence of basic moves fixed on R*

$$\Pi = \Pi_0 \mapsto \Pi_1 \mapsto \dots \mapsto \Pi_N = \Pi'$$

such that the composition of the corresponding morphisms is $[\phi]$.

The proof of Theorem 2.9 will be given in full generality in Section 7. The hardest part of the proof is the subject of Section 6 (Proposition 6.1), where we restrict ourselves to the case $R = \partial\Pi$.

3. BUBBLE MOVES

Definition 3.1. Let $r = [\theta_1; \theta_2] \times [\varphi_1; \varphi_2]$ be a rectangle of a rectangular diagram of a surface Π , and the points $\theta_3, \theta_4 \in (\theta_1; \theta_2)$ be such that no occupied meridian of Π lies in the domain $[\theta_3; \theta_4] \times \mathbb{S}^1$, and the rectangle $[\theta_3; \theta_4] \times [\varphi_1; \varphi_2]$ is not contained in the interior of any rectangle in Π . Then the passage from Π to $\Pi' = \Pi \cup \{r_1, r_2, r_3\} \setminus \{r\}$ (assigned with the morphism $\hat{\Pi} \rightarrow \hat{\Pi}'$ described below), where

$$r_1 = [\theta_1; \theta_3] \times [\varphi_1; \varphi_2], \quad r_2 = [\theta_3; \theta_4] \times [\varphi_2; \varphi_1], \quad r_3 = [\theta_4; \theta_2] \times [\varphi_1; \varphi_2],$$

is called a *vertical bubble creation move*, and the inverse passage $\Pi' \mapsto \Pi$ a *vertical bubble reduction move*.

One can see that the transition from $\hat{\Pi}$ to $\hat{\Pi}'$ is a replacement of the interior of the two-disc \hat{r} with the interior of the two-disc $d = \hat{r}_1 \cup \hat{r}_2 \cup \hat{r}_3$, and we have $\hat{r} \cap d = \partial\hat{r} = \partial d$. The interior of both discs lies in the domain $\theta \in (\theta_1; \theta_2)$. For any $\theta \in (\theta_1; \theta_2)$, the intersection of each of the discs with \mathcal{P}_θ is an arc, and the two arcs $\hat{r} \cap \mathcal{P}_\theta$ and $d \cap \mathcal{P}_\theta$ enclose a 2-disc whose interior is disjoint from $\hat{\Pi}$. This implies that there exists a self-homeomorphism ϕ of \mathbb{S}^3 preserving each page \mathcal{P}_θ and taking $\hat{\Pi}$ to $\hat{\Pi}'$. The morphism $[\phi]$ is the one assigned to the bubble creation move $\Pi \mapsto \Pi'$.

Horizontal bubble creation and reduction moves are defined similarly with the roles of θ and φ exchanged.

Bubble moves are illustrated in Figure 3.1.

Lemma 3.2. *Any bubble move of rectangular diagrams of surfaces can be decomposed into basic moves.*

Proof. Let Π, Π' be as in Definition 3.1. Pick $(\theta_5; \theta_6) \subset (\theta_1; \theta_3)$ so close to θ_1 that no occupied meridian of Π lies in the domain $(\theta_1; \theta_6]$. Define Π'' to be $\Pi \cup \{r'_1, r'_2, r'_3\}$, where

$$r'_1 = [\theta_1; \theta_5] \times [\varphi_1; \varphi_2], \quad r'_2 = [\theta_5; \theta_6] \times [\varphi_2; \varphi_1], \quad r'_3 = [\theta_6; \theta_2] \times [\varphi_1; \varphi_2].$$

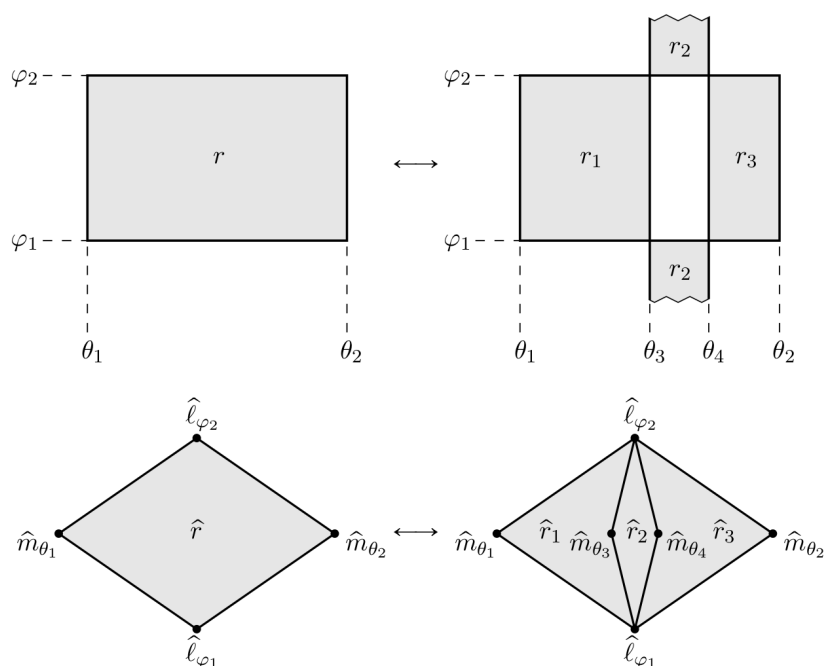


FIGURE 3.1. Bubble moves

Then $\Pi \mapsto \Pi''$ is a wrinkle creation move, and $\Pi'' \mapsto \Pi'$ is an exchange move. It is elementary to verify that the composition of the respective morphisms yields the morphism assigned to the bubble creation move $\Pi \mapsto \Pi'$. \square

4. MOVIE DIAGRAMS

Definition 4.1. By a *non-crossing chord diagram* we mean a finite set

$$\{\{\varphi'_1, \varphi''_1\}, \{\varphi'_2, \varphi''_2\}, \dots, \{\varphi'_m, \varphi''_m\}\}$$

of unordered pairs (which are referred as *chords*) of points of \mathbb{S}^1 such that, for any $i, j = 1, \dots, m$, $i \neq j$, we have either $\{\varphi'_i, \varphi''_i\} \subset (\varphi'_j; \varphi''_j)$ or $\{\varphi'_i, \varphi''_i\} \subset (\varphi''_j; \varphi'_j)$. The set of all non-crossing chord diagrams is denoted by \mathcal{C} and endowed with a discrete topology.

Definition 4.2. Let C_1 and C_2 be two non-crossing chord diagrams. We say that the passage from C_1 to C_2 is an *admissible event* if, for some $\varphi_1, \varphi_2, \dots, \varphi_k \in \mathbb{S}^1$, $k \geq 2$, following on \mathbb{S}^1 in the same circular order as listed, one of the diagrams (whichever of the two) can be obtained from the other in one of the following ways:

(1) replacing

$$\{\varphi_1, \varphi_2\}, \{\varphi_3, \varphi_4\}, \dots, \{\varphi_{k-1}, \varphi_k\}$$

with

$$\{\varphi_2, \varphi_3\}, \{\varphi_4, \varphi_5\}, \dots, \{\varphi_{k-2}, \varphi_{k-1}\}$$

(provided k is even);

(2) replacing

$$\{\varphi_1, \varphi_2\}, \{\varphi_3, \varphi_4\}, \dots, \{\varphi_{k-2}, \varphi_{k-1}\}$$

with

$$\{\varphi_2, \varphi_3\}, \{\varphi_4, \varphi_5\}, \dots, \{\varphi_{k-1}, \varphi_k\}$$

(provided k is odd);

(3) replacing

$$\{\varphi_1, \varphi_2\}, \{\varphi_3, \varphi_4\}, \dots, \{\varphi_{k-1}, \varphi_k\}$$

with

$$\{\varphi_2, \varphi_3\}, \{\varphi_4, \varphi_5\}, \dots, \{\varphi_{k-2}, \varphi_{k-1}\}, \{\varphi_k, \varphi_1\}$$

(provided $k \geq 4$ is even).

The points $\varphi_1, \dots, \varphi_k \in \mathbb{S}^1$ are then said to be *involved* in the event.

To specify an admissible event we indicate which chords are removed and which are added, separating the two lists by a ‘ \rightsquigarrow ’ sign. The two extreme cases, when no chord is removed and just one added, or no one is added and just one removed, are allowed. For instance, the following are valid specifications of admissible events (provided that $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ follow on \mathbb{S}^1 in the same circular order as listed):

$$\begin{aligned} \{\varphi_1, \varphi_2\} &\rightsquigarrow \emptyset, & \emptyset &\rightsquigarrow \{\varphi_1, \varphi_2\}, \\ \{\varphi_1, \varphi_2\} &\rightsquigarrow \{\varphi_2, \varphi_3\}, & \{\varphi_1, \varphi_2\}, \{\varphi_3, \varphi_4\} &\rightsquigarrow \{\varphi_2, \varphi_3\}, \{\varphi_4, \varphi_1\}. \end{aligned}$$

Definition 4.3. A *movie diagram* is a left continuous map $\Psi : \mathbb{S}^1 \rightarrow \mathcal{C}$ with finitely many discontinuities, such that, for any discontinuity point $\theta_0 \in \mathbb{S}^1$, the passage from $C_0 = \Psi(\theta_0)$ to $C_1 = \lim_{\theta \rightarrow \theta_0+0} \Psi(\theta)$ is an admissible event.

When Ψ is a movie diagram, and $\theta_0 \in \mathbb{S}^1$, we denote $\lim_{\theta \rightarrow \theta_0+0} \Psi(\theta)$ by $\Psi(\theta_0 + 0)$ for brevity. We also denote by $\Phi(\Psi)$ the set of all $\varphi_0 \in \mathbb{S}^1$ such that for some $\theta_0, \varphi_1 \in \mathbb{S}^1$ the non-crossing chord diagram $\Psi(\theta_0)$ contains the chord $\{\varphi_0, \varphi_1\}$.

Definition 4.4. Let Ψ be a movie diagram. We say that a surface $F \subset \mathbb{S}^3$ *represents* Ψ if the following conditions hold:

- (1) $F \cap \mathbb{S}_{\tau=0}^1 = \Phi(\Psi)$;
- (2) for any $\theta \in \mathbb{S}^1$, each connected component of the intersection $\mathcal{P}_\theta \cap F$ is either an arc or a star-like graph whose edges join a few points in $\mathbb{S}_{\tau=0}^1$ with a vertex located in the interior of \mathcal{P}_θ ;
- (3) for any $\theta \in \mathbb{S}^1$, the points $\varphi', \varphi'' \in \mathbb{S}_{\tau=0}^1$ are in the same connected component of $\mathcal{P}_\theta \cap F$ if and only if one of the following occurs:
 - $\{\varphi', \varphi''\} \in \Psi(\theta)$;

- Ψ has an event at θ and φ', φ'' are involved in it;

(4) there are only finitely many points at which F is tangent to some page \mathcal{P}_θ .

Definition 4.5. For every rectangular diagram of a surface Π we define *the associated movie diagram* Ψ_Π by requesting that, whenever m_{θ_0} is not an occupied meridian of Π , we have $\{\varphi_1, \varphi_2\} \in \Psi_\Pi(\theta_0)$ if and only if the following holds: there exist $\theta_1, \theta_2 \in \mathbb{S}^1$ such that $\theta_0 \in (\theta_1; \theta_2)$ and either $[\theta_1; \theta_2] \times [\varphi_1; \varphi_2] \in \Pi$ or $[\theta_1; \theta_2] \times [\varphi_2; \varphi_1] \in \Pi$.

Proposition 4.6 below, which follows easily from definitions, gives another characterization of Ψ_Π .

Proposition 4.6. *Let Π be a rectangular diagram of a surface, and let Ψ be a movie diagram. Then the surface $\hat{\Pi}$ represents Ψ if and only if $\Psi = \Psi_\Pi$.*

We omit the easy proof. For the reason which is clear from this statement, we say that a rectangular diagram of a surface Π represents a movie diagram Ψ if $\Psi = \Psi_\Pi$ or, equivalently, if $\hat{\Pi}$ represents Ψ .

Example 4.7. Shown in Figure 4.1 are a rectangular diagram of a surface Π and the intersections of the pages \mathcal{P}_θ with the surface $\hat{\Pi}$. The topology of these intersections is the information encoded in the movie diagram Ψ_Π .

Note that if m_{θ_0} is an occupied meridian of Π that contains exactly two vertices of Π and no vertices from $\partial\Pi$, then, from the topological point of view, the intersection $\mathcal{P}_{\theta_0} \cap \hat{\Pi}$ has no singularity, and there is no event of Ψ_Π at the moment θ_0 . (For instance, such a situation occurs in the case shown in Figure 4.1 for the midpoint of the interval $(\theta_7; \theta_8)$. What does change in $\mathcal{P}_\theta \cap \hat{\Pi}$ at this moment is the relative position of the arc joining φ_1 with φ_7 and the point $\mathcal{P}_\theta \cap \mathbb{S}_{\tau=1}^1$.) In particular, a vertical bubble move applied to Π does not change Ψ_Π .

Definition 4.8. Let F_1 and F_2 be two surfaces representing the same movie diagram. A morphism from F_1 to F_2 is said to be *canonical* if it can be represented by a self-homeomorphism of \mathbb{S}^3 that fixes $\mathbb{S}_{\tau=0}^1$ pointwise and preserves every page \mathcal{P}_θ . Such a morphism is clearly unique, so it will be referred to as *the canonical morphism* from F_1 to F_2 .

Proposition 4.9. (i) *Any movie diagram has the form Ψ_Π for some rectangular diagram of a surface Π .*

(ii) *We have $\Psi_\Pi = \Psi_{\Pi'}$ if and only if the diagrams Π and Π' are related by a finite sequence of vertical bubble moves and vertical half-wrinkle moves*

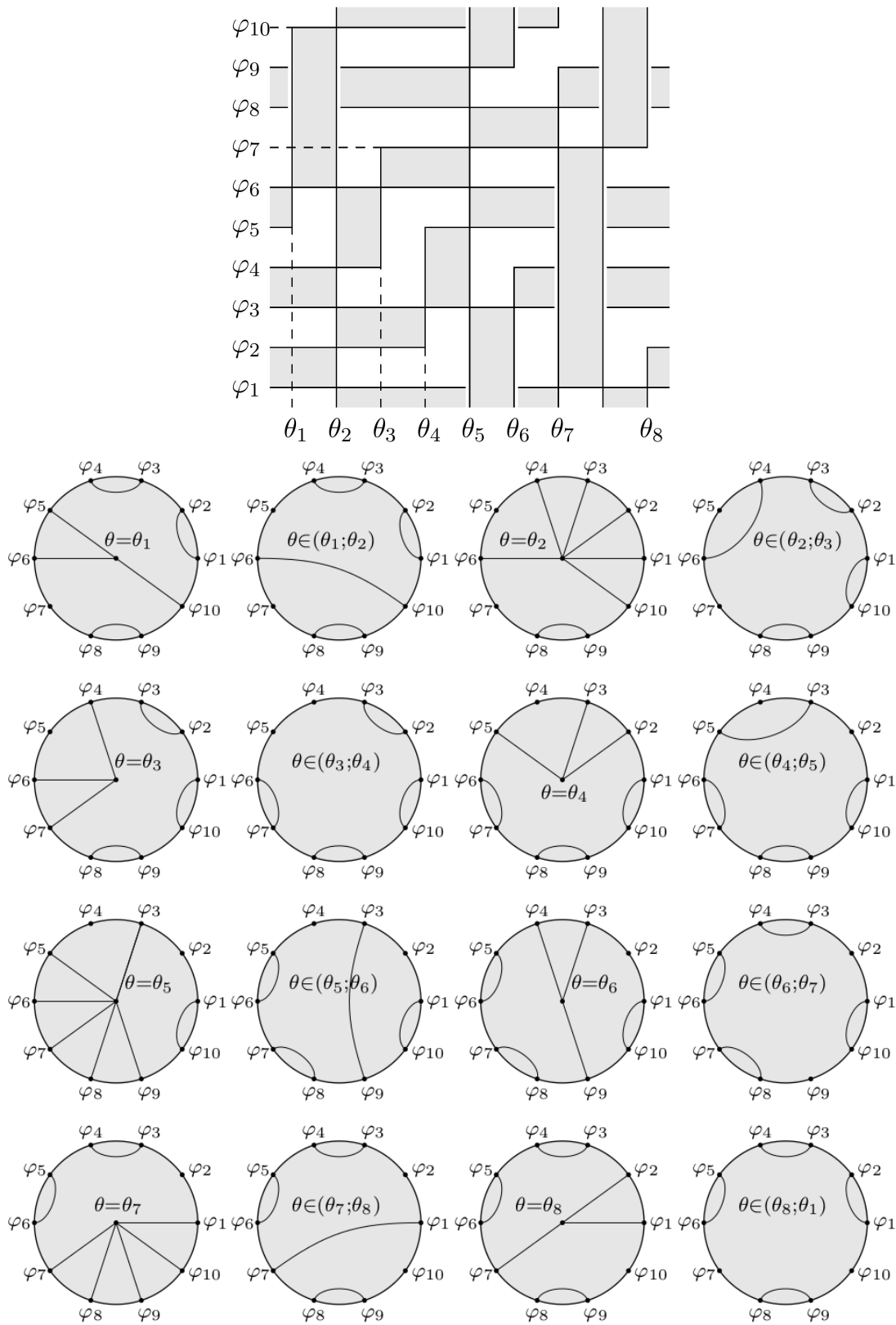


FIGURE 4.1. A rectangular diagram of a surface Π and the topology of the intersections $\hat{\Pi} \cap \mathcal{P}_\theta$ for various θ

preserving the boundary of the diagram. Any such sequence induces the canonical morphism $\widehat{\Pi} \rightarrow \widehat{\Pi}'$.

Proof. The fact that vertical bubble and half-wrinkle moves induce the canonical morphisms between the respective surfaces follows easily from the definitions of the moves (Definitions 3.1 and 2.4). So, we need only to explain how to recover a rectangular diagram of a surface Π from Ψ_Π , and to show that the arbitrariness in this procedure amounts to an application of a sequence of vertical bubble and half-wrinkle moves to Π . We provide a sketch only, since the details are pretty elementary.

We need some preparation before describing the procedure. For a non-crossing chord diagram C , by a *region of C* we mean a union of intervals

$$R = (\varphi_1; \varphi_2) \cup \dots \cup (\varphi_{2k-1}; \varphi_{2k}) \subset \mathbb{S}^1$$

such that:

- (1) R is disjoint from any chord of C ;
- (2) $\{\varphi_2, \varphi_3\}, \dots, \{\varphi_{2k-2}, \varphi_{2k-1}\}, \{\varphi_{2k}, \varphi_1\}$ are chords of C .

The subset $\{\varphi_1, \varphi_2, \dots, \varphi_{2k}\} \subset \mathbb{S}^1$ will be referred to as *the boundary of R* and denoted ∂R .

In other words, a region of C is a maximal subset R of \mathbb{S}^1 such that, for any two distinct points φ', φ'' of R , the addition of $\{\varphi', \varphi''\}$ to C yields a non-crossing chord diagram.

Clearly, any two distinct regions of C are disjoint, and their boundaries are either disjoint or have exactly two points in common, which form a chord of C . In the latter case the regions are called *neighboring*.

Denote by $T(C)$ the graph whose vertices are regions of C , and edges are pairs of neighboring regions. This graph is obviously a tree. The set of all subsets of \mathbb{S}^1 that have the form of a union of finitely many pairwise disjoint open intervals is denoted by \mathcal{R} .

Let Π be a rectangular diagram of a surface. For every $\theta_0 \in \mathbb{S}^1$ such that m_{θ_0} is not an occupied meridian of Π , define $R_\Pi(\theta_0)$ to be the region $\Omega(\theta_0) \cap \mathbb{S}_{\tau=0}^1$ of $\Psi_\Pi(\theta_0)$, where $\Omega(\theta_0)$ is the connected component of $\mathcal{P}_{\theta_0} \setminus \widehat{\Pi}$ that contains the point $\mathcal{P}_{\theta_0} \cap \mathbb{S}_{\tau=1}^1$. One can see that the following equality is an equivalent characterization of $R_\Pi(\theta_0)$:

$$\{\theta_0\} \times R_\Pi(\theta_0) = m_{\theta_0} \setminus \bigcup_{r \in \Pi} r. \quad (4.1)$$

The function R_Π is clearly constant on every open interval between two occupied meridians of Π . We extend R_Π to be a left continuous map $\mathbb{S}^1 \rightarrow \mathcal{R}$ (the set \mathcal{R} is endowed with a discrete topology).

The diagram Π can be uniquely recovered from the union $\bigcup_{r \in \Pi} r$. Indeed, Π is the set of all maximal rectangles (with respect to inclusion) contained

in $\bigcup_{r \in \Pi} r$. Therefore, it can also be uniquely recovered from R_Π . Indeed, due to (4.1), the union $\bigcup_{r \in \Pi} r$ is the closure of $\mathbb{T}^2 \setminus \bigcup_{\theta \in \mathbb{S}^1} (\{\theta\} \times R_\Pi(\theta))$.

Suppose that m_{θ_0} is an occupied meridian of Π containing at least three vertices of Π . Then $\Psi_\Pi(\theta_0) \mapsto \Psi_\Pi(\theta_0 + 0)$ is an admissible event involving at least three points of \mathbb{S}^1 . One can see that there are unique regions of $\Psi_\Pi(\theta_0)$ and $\Psi_\Pi(\theta_0 + 0)$ whose closure contains all the points of \mathbb{S}^1 involved in the event, and these regions are $R_\Pi(\theta_0)$ and $R_\Pi(\theta_0 + 0)$, respectively.

Now suppose that a meridian m_{θ_0} contains exactly two vertices of Π , and they belong to $\partial\Pi$. This means that the event $\Psi_\Pi(\theta_0) \mapsto \Psi_\Pi(\theta_0 + 0)$ either has the form $\{\varphi', \varphi''\} \rightsquigarrow \emptyset$ or $\emptyset \rightsquigarrow \{\varphi', \varphi''\}$. In the former case, there are two regions of $\Psi_\Pi(\theta_0)$ whose boundary contain φ' and φ'' , and one of them is $R_\Pi(\theta_0)$. By applying a vertical half-wrinkle creation move keeping Ψ_Π unaltered we can change $R_\Pi(\theta_0)$ to the other one. The region $R_\Pi(\theta_0 + 0)$ is prescribed by Ψ_Π , and this is the unique region of $\Psi_\Pi(\theta_0 + 0)$ that contains φ' and φ'' .

Similarly, if the event $\Psi_\Pi(\theta_0) \mapsto \Psi_\Pi(\theta_0 + 0)$ has the form $\emptyset \rightsquigarrow \{\varphi', \varphi''\}$, then $R_\Pi(\theta_0)$ is prescribed by Ψ_Π , whereas there are two regions of $\Psi_\Pi(\theta_0 + 0)$ that are eligible for $R_\Pi(\theta_0 + 0)$, and the choice can be changed by a vertical half-wrinkle creation move applied to Π .

We are ready to give a recipe for constructing a rectangular diagram of a surface representing a given movie diagram. Let Ψ be an arbitrary movie diagram. Pick a function $R : \mathbb{S}^1 \rightarrow \mathcal{R}$ having the following properties:

- (1) R is left continuous and has finitely many discontinuity points;
- (2) $R(\theta)$ is a region of $\Psi(\theta)$ for all $\theta \in \mathbb{S}^1$;
- (3) if no admissible event of Ψ occurs at $\theta_0 \in \mathbb{S}^1$ and $R(\theta_0) \neq R(\theta_0 + 0)$, then $R(\theta_0)$ and $R(\theta_0 + 0)$ are neighboring regions of $\Psi(\theta_0)$;
- (4) if an admissible event of Ψ occurs at θ_0 , $\theta_0 \in \mathbb{S}^1$, then $\overline{R(\theta_0)}$ and $\overline{R(\theta_0 + 0)}$ contain all points involved in the event;
- (5) we have $\overline{\bigcup_{\theta \in \mathbb{S}^1} R(\theta)} = \mathbb{S}^1$.

To see that such R does exist, let θ', θ'' be the moments of two successive events of Ψ . We start by defining $R(\theta' + 0)$ and $R(\theta'')$ to comply with condition (4) above. These are two regions of $\Psi(\theta'') = \Psi(\theta' + 0)$, and R can be defined on the whole interval $(\theta'; \theta'']$ to satisfy conditions (1)-(3) due to the fact that the graph $T(\Psi(\theta''))$ is connected, and $\Psi(\theta)$ is constant for $\theta \in (\theta'; \theta'']$. This is done independently for all intervals between any two successive events.

To see that condition (5) can also be met by R , note that the graph $T(C)$ is connected for any non-crossing chord diagram C , and the closure of the union of all regions of C is the entire circle \mathbb{S}^1 . So, if R satisfies

conditions (1)-(4), but not (5), we choose an interval $[\theta_1; \theta_2)$ on which Ψ is constant and modify the values of R in it so as to let $R(\theta)$, $\theta \in [\theta_1; \theta_2)$ visit all vertices of the tree $T(\Psi(\theta_1))$. This can clearly be done without violating conditions (1)-(4).

Now we let Π be the collection of maximal rectangles contained in the closure of $\mathbb{T}^2 \setminus \bigcup_{\theta \in \mathbb{S}^1} (\{\theta\} \times R_\Pi(\theta))$. We leave it to the reader to verify that Π is a rectangular diagram of a surface such that $\Psi = \Psi_\Pi$.

The arbitrariness in the construction of Π has the following two sources. First, if at some moment θ_0 , an admissible event $\{\varphi', \varphi''\} \rightsquigarrow \emptyset$ (or, respectively, $\emptyset \rightsquigarrow \{\varphi', \varphi''\}$) occurs in Ψ , then $R(\theta_0)$ (respectively, $R(\theta_0 + 0)$) can be chosen in two different ways. As mentioned above, a vertical half-wrinkle move can be used to change the choice.

Second, there is a large freedom in defining R on an interval $(\theta'; \theta'')$ between two successive events of Ψ , provided that $R(\theta' + 0)$ and $R(\theta'')$ are already fixed. Different choices correspond to different paths from $R(\theta' + 0)$ to $R(\theta'')$ in the graph $T(\Psi(\theta''))$. Since this graph is a tree, different choices are related by a sequence of the following operations and their inverses.

Let Ψ and R be constant on an interval $[\theta_1; \theta_2]$, and let $[\theta_3; \theta_4]$ be a subinterval of $(\theta_1; \theta_2)$. Change the value of R on $(\theta_3; \theta_4]$ to any region of $\Psi(\theta_1)$ which is neighboring to $R(\theta_1) = R(\theta_2)$.

One can see that such an operation results in a vertical bubble creation move performed on Π . This completes the proof of the proposition. \square

It is often useful to look at a movie diagram Ψ ‘from inside a surface representing Ψ ’. Let F be such a surface. We denote by \mathcal{F}_F the singular foliation induced on F by the open book decomposition $\{\mathcal{P}_\theta\}_{\theta \in \mathbb{S}^1}$. More precisely, the foliation \mathcal{F}_F is defined on $F \setminus \mathbb{S}_{\tau=0}^1$. The leaves of \mathcal{F}_F are connected components of the intersections $F \cap (\mathcal{P}_\theta \setminus \mathbb{S}_{\tau=0}^1)$, $\theta \in \mathbb{S}^1$.

The points in $F \cap \mathbb{S}_{\tau=0}^1$, where \mathcal{F}_F is not defined, are referred to as *vertices of \mathcal{F}_F* . Unless otherwise specified, we adopt a purely topological point of view on \mathcal{F}_F and regard all points $p \in F \setminus \mathbb{S}_{\tau=0}^1$ such that a sufficiently small neighborhood of p is foliated by open arcs as *regular*. So, *singularities of \mathcal{F}_F* are such points in $F \setminus \mathbb{S}_{\tau=0}^1$ that do not have an open neighborhood foliated by open arcs. Leaves containing a singularity are called *singular* (and otherwise *regular*). Connected components of the set of regular points of a singular leaf are called *separatrices*. The behavior of \mathcal{F}_F near vertices and some singularities is shown in Figure 4.2.

If F and F' are two surfaces representing the same movie diagram Ψ , then there is a homeomorphism representing the canonical morphism $F \rightarrow F'$ that takes \mathcal{F}_F to $\mathcal{F}_{F'}$. In other words, the topological structure of \mathcal{F}_F does not depend on the concrete choice of F . For this reason we will use the notation \mathcal{F}_Ψ for any foliation \mathcal{F}_F with F representing Ψ .

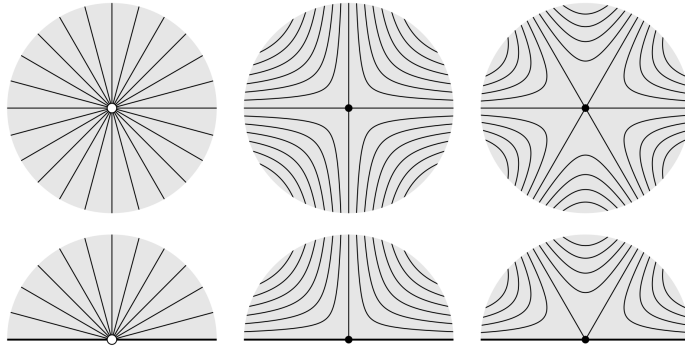


FIGURE 4.2. Foliation \mathcal{F}_F around vertices and singularities

Let Ψ be a movie diagram. Quite clearly, the topological type of \mathcal{F}_Ψ with a little more data allows to completely recover Ψ . The additional data include the value of φ at the vertices of \mathcal{F}_Ψ , the value of θ at the singularities of \mathcal{F}_Ψ , and the coorientation of \mathcal{F}_Ψ defined by $d\theta$. So, to illustrate an alteration of Ψ it is sometimes more convenient to show the alteration of \mathcal{F}_Ψ (and of the additional data).

5. MOVES OF MOVIE DIAGRAMS

Definition 5.1. Let Ψ and Ψ' be movie diagrams. A *morphism* from Ψ to Ψ' is a maximal collection χ of morphisms of surfaces $F \rightarrow F'$ representing Ψ and Ψ' , respectively, such that if $\mu : F \rightarrow F'$ and $\mu_1 : F_1 \rightarrow F'_1$ are two morphisms belonging to χ , then $\sigma' \circ \mu = \mu_1 \circ \sigma$, where $\sigma : F \rightarrow F_1$ and $\sigma' : F' \rightarrow F'_1$ are the canonical morphisms.

By a *move* of movie diagrams we mean a pair (Ψ, Ψ') of movie diagrams assigned with a morphism χ from Ψ to Ψ' . For such a move we use the notation $\Psi \mapsto \Psi'$ or $\Psi \xrightarrow{\chi} \Psi'$ or $\Psi \xrightarrow{\mu} \Psi'$ if $\mu \in \chi$.

Let $F, F' \subset \mathbb{S}^3$ be two surfaces, and let ϕ be a self-homeomorphism of \mathbb{S}^3 taking F to F' . If ϕ is identical outside of a 3-ball B intersecting each of F and F' in an open disc or an open half-disc we call ϕ an *elementary isotopy from F to F' supported on B* .

We define below several types of moves of movie diagrams. In each case except the one of a rescaling move, for which the corresponding morphism is described explicitly, the morphism assigned to the move is induced either by an elementary isotopy or by a composition of two elementary isotopies supported on two disjoint 3-balls. So, to define the morphisms assigned to the moves it suffices to specify the altered part of the respective surfaces.

Definition 5.2. Let Ψ and Ψ' be two movie diagrams such that there exist pairwise distinct $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \mathbb{S}^1$ and distinct $\theta_1, \theta_2 \in \mathbb{S}^1$ satisfying the following conditions:

- (1) the intersection of $[\varphi_1; \varphi_2]$ with $\Phi(\Psi)$ is empty;
- (2) there are no events of Ψ in the interval $[\theta_1; \theta_2]$
- (3) if $\theta \in (\theta_1; \theta_2]$, then $\Psi'(\theta)$ is obtained from $\Psi(\theta)$ by replacing $\{\varphi_3, \varphi_4\}$ with $\{\varphi_1, \varphi_4\}$ and $\{\varphi_2, \varphi_3\}$;
- (4) if $\theta \in (\theta_2; \theta_1]$, then $\Psi'(\theta)$ is obtained from $\Psi(\theta)$ by adding the chord $\{\varphi_1, \varphi_2\}$.

Then we say that $\Psi \mapsto \Psi'$ is a *finger move*, and $\Psi' \mapsto \Psi$ is an *inverse finger move*. Surfaces representing Ψ and Ψ' can be chosen to be related by an elementary isotopy; encircled in the dashed line in Figure 5.1 are the intersections of the surfaces with the 3-ball on which the elementary isotopy is supported.

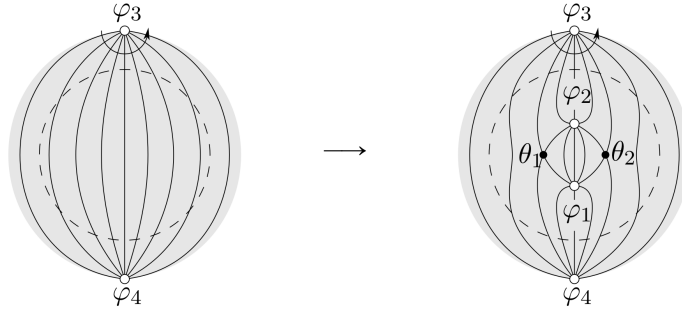


FIGURE 5.1. Change of the foliation \mathcal{F}_Ψ under a finger move. The arrows show the coorientation of the foliation

Definition 5.3. Let Ψ and Ψ' be two movie diagrams with $\Phi(\Psi) = \Phi(\Psi')$ and, for some distinct $\theta_1, \theta_2 \in \mathbb{S}^1$, the following conditions hold:

- (1) for any $\theta \in (\theta_2; \theta_1]$ we have $\Psi(\theta) = \Psi'(\theta)$;
- (2) there is exactly one event of Ψ in the interval $[\theta_1; \theta_2]$;
- (3) there are two events e_1, e_2 of Ψ' at the moments θ_1 and θ_2 , respectively, and no events in $(\theta_1; \theta_2)$;
- (4) the non-crossing chord diagram $\Psi'(\theta_2)$ contains an arc $\{\varphi_1, \varphi_2\}$ not present in $\Psi'(\theta_1)$ and $\Psi'(\theta_2 + 0)$;
- (5) all points of \mathbb{S}^1 involved in e_1 (respectively, e_2) are contained in $[\varphi_2; \varphi_1]$ (respectively, $[\varphi_1; \varphi_2]$).

Then we say that $\Psi \mapsto \Psi'$ is a *splitting of an event*, and $\Psi' \mapsto \Psi$ is a *merging of events*. The morphisms assigned to these moves are again induced by an elementary isotopy supported on a 3-ball B that contains no vertices and only those singularities of $\mathcal{F}_\Psi, \mathcal{F}_{\Psi'}$ that correspond to the events in which Ψ and Ψ' differ.

A few examples of how the foliation \mathcal{F}_Ψ changes under a splitting of an event is shown in Figure 5.2. Encircled in the dashed line are the intersections of the surfaces with B .

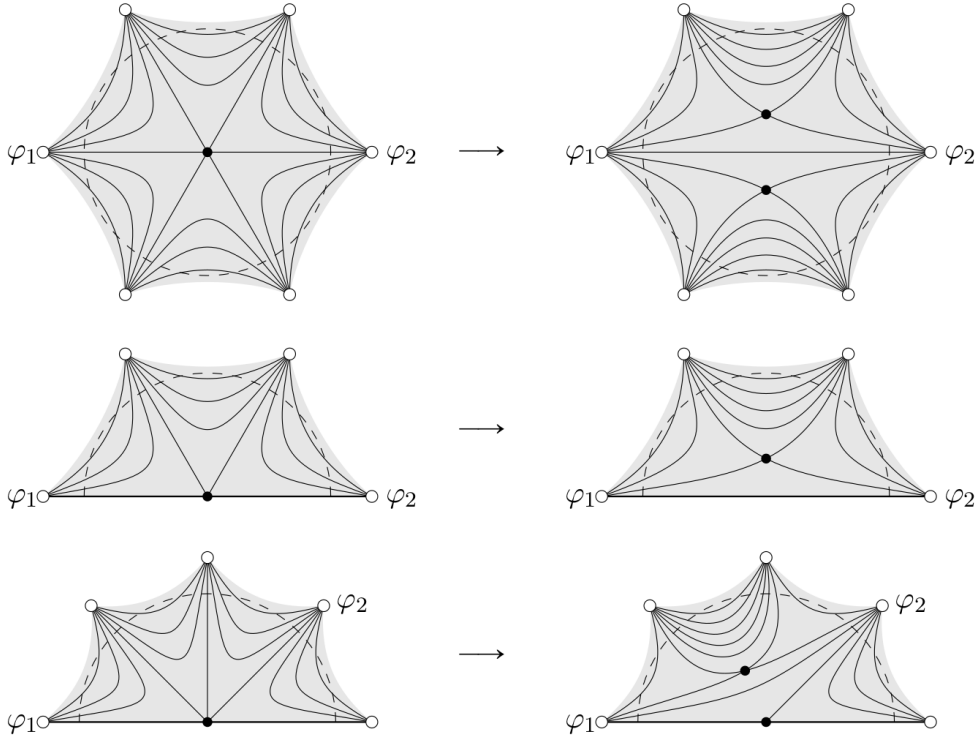


FIGURE 5.2. Change of the foliation \mathcal{F}_Ψ under a splitting of an event

Definition 5.4. Let Ψ and Ψ' be two movie diagrams such that, for some $\theta_1, \theta_2, \theta_3 \in \mathbb{S}^1$, $\theta_2 \in (\theta_1; \theta_3)$, the non-crossing chord diagrams $\Psi(\theta)$ and $\Psi'(\theta)$ coincide for all $\theta \in (\theta_3; \theta_1]$, and no event of Ψ or Ψ' occur in the union of intervals $(\theta_1; \theta_2) \cup (\theta_2; \theta_3)$. Suppose also that there are pairwise distinct $\varphi_1, \varphi_2, \varphi_3, \varphi_6, \varphi_5, \varphi_4 \in \mathbb{S}^1$ following on \mathbb{S}^1 in the same or opposite circular order as listed (observe that it is not the natural one) such that the following events occur at the moments $\theta_1, \theta_2, \theta_3$:

	Ψ	Ψ'
θ_1	no event	$\{\varphi_3, \varphi_4\}, \{\varphi_6, \varphi_5\} \rightsquigarrow$ $\{\varphi_3, \varphi_6\}, \{\varphi_5, \varphi_4\}$
θ_2	$\{\varphi_1, \varphi_2\}, \{\varphi_3, \varphi_4\} \rightsquigarrow$ $\{\varphi_1, \varphi_4\}, \{\varphi_2, \varphi_3\}$	$\{\varphi_1, \varphi_2\}, \{\varphi_3, \varphi_6\} \rightsquigarrow$ $\{\varphi_1, \varphi_6\}, \{\varphi_2, \varphi_3\}$
θ_3	$\{\varphi_1, \varphi_4\}, \{\varphi_6, \varphi_5\} \rightsquigarrow$ $\{\varphi_1, \varphi_6\}, \{\varphi_5, \varphi_4\}$	no event

Then $\Psi \mapsto \Psi'$ is called a *special commutation*.

We also use this term for a move whose definition is similar with the only distinction that φ_2 is not involved. Thus, the events in Ψ and Ψ' at the moments $\theta_1, \theta_2, \theta_3$ are:

	Ψ	Ψ'
θ_1	no event	$\{\varphi_3, \varphi_4\}, \{\varphi_6, \varphi_5\} \rightsquigarrow$ $\{\varphi_3, \varphi_6\}, \{\varphi_5, \varphi_4\}$
θ_2	$\{\varphi_3, \varphi_4\} \rightsquigarrow \{\varphi_1, \varphi_4\}$	$\{\varphi_3, \varphi_6\} \rightsquigarrow \{\varphi_1, \varphi_6\}$
θ_3	$\{\varphi_1, \varphi_4\}, \{\varphi_6, \varphi_5\} \rightsquigarrow$ $\{\varphi_1, \varphi_6\}, \{\varphi_5, \varphi_4\}$	no event

The morphism assigned to the special commutation is induced by an elementary isotopy supported on a 3-ball B similarly to Definition 5.3. Figure 5.3 illustrates the difference between the foliations \mathcal{F}_Ψ and \mathcal{F}'_Ψ , where the intersections of the surfaces with B are encircled in the dashed line.

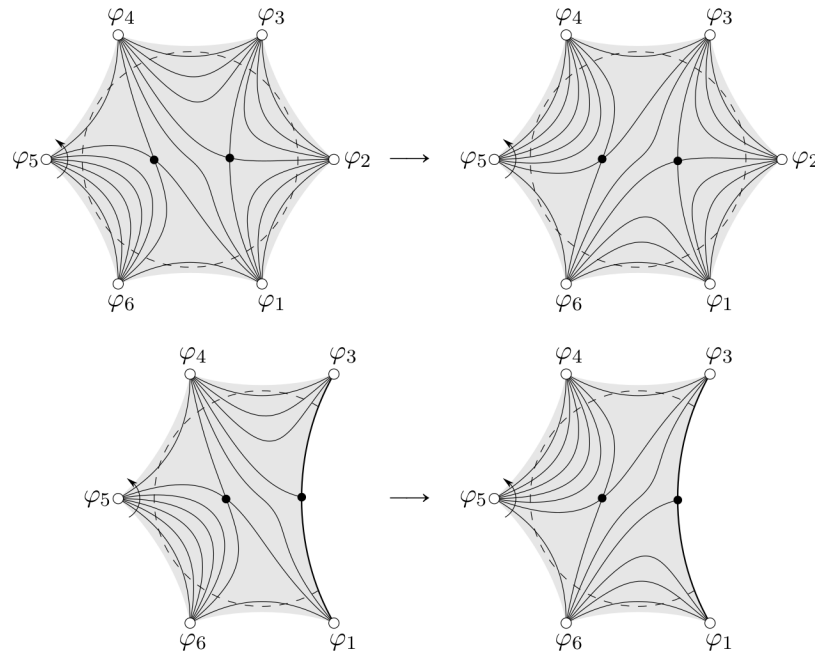


FIGURE 5.3. Change of the foliation under a special commutation

Two admissible events $C_1 \mapsto C_2$ and $C'_1 \mapsto C'_2$ are considered equal if they ‘do the same’, that is, remove the same chords, and add the same chords:

$$C_1 \setminus C_2 = C'_1 \setminus C'_2, \quad C_2 \setminus C_1 = C'_2 \setminus C'_1.$$

Definition 5.5. Let Ψ and Ψ' be two movie diagrams such that, for some distinct $\theta_1, \theta_3 \in \mathbb{S}^1$, the following holds:

- (1) $\Psi(\theta) = \Psi'(\theta)$ for all $\theta \in [\theta_3; \theta_1]$;

- (2) each of Ψ and Ψ' has exactly two events in the interval $(\theta_1; \theta_3)$, these events for Ψ' are the same as for Ψ , but follow in the opposite order.

Then we say that $\Psi \mapsto \Psi'$ is a (*non-special*) *commutation*.

The morphism assigned to a commutation $\Psi \mapsto \Psi'$ is a composition of two morphisms that can be represented by elementary isotopies supported on two 3-balls which are disjoint from one another and from $\mathbb{S}_{\tau=0}^1$ such that each of them contains exactly one singularity of each of \mathcal{F}_Ψ and $\mathcal{F}'_{\Psi'}$.

Definition 5.6. Let Ψ be a movie diagram, and let $F \subset \mathbb{S}^3$ be a surface representing Ψ . Let also f, g be orientation preserving homeomorphisms of \mathbb{S}^1 . There is a unique movie diagram Ψ' represented by $(f * g)(F)$. This diagram can be formally written as

$$\Psi' = (S^2 f) \circ \Psi \circ g^{-1},$$

where $S^2 f$ is the self-homeomorphism of the symmetric square $S^2 \mathbb{S}^1$ induced by f .

In this situation, the passage $\Psi \xrightarrow{[f * g]} \Psi'$ is called a *rescaling*.

Lemma 5.7. *Let $\Psi \xrightarrow{\chi} \Psi'$ be one of the moves introduced by any of the Definitions 5.2–5.6. Then there exist rectangular diagrams of surfaces Π and Π' representing Ψ and Ψ' , respectively, and a sequence of basic moves not including half-wrinkle moves*

$$\Pi = \Pi_0 \mapsto \Pi_1 \mapsto \dots \mapsto \Pi_k = \Pi' \quad (5.1)$$

such that the move $\Pi_{i-1} \mapsto \Pi_i$ is fixed on $\partial \Pi \cap \partial \Pi'$ for all $i = 1, \dots, k$, and the morphism $\hat{\Pi} \rightarrow \hat{\Pi}'$ obtained by composing the moves (5.1) represents χ .

Proof. We consider all types of moves one by one. The fact that the morphism obtained from the constructed sequence of basic moves is χ , is pretty obvious in each case. So, we concentrate on the description of the decomposition.

Case 1: $\Psi \xrightarrow{\chi} \Psi'$ is a finger move. We use the notation from Definition 5.2. Since Ψ has no events in the interval $[\theta_1; \theta_2]$, we may choose a rectangular diagram of a surface Π so that $\Psi = \Psi_\Pi$ and Π has no occupied meridians in $(\theta_1; \theta_2) \times \mathbb{S}^1$. Since $\{\varphi_3, \varphi_4\} \in \Psi(\theta)$ for all $\theta \in [\theta_1; \theta_2]$ we may additionally ensure that Π contains the rectangle $r = [\theta_1; \theta_2] \times [\theta_4; \theta_3]$.

Let Π' be the rectangular diagram of a surface obtained from Π by replacing r with the rectangles

$$[\theta_1; \theta_2] \times [\varphi_4; \varphi_1], \quad [\theta_1; \theta_2] \times [\varphi_2; \varphi_3], \quad [\theta_2; \theta_1] \times [\varphi_1; \varphi_2].$$

One can see that $\Psi' = \Psi_{\Pi'}$, and $\Pi \mapsto \Pi'$ is a horizontal bubble creation move, which completes the proof in this case.

Case 2: $\Psi \xrightarrow{\chi} \Psi'$ is a splitting of an event. We use the notation from Definition 5.3. A rectangular diagram of a surface Π' such that $\Psi' = \Psi_{\Pi'}$ can be chosen so that the following two rectangles belong to Π' :

$$[\theta_1; \theta_3] \times [\varphi_1; \varphi_2], \quad [\theta_3; \theta_2] \times [\varphi_2; \varphi_1],$$

where θ_3 is the midpoint of the interval $[\theta_1; \theta_2]$. We may also ensure that Π' has no occupied meridians in the domain $(\theta_1; \theta_2) \times \mathbb{S}^1$ except m_{θ_3} .

There is then a vertical wrinkle reduction move $\Pi' \mapsto \Pi$ reducing these two rectangles and producing a diagram Π such that $\Psi = \Psi_{\Pi}$. This case is also done.

Case 3: $\Psi \xrightarrow{\chi} \Psi'$ is a special commutation. We will use the notation from Definition 5.4. Without loss of generality, assume that $\varphi_1, \varphi_2, \varphi_3, \varphi_6, \varphi_5, \varphi_4$ follow on \mathbb{S}^1 in the same cyclic order as listed. We consider the first version of the move (with φ_2 involved).

We may choose a rectangular diagram of a surface Π such that $\Psi = \Psi_{\Pi}$, and Π has no occupied meridians in the domain $([\theta_1; \theta_2] \cup (\theta_2; \theta_3)) \times \mathbb{S}^1$. By applying a vertical bubble creation move, we may achieve that Π has rectangles of the form

$$r_0 = [\theta_4; \theta_1] \times [\theta_4; \theta_3], \quad r_1 = [\theta_1; \theta_2] \times [\theta_3; \theta_4], \quad \theta_4 \in (\theta_3; \theta_1).$$

It follows from Definition 5.4 that there are also the following rectangles in Π :

$$\begin{aligned} r_2 &= [\theta_2; \theta_3] \times [\varphi_4; \varphi_1], & r_3 &= [\theta_3; \theta_5] \times [\varphi_5; \varphi_4], & r_4 &= [\theta_6; \theta_3] \times [\varphi_6; \varphi_5], \\ r_5 &= [\theta_3; \theta_7] \times [\varphi_1; \varphi_6], & r_6 &= [\theta_8; \theta_2] \times [\varphi_1; \varphi_2], & r_7 &= [\theta_2; \theta_9] \times [\varphi_2; \varphi_3] \end{aligned}$$

for some $\theta_5, \dots, \theta_9 \in \mathbb{S}^1$ (see Figure 5.4).

Let Π' be a rectangular diagram of a surface obtained by replacing the rectangles r_1, r_2, r_3, r_4 with

$$\begin{aligned} r'_1 &= [\theta_1; \theta_2] \times [\varphi_3; \varphi_6], & r'_2 &= [\theta_2; \theta_3] \times [\varphi_6; \varphi_1], \\ r'_3 &= [\theta_1; \theta_5] \times [\varphi_5; \varphi_4], & r'_4 &= [\theta_6; \theta_1] \times [\varphi_6; \varphi_5]. \end{aligned}$$

Then $\Pi \mapsto \Pi'$ is a flype (the roles of r_i, r'_i , $i = 1, 2, 3, 4$, here are the same as in Definition 2.7) and Π' represents Ψ' .

For the ‘simplified’ version of the move the argument is exactly the same, one only has to ignore rectangles r_6 and r_7 , which are now not present in the diagrams.

Case 4: $\Psi \xrightarrow{\chi} \Psi'$ is a non-special commutation. We use the notation from Definition 5.5 and the construction from the proof of Proposition 4.9.

Let $\varphi'_1, \varphi'_2, \dots, \varphi'_k \in \mathbb{S}^1$ (respectively, $\varphi''_1, \varphi''_2, \dots, \varphi''_l \in \mathbb{S}^1$) be the points involved in the first (respectively, the second) event of Ψ occurring in the interval $[\theta_1; \theta_3]$, and let θ' (respectively, θ'') be the exact moment of this

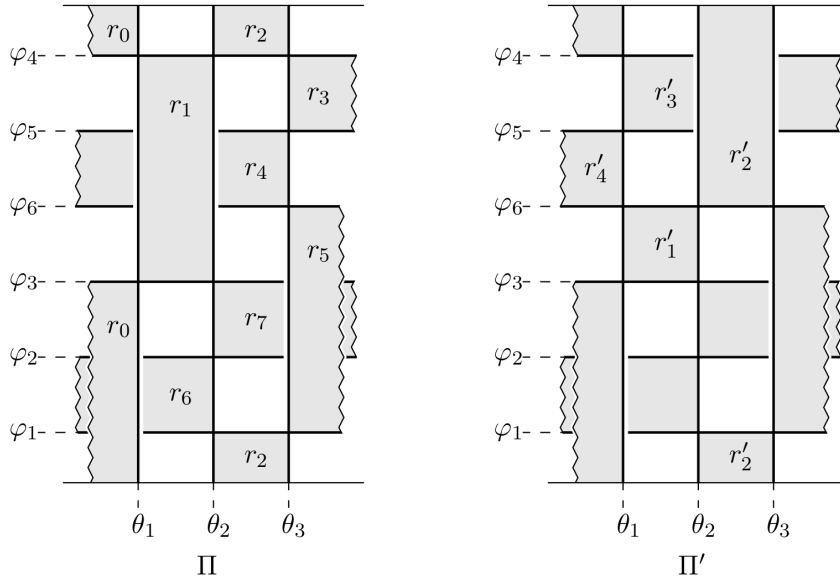


FIGURE 5.4. Realizing a special commutation by means of a flype

event. Exchangeability of the events means that there are a region R_2 of $\Psi(\theta'') = \Psi(\theta' + 0)$ and points $\varphi_1, \varphi_2 \in R_2$ such that

$$\varphi'_1, \varphi'_2, \dots, \varphi'_k \in (\varphi_2; \varphi_1), \quad \varphi''_1, \varphi''_2, \dots, \varphi''_l \in (\varphi_1; \varphi_2).$$

Moreover, there are regions R_1 of $\Psi(\theta') = \Psi(\theta_1)$ and R_3 of $\Psi(\theta'' + 0) = \Psi(\theta_3)$ also containing φ_1, φ_2 .

Choose $\theta_2 \in (\theta'; \theta'')$ and a rectangular diagram of a surface Π representing Ψ such that:

- (1) $R_\Pi(\theta_i) = R_i$, $i = 1, 2, 3$;
- (2) on the intervals $(\theta_1; \theta')$, $(\theta'; \theta_2)$, $(\theta_2; \theta'')$, $(\theta''; \theta_3)$, the diagram Π realizes the shortest paths:
 - from R_1 to $R_\Pi(\theta')$ on the tree $T(\Psi(\theta'))$,
 - from $R_\Pi(\theta' + 0)$ to R_2 on the tree $T(\Psi(\theta''))$,
 - from R_2 to $R_\Pi(\theta'')$ on the tree $T(\Psi(\theta''))$, and
 - from $R_\Pi(\theta'' + 0)$ to R_3 on the tree $T(\Psi(\theta_3))$,
respectively.

Then Π satisfies all the conditions of Definition 2.6 (with $\theta_1, \theta_2, \theta_3, \varphi_1, \varphi_2$ playing the same role), so, an exchange move can be applied to Π , which will exchange the events that occur at θ_1 and θ_2 in Ψ . This exchange move can be chosen so as to obtain Π' representing Ψ' .

Case 5: $\Psi \xrightarrow{\chi} \Psi'$ is a rescaling. We use the notation from Definition 5.6. Choose any rectangular diagram Π with $\Psi_\Pi = \Psi$. It suffices to consider

the case when the homeomorphism $f \times g : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is identical outside an annulus containing the whole of exactly one occupied level x of Π , and such that x is the midline of the annulus. Indeed, a general rescaling can be decomposed into a sequence of such, ‘elementary’, rescalings.

Let $\Pi' = (f \times g)(\Pi)$. The transformation $\Pi \mapsto \Pi'$ can be decomposed into two basic moves: first, a wrinkle creation, and second, a wrinkle reduction. The arbitrariness in the definition of these moves allows the second one to be inverse to the first one in the combinatorial, but not in the geometrical, sense. \square

6. THE FIXED BOUNDARY CASE

Proposition 6.1. *Let Π and Π' be rectangular diagrams of surfaces having common boundary, $\partial\Pi = \partial\Pi'$. Suppose that there are a self-homeomorphism ϕ_1 of \mathbb{S}^3 that takes $\widehat{\Pi}$ to $\widehat{\Pi}'$, and an isotopy $(\phi_t)_{t \in [0;1]}$ from $\phi_0 = \text{id}|_{\mathbb{S}^3}$ to ϕ_1 fixed on the boundary $\partial\widehat{\Pi}$ and keeping fixed the tangent plane to the surface at every point of $\partial\widehat{\Pi}$.*

Then there exists a sequence of basic moves producing Π' from Π such that all the moves in it are fixed on $\partial\Pi$, and the composition of all the moves induces the morphism $[\phi_1]$.

Proof. For brevity, we denote $\phi_t(\widehat{\Pi})$ by F_t , $t \in [0;1]$. In particular, $F_0 = \widehat{\Pi}$. The singular foliation induced by the open book decomposition $\{\mathcal{P}_\theta\}_{\theta \in \mathbb{S}^1}$ on the surface $F_t \setminus \mathbb{S}_{\tau=0}^1$ will be denoted by \mathcal{F}_t .

We are going to convert the isotopy ϕ into a sequence of basic moves. This is done in the following four steps.

Step 1: Alter ϕ slightly to make it ‘as generic as possible’ with respect to the induced family of foliations \mathcal{F}_t .

Namely, we disturb ϕ slightly so that, for all but finitely many values of t , the following holds (in which case the foliation \mathcal{F}_t is said to be *generic*):

- (1) the surface F_t is transverse to $\mathbb{S}_{\tau=0}^1$,
- (2) the foliation \mathcal{F}_t , geometrically, has only Morse type singularities in the interior of F_t ,
- (3) each page contains at most one of the following:
 - a singularity of \mathcal{F}_t at an interior point of F_t ,
 - an arc contained in $\partial F_t = \partial F_0$,
- (4) at each point in $\partial F_t \cap \mathbb{S}_{\tau=1}^1$ (where the boundary of the surface usually has a singularity, and the surface is tangent to a page \mathcal{P}_θ), the foliation \mathcal{F}_t either has no topological singularity or has a ‘half-saddle’ singularity (illustrated by the central picture in the bottom row in Figure 4.2).

Moreover, we assume that the violations of these rules that occur at exceptional moments t cannot be resolved in a one-parametric family by a small perturbation. Here is the full list of what can happen at such moments:

- (1) there is a tangency point of F_t and $\mathbb{S}_{\tau=0}^1$,
- (2) \mathcal{F}_t has multiple saddle singularities, $t = 0$ or 1 ,
- (3) there is a page \mathcal{P}_θ containing either two singularities of \mathcal{F}_t or an arc of ∂F_t and a singularity of \mathcal{F}_t outside of this arc,
- (4) the foliation \mathcal{F}_t has an index zero non-Morse-type geometric singularity in an interior point.

Consider these situations one by one in more detail.

Tangency with $\mathbb{S}_{\tau=0}^1$. A tangency of F_t and $\mathbb{S}_{\tau=0}^1$ is persistent under small perturbations if two vertices of \mathcal{F}_t are created or disappear at this moment. We may assume that the second fundamental form of the surface is non-degenerate at the tangency point. If it is sign-definite, then a creation of two vertices of \mathcal{F}_t is accompanied by an annihilation of two center singularities. If the second fundamental form is indefinite, then a creation of two vertices is accompanied by a creation of two saddle singularities. This is illustrated in Figure 6.1.

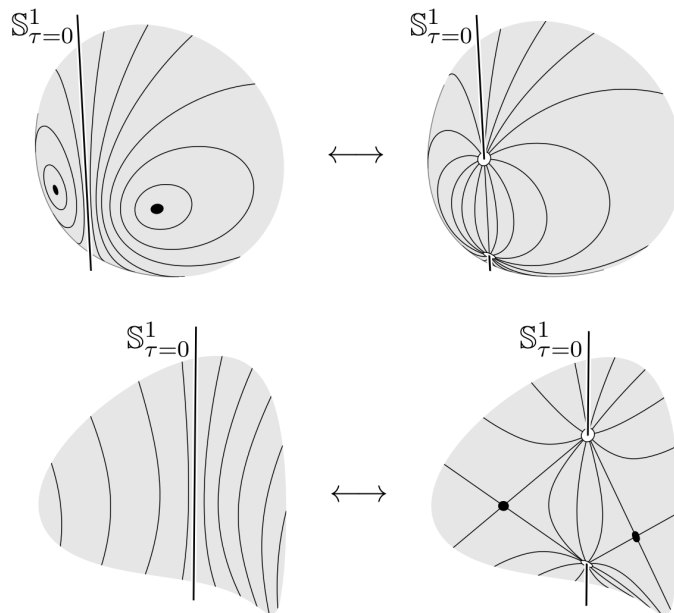


FIGURE 6.1. Creation and cancellation of two vertices of \mathcal{F}_t

Multiple saddles. By definition, the surfaces F_0 and F_1 are tangent to the respective pages at all points of their intersections with $\mathbb{S}_{\tau=1}^1$. At all such points, the foliation \mathcal{F}_t , $t = 0, 1$, has geometrical singularities, which are,

in general, multiple saddles. The multiplicity of a saddle s is defined as $(m - 2)/2$ if s is an interior point of the surface and otherwise as $(m - 2)$, where m is the number of separatrices approaching s . Zero multiplicity saddles are regular points from the topological point of view.

By a small perturbation of the surface all multiple saddles of multiplicities greater than one can be resolved into ordinary (multiplicity one) saddles, and zero multiplicity saddles at interior points can be smoothed out to become geometrically regular points.

Note that to specify a resolution it suffices to show how the foliation \mathcal{F}_t changes near the singularity. Indeed, for a fixed t , the embedding $\phi_t|_{F_0}$ can be viewed as three functions, θ , φ , and τ defined on F_0 (with θ and φ taking values in \mathbb{S}^1 and not everywhere defined). At a singularity, the surface is tangent to a page \mathcal{P}_θ , so, small deformations in the θ -direction are safe in the sense that they will not prevent the respective map $F_0 \rightarrow \mathbb{S}^3$ from being an embedding. Deformation in the θ -direction means keeping the functions φ and τ fixed while altering θ . And to specify the alteration of the latter it suffices to specify how the foliation \mathcal{F}_t changes.

All multiple saddles must be resolved at the beginning of the isotopy and turned into multiple saddles back (probably in a different way) at the last moment.

Two singularities in a single page. At each moment t , the values of θ at singularities of \mathcal{F}_t and on ∂F_t are called *critical*. Critical values of θ may eventually coincide. In a one-parametric family of surfaces this situation is persisting if two critical values are going to be exchanged. Such an unavoidable coincidence of two critical values of θ will be referred to as a *collision*.

A non-Morse-type geometrical singularity. Finally, there could be moments at which a saddle-center pair of singularities is born or cancelled. At such moments the foliation \mathcal{F}_t has an index zero non-Morse-type singularity at an interior point.

Step 2: Turn the isotopy into a sequence of moves of movie diagrams. Let $t_0 = 0 < t_1 < t_2 < \dots < t_m = 1$ be the moments at which \mathcal{F}_t is not generic, and let $\varepsilon > 0$ be smaller than half the distance between t_{i-1} and t_i for any $i = 1, 2, \dots, m$.

By taking ε sufficiently small and modifying ϕ slightly if necessary, we may assume that $\phi_{t_i+\varepsilon} \circ \phi_{t_i-\varepsilon}^{-1}$ is an elementary isotopy from $\phi_{t_i-\varepsilon}(F_0)$ to $\phi_{t_i+\varepsilon}(F_0)$, $i = 1, \dots, m - 1$, supported on a small ball containing only those singularities and vertices of $\mathcal{F}_{t_i \pm \varepsilon}$ that are directly involved in the combinatorial change of the foliation (in the case of a collision, just one of the two collided singularities).

For $t \in (t_1; t_{m-1})$ the foliation \mathcal{F}_t might have closed leaves, in which case it cannot be encoded by a movie diagram. We overcome this as follows.

In the proof of [3, Proposition 5] we described a procedure allowing to deform an individual generic surface so that all intersections of the obtained surface with every page become simply connected. It is based on an idea borrowed from [4] and consists of operations that are called finger moves in [3]. The operations we define below are slightly more general than those in [3]. We call them geometric finger moves to distinguish from finger moves of movie diagrams (which are a combinatorial version of a particular case of geometric finger moves).

Definition 6.2. Let F and F' be two surfaces related by an elementary isotopy ζ supported on an open 3-ball B . Suppose that there is a closed 3-ball $D \subset B$ with the following properties:

- (1) D intersects $\mathbb{S}_{\tau=0}^1$ in an arc, and each page \mathcal{P}_θ in a 2-disc;
- (2) $D \cap F = \partial D \cap F = d$ is a 2-disc disjoint from $\mathbb{S}_{\tau=0}^1$ and from some page \mathcal{P}_{θ_0} ;
- (3) the boundary of d consists of two smooth arcs transverse to \mathcal{F}_F whose endpoints are regular points or vertices of \mathcal{F}_F ;
- (4) there is a self-homeomorphism of \mathbb{S}^3 identical outside B which preserve each page \mathcal{P}_θ and takes $\overline{(\partial D) \Delta F}$ to F' , where Δ stands for the symmetric difference.

Then the passage $F \mapsto F'$ assigned with the morphism $[\zeta]$ is called a *geometric finger move*. The leaves of \mathcal{F}_F intersecting the interior of d are said to be *broken up* by this finger move.

If F and F' represent movie diagrams Ψ and Ψ' , respectively, then the transformation $\Psi \xrightarrow{[\zeta]} \Psi'$ is also called a geometric finger move.

The effect of a geometric finger move $F \mapsto F'$ on the foliation \mathcal{F}_F depends on the number of center singularities contained in ∂d (we use the notation from Definition 6.2). This can be equal to zero, one, or two. The corresponding changes of the foliation are shown in Figure 6.2, where ∂d is shown in the dashed line.

In all three cases, the leaves that are said to be broken up by the geometric finger move are, indeed, modified in a way justifying this terminology. If such a leaf is a closed curve it is turned into an arc, and if it is an arc it is split into two arcs. The leaves that do not intersect d are untouched or just deformed. The leaves passing through the endpoints of the two smooth arcs transverse to \mathcal{F}_F that form ∂d are modified by a homotopy equivalence.

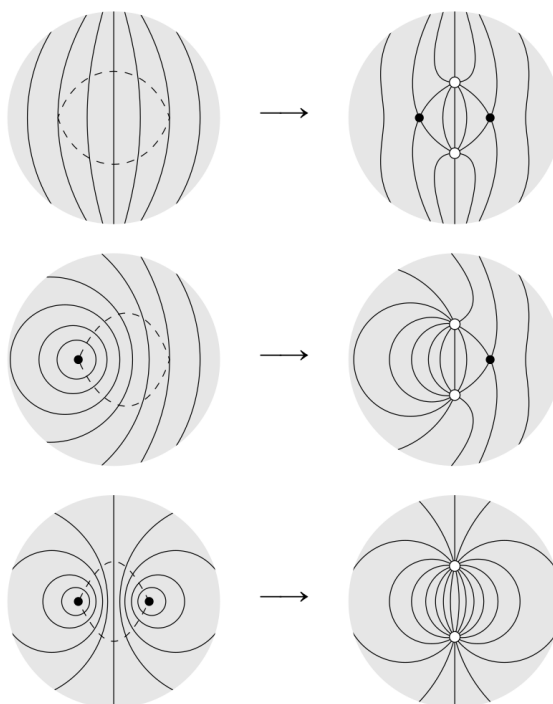


FIGURE 6.2. Change of the foliation under a geometric finger move

For $t \in (0; 1)$, denote by \mathcal{Z}_t the set of all homeomorphisms $\iota : \mathbb{S}^3 \leftrightarrow \mathbb{S}^3$ such that:

- (1) the transition from F_t to $\iota(F_0)$ assigned with the morphism $[\iota \circ \phi_t^{-1}]$ is the composition of a sequence of geometric finger moves and ambient isotopies preserving the open book decomposition and fixed on ∂F_0 ;
- (2) $\mathcal{F}_{\iota(F_0)}$ is generic and has no closed leaves.

As follows from the construction in the proof of [3, Proposition 5] the set \mathcal{Z}_t is non-empty for all $t \notin \{t_1, \dots, t_m\}$. One can also show that the large arbitrariness in the choice of a way of breaking up closed leaves of \mathcal{F}_t allows one to proceed from one choice to another by means of geometric finger moves, their inverses, and rescalings without letting any of the closed leaves to recover. In other words, for any two elements ι, ι' from \mathcal{Z}_t the transition from $\iota(F_0)$ to $\iota'(F_0)$ assigned with the morphism $[\iota' \circ \iota^{-1}]$ can be decomposed into geometric finger moves, their inverses, and rescalings performed *within* \mathcal{Z}_t .

For each element $\iota \in \mathcal{Z}_t$, the surface $\iota(F_0)$ represents a unique movie diagram, which we denote by Ψ_ι .

For any $i = 1, \dots, m$, small enough ε , and $t \in (t_{i-1} + \varepsilon; t_i - \varepsilon)$, the foliation \mathcal{F}_t is generic, which implies that $\phi_{t_i - \varepsilon}$ can be obtained from $\phi_{t_{i-1} + \varepsilon}$ by an ambient isotopy preserving the open book decomposition of \mathbb{S}^3 and

fixed on ∂F_0 . Therefore, for any $\iota \in \mathcal{Z}_{t_{i-1}+\varepsilon}$ there exists $\iota' \in \mathcal{Z}_{t_i-\varepsilon}$ such that $\Psi_\iota \xrightarrow{\iota' \circ \iota^{-1}} \Psi_{\iota'}$ is a rescaling, which means $\mathcal{Z}_{t_{i-1}+\varepsilon} = \mathcal{Z}_{t_i-\varepsilon}$.

Suppose that, for each $i = 1, \dots, m$, two elements $\iota'_i, \iota''_i \in \mathcal{Z}_{t_{i-1}+\varepsilon} = \mathcal{Z}_{t_i-\varepsilon}$ has been chosen. Then the transformation $\Psi_\Pi \xrightarrow{[\phi_1]} \Psi_{\Pi'}$ is the composition of the following ones:

$$\begin{aligned} \Psi_\Pi \xrightarrow{[\iota'_1]} \Psi_{\iota'_1} \xrightarrow{[\iota''_1 \circ \iota'_1{}^{-1}]} \Psi_{\iota''_1} \xrightarrow{[\iota'_2 \circ \iota''_1{}^{-1}]} \Psi_{\iota'_2} \xrightarrow{[\iota''_2 \circ \iota'_2{}^{-1}]} \\ \Psi_{\iota''_2} \xrightarrow{[\iota'_3 \circ \iota''_2{}^{-1}]} \dots \xrightarrow{[\iota''_m \circ \iota'_m{}^{-1}]} \Psi_{\iota''_m} \xrightarrow{[\phi_1 \circ \iota''_m{}^{-1}]} \Psi_{\Pi'}. \end{aligned}$$

It follows from what was just said that each transformation

$$\Psi_{\iota'_i} \xrightarrow{[\iota''_i \circ \iota'_i{}^{-1}]} \Psi_{\iota''_i}, \quad i = 1, \dots, m,$$

admits a decomposition into geometric finger moves and rescalings.

Observe that \mathcal{F}_t has no closed leaves for $t = \varepsilon$ or $t = 1 - \varepsilon$, hence, $\phi_\varepsilon \in \mathcal{Z}_\varepsilon$ and $\phi_{1-\varepsilon} \in \mathcal{Z}_{1-\varepsilon}$. We take ϕ_ε and $\phi_{1-\varepsilon}$ for ι'_1 and ι''_m , respectively. We will have that the passage $\Psi_\Pi \xrightarrow{[\iota'_1]} \Psi_{\iota'_1}$ can be decomposed into a sequence of splittings of an event, and the passage $\Psi_{\iota''_m} \xrightarrow{[\phi_1 \circ \iota''_m{}^{-1}]} \Psi_{\Pi'}$ into a sequence of mergings of events.

Thus, it remains to do the following two things:

- (1) show how to implement a geometric finger move within a class \mathcal{Z}_t by means of allowed moves of movie diagrams;
- (2) show how to choose, for each $i = 1, \dots, m - 1$, the elements ι''_i, ι'_{i+1} so that the transformation $\Psi_{\iota''_i} \xrightarrow{[\iota'_{i+1} \circ \iota''_i{}^{-1}]} \Psi_{\iota'_{i+1}}$ can be decomposed into a sequence of allowed moves of movie diagrams.

We start from the latter.

Suppose that a collision occurs at a moment $t = t_i$, and s_1, s_2 are the two singularities of \mathcal{F}_t that belong to the same page. There are the following five cases of collisions treated differently.

Case 1: one of s_1 and s_2 is a center singularity of \mathcal{F}_{t_i} . Without loss of generality we may assume that s_1 is a center, and s_2 is unaltered when t goes from $t_i - \varepsilon$ to $t_i + \varepsilon$. We may also assume that s_1 is a local maximum of $\theta|_{F_i}$, and it moves ‘forward’ when t goes from $t_i - \varepsilon$ to $t_i + \varepsilon$, from a page \mathcal{P}_{θ_0} to a page $\mathcal{P}_{\theta_0+\delta}$, where $\delta > 0$ is small (the other cases are symmetric to this one).

Clearly, there is an element $\iota \in \mathcal{L}_{t_i+\varepsilon}$ such that $\iota(F_0)$ is obtained from $\phi_{t_i+\varepsilon}(F_0)$ by a sequence of geometric finger moves in which the move eliminating s_1 breaks up a leaf in each page \mathcal{P}_θ with $\theta \in [\theta_0; \theta_0 + \delta)$. One can see that such ι belongs also to $\mathcal{L}_{t_i-\varepsilon}$.

Thus, in this case the sets $\mathcal{L}_{t_i\pm\varepsilon}$ have a non-empty intersection, and we choose any $\iota''_i = \iota'_{i+1} \in \mathcal{L}_{t_i-\varepsilon} \cap \mathcal{L}_{t_i+\varepsilon}$.

Case 2: s_1 and s_2 are saddles (one of them may be at the boundary and have multiplicity zero or one) not connected by a separatrix. Recall that the surfaces $\phi_{t_i-\varepsilon}(F_0)$ and $\phi_{t_i+\varepsilon}(F_0)$ coincide outside of a small ball B containing one of the saddles s_1, s_2 . This implies that elements $\iota_0 \in \mathcal{L}_{t_i-\varepsilon}$ and $\iota_1 \in \mathcal{L}_{t_i+\varepsilon}$ can be chosen so that:

- (1) $\phi_{t_i-\varepsilon} \circ \iota_0^{-1}$ and $\phi_{t_i+\varepsilon} \circ \iota_1^{-1}$ are identical on B ;
- (2) $\iota_0 \circ \iota_1^{-1}$ is identical outside B ;
- (3) there is an isotopy $\{\iota_t\}_{t \in [0;1]}$ from ι_0 to ι_1 such that for all $t \in [0;1] \setminus \{\frac{1}{2}\}$ the foliation $\mathcal{F}_{\iota_t(F_0)}$ is generic, and at $t = \frac{1}{2}$ a collision of s_1 and s_2 occurs.

We let ι''_i and ι'_{i+1} be ι_0 and ι_1 , respectively. One can see that

$$\Psi_{\iota''_i} \xrightarrow{[\iota'_{i+1} \circ \iota''_i{}^{-1}]} \Psi_{\iota'_{i+1}}$$

is a non-special commutation.

Special subcase of Case 2: the foliations $\mathcal{F}_{t_i\pm\varepsilon}$ have no closed leaves. We simply put $\iota''_i = \phi_{t_i-\varepsilon}$ and $\iota'_{i+1} = \phi_{t_i+\varepsilon}$.

Cases 3 and 4: s_1 and s_2 are saddles which are connected by a separatrix at the moment of collision, and neither of $\mathcal{F}_{\phi_{t_i-\varepsilon}(F_0)}$, $\mathcal{F}_{\phi_{t_i+\varepsilon}(F_0)}$ has closed leaves, so, $\phi_{t_i\pm\varepsilon} \in \mathcal{L}_{t_i\pm\varepsilon}$. In these cases, we again put $\iota''_i = \phi_{t_i-\varepsilon}$ and $\iota'_{i+1} = \phi_{t_i+\varepsilon}$. Denote by α the separatrix connecting s_1 and s_2 at the collision moment.

Assume for the moment that none of s_1 and s_2 lies at the boundary, which means that they are ordinary saddles of $\mathcal{F}_{\phi_{t_i}}$.

Let β_i , $i = 1, 2, 3, 4, 5, 6$, be the six separatrices other than α approaching s_1 and s_2 , numbered in the order in which they intersect the boundary of a small regular neighborhood of α , and so that $\beta_1, \beta_2, \beta_3$ approach s_1 . Since $\mathcal{F}_{\phi_{t_i-\varepsilon}(F_0)}$ and $\mathcal{F}_{\phi_{t_i+\varepsilon}(F_0)}$ have no closed leaves, the foliation $\mathcal{F}_{\phi_{t_i}(F_0)}$ has no saddle connection cycles. Therefore, the separatrices β_i are pairwise distinct, and each β_i approaches a vertex of $\mathcal{F}_{\phi_{t_i}(F_0)}$, which we denote by v_i .

There are two cases here depending on whether the tangent plane to $\phi_{t_i}(F_0)$ at $p \in \alpha$ makes a half-twist around α relative to \mathcal{P}_{θ_0} when p proceeds from one endpoint of α to the other.

Case 3: No half-twist. The vertices v_1, \dots, v_6 follow on $\mathbb{S}_{\tau=0}^1$ in the same or opposite cyclic order as their numeration suggests. In this case, by an isotopy fixed outside a small neighborhood of α , the separatrix α can be collapsed to a point producing a double saddle out of two ordinary ones.

This means that the transition $\Psi_{\iota_i''} \xrightarrow{[\iota_{i+1}' \circ \iota_i''^{-1}]} \Psi_{\iota_{i+1}'}$ can be decomposed into two moves: a merging of events, and a splitting of an event.

Case 4: There is a half-twist. The cyclic order of v_1, \dots, v_6 in $\mathbb{S}_{\tau=0}^1$ is either

$$v_1, v_2, v_3, v_6, v_5, v_4$$

or the opposite one. In this case, the passage $\Psi_{\iota_i''} \xrightarrow{[\iota_{i+1}' \circ \iota_i''^{-1}]} \Psi_{\iota_{i+1}'}$ is a special commutation.

In Cases 3 and 4, if s_1 lies at the boundary of the surface, the reasoning is exactly the same with five separatrices instead of six ones (β_2 and v_2 should be omitted).

Case 5: s_1 and s_2 are saddles connected by a separatrix, and at least one of $\mathcal{F}_{\phi_{t_i-\varepsilon}}$ and $\mathcal{F}_{\phi_{t_i+\varepsilon}}$ has closed leaves. Geometric finger moves allow to break up not only closed leaves of the foliations \mathcal{F}_t but also saddle connections. So, we can proceed as in Case 2 by choosing the elements $\iota_0 \in \mathcal{L}_{t_i-\varepsilon}$ and $\iota_1 \in \mathcal{L}_{t_i+\varepsilon}$ satisfying the additional requirement that, at the moment of the collision that occurs during the isotopy between them, no saddle connection between s_1 and s_2 is present.

We are done with collisions.

Now suppose that a tangency of F_t and $\mathbb{S}_{\tau=0}^1$ occurs at a moment $t = t_i$ and two vertices of \mathcal{F}_t are created. In this case, the passage from $\phi_{t_i-\varepsilon}$ to $\phi_{t_i+\varepsilon}$ is a geometric finger move, therefore $\mathcal{L}_{t_i+\varepsilon} \subset \mathcal{L}_{t_i-\varepsilon}$. Similarly, If a tangency of F_t and $\mathbb{S}_{\tau=0}^1$ occurs at a moment $t = t_i$ and two vertices of \mathcal{F}_t disappear, we have $\mathcal{L}_{t_i+\varepsilon} \supset \mathcal{L}_{t_i-\varepsilon}$. In these cases, we again pick any $\iota_i'' = \iota_{i+1}' \in \mathcal{L}_{t_i-\varepsilon} \cap \mathcal{L}_{t_i+\varepsilon}$.

Suppose that at a moment $t = t_i$ a saddle-center pair of \mathcal{F}_t is being born. We may assume that $\phi_{t_i+\varepsilon} \circ \phi_{t_i-\varepsilon}^{-1}$ is identical outside a small ball B in which the change of the foliation occurs.

We can find an element $\iota \in \mathcal{L}_{t_i-\varepsilon}$ obtained from $\phi_{t_i-\varepsilon}$ by a sequence of geometric finger moves such that $\iota \circ \phi_{t_i-\varepsilon}^{-1}$ is identical on B . Then the same finger moves can be applied to $\phi_{t_i+\varepsilon}$ resulting in a self-homeomorphism ι' of \mathbb{S}^3 such that $\iota' \circ \iota^{-1}$ is identical outside B , and $\iota' \circ \phi_{t_i+\varepsilon}^{-1}$ is identical on B .

The foliation $\mathcal{F}_{\iota'(F_0)}$ has closed leaves, and all them are the result of the creation of the saddle–center pair. Let $\iota'(F_0) \mapsto \tilde{F}$ be a geometric finger move that breaks up all these leaves, and $\tilde{\phi}$ be the corresponding elementary isotopy. Put $\iota'' = \tilde{\phi} \circ \iota'$. One can see that $\iota(F_0) \mapsto \iota''(F_0)$ is also a geometric finger move. This means $\iota'' \in \mathcal{Z}_{t_i-\varepsilon} \cap \mathcal{Z}_{t_i+\varepsilon}$, so, we put $\iota''_i = \iota'_{i+1} = \iota''$.

The situation when a saddle–center pair of \mathcal{F}_t is being cancelled at $t = t_i$ is symmetric to this one.

Now let $\iota, \iota_1 \in \mathcal{Z}_t$, $t \notin \{t_i\}_{i=0}^m$, be such that $\Psi_\iota \xrightarrow{\iota_1 \circ \iota^{-1}} \Psi_{\iota_1}$ is a geometric finger move. Let $\gamma \subset \iota(F_0)$ be an arc transverse to $\mathcal{F}_{\iota(F_0)}$ intersecting (except at the endpoints) exactly those leaves that are broken up by this move, and let $[\theta_1; \theta_2]$ be the interval in which $\theta|_\gamma$ takes values.

If there are no singularities of $\mathcal{F}_{\iota(F_0)}$ in the domain $\bigcup_{\theta \in [\theta_1; \theta_2]} \mathcal{P}_\theta$, then the passage $\Psi_\iota \xrightarrow{\iota_1 \circ \iota^{-1}} \Psi_{\iota_1}$ fits into the definition of a finger move of movie diagrams.

In general, one can always find a geometric finger move $\iota(F_0) \mapsto \iota_0(F_0)$ with $\iota_0 \in \mathcal{Z}_t$ such that the leaves of $\mathcal{F}_{\iota_0(F_0)}$ broken up by this move intersect only a small portion of γ , and $\Psi_\iota \xrightarrow{\iota_0 \circ \iota^{-1}} \Psi_{\iota_0}$ is a finger move of movie diagrams. There is then an isotopy $\{\iota_u\}_{u \in [0;1]}$ from ι_0 to ι_1 such that, for all u , $\iota(F_0) \mapsto \iota_u(F_0)$ is a geometric finger move such that the leaves of $\mathcal{F}_{\iota_u(F_0)}$ broken up by this move intersect a portion of γ that is lengthening when u grows. This is illustrated in Figure 6.3.

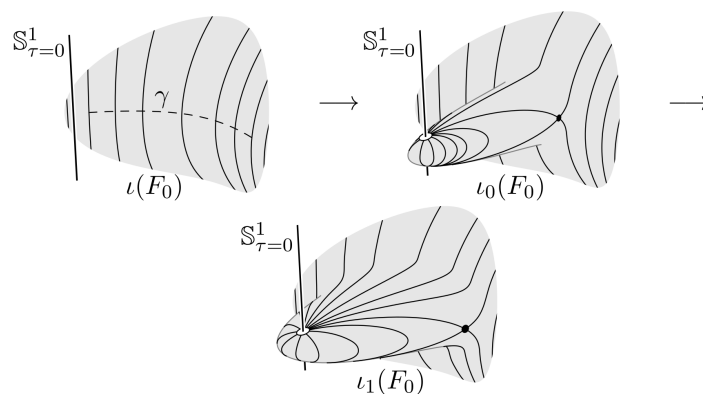


FIGURE 6.3. Realizing a geometric finger move

During this isotopy, the foliation \mathcal{F}_{ι_u} will remain generic except at finitely many moments when a collision of one of the two saddles created by the geometric finger move $\iota \mapsto \iota_u$ with another singularity of \mathcal{F}_{ι_u} occurs. For each of these collisions, Case 3, Case 4, or Special subcase of Case 2 considered above takes place, whereas the effect of the isotopy between the collisions amounts to a rescaling. Hence, there is a decomposition of the

transformation $\Psi_{\iota_0} \xrightarrow{\iota_1 \circ \iota_0^{-1}} \Psi_{\iota_1}$ into a sequence of allowed moves of movie diagrams.

Thus, we have shown that there is a decomposition of $\Psi_{\Pi} \xrightarrow{[\phi_1]} \Psi_{\Pi'}$ into a sequence of moves of movie diagrams introduced by Definitions 5.2–5.6, which completes Step 2.

Step 3: Represent the evolution of the movie diagram by basic moves of rectangular diagrams of surfaces. It follows from Proposition 4.9 and Lemma 5.7 that a sequence of allowed moves of movie diagrams can be converted into a sequence of basic moves of rectangular diagrams of surfaces not including stabilizations and horizontal half-wrinkle moves and preserving $\partial\Pi$. So, a sequence of such moves produces Π' from Π and induces the morphism $[\phi_1]$ from $\widehat{\Pi}$ to $\widehat{\Pi}'$.

Step 4: Exclude vertical half-wrinkle moves from the constructed sequence of basic moves. Let

$$\Pi = \Pi_0 \mapsto \Pi_1 \mapsto \Pi_2 \mapsto \dots \mapsto \Pi_N = \Pi'$$

be the sequence of basic moves obtained at the previous step. By construction, the boundaries of all Π_j are the same, but the type of each boundary vertex ('/' or '\') may vary. Observe that the type of a boundary vertex changes under the move $\Pi_j \mapsto \Pi_{j+1}$ only if this move is a vertical half-wrinkle move (since horizontal half-wrinkle moves are not involved), which changes simultaneously the types of two boundary vertices forming a vertical edge.

Pick $\delta > 0$ smaller than half the distance between any two distinct θ', θ'' such that $m_{\theta'}$ and $m_{\theta''}$ are occupied meridians of $\Pi_{j'}$, $\Pi_{j''}$, respectively, for some $j', j'' \in \{0, 1, \dots, N\}$.

For each $j = 0, 1, 2, \dots, N$ we construct a rectangular diagram of a surface Π'_j as follows. If the types of any $v \in \partial\Pi$ is the same in Π_j and Π , we put $\Pi'_j = \Pi_j$. In particular, $\Pi'_0 = \Pi_0 = \Pi$ and $\Pi'_N = \Pi_N = \Pi'$.

Suppose there is a vertical edge $\{v_1, v_2\} \in \partial\Pi$ such that the types of v_1, v_2 in Π and Π_j disagree. Apply a vertical half-wrinkle creation move to Π_j so that:

- (1) the boundary of the diagram is preserved;
- (2) the types of v_1, v_2 change to the opposite;
- (3) the new occupied meridian is at distance δ from the meridian containing v_1 and v_2 .

Repeat this procedure with the new diagram in place of Π_j until all boundary vertices have the same type in it as they have in Π . Let Π'_j be the obtained diagram.

One can see the following:

- (1) if $\Pi_j \mapsto \Pi_{j+1}$ is a vertical half-wrinkle creation (respectively, reduction) move, then $\Pi'_j \mapsto \Pi'_{j+1}$ is a vertical wrinkle creation (respectively, reduction) move or a rescaling;
- (2) if $\Pi_j \mapsto \Pi_{j+1}$ is not a vertical half-wrinkle move, then $\Pi'_j \mapsto \Pi'_{j+1}$ is a basic move of the same kind as $\Pi_j \mapsto \Pi_{j+1}$;
- (3) all the moves $\Pi'_j \mapsto \Pi'_{j+1}$, $j = 0, 1, \dots, N - 1$, are fixed on $\partial\Pi$;
- (4) for all $j = 0, 1, \dots, N$, we have $\Psi_{\Pi'_j} = \Psi_{\Pi_j}$, which implies that the morphism from Π to Π' induced by the sequence of moves

$$\Pi = \Pi'_0 \mapsto \Pi'_1 \mapsto \Pi'_2 \mapsto \dots \mapsto \Pi'_N = \Pi'$$

is still $[\phi_1]$. □

7. MODIFYING THE BOUNDARY

Lemma 7.1. *Let Π be a rectangular diagram of a surface, and let $r \in \Pi$ be a rectangle having exactly one, two, or three consecutive vertices at $\partial\Pi$ and such that $\Pi' = \Pi \setminus \{r\}$ is also a rectangular diagram of a surface. Let also $\phi : (\mathbb{S}^3, \hat{\Pi}') \rightarrow (\mathbb{S}^3, \hat{\Pi})$ be a homeomorphism isotopic to the identity (in the class of maps $(\mathbb{S}^3, \hat{\Pi}') \rightarrow (\mathbb{S}^3, \hat{\Pi})$). Then the transformation $\Pi \mapsto \Pi'$ assigned with the morphism $[\phi]$ can be decomposed into basic moves fixed on the boundary components disjoint from $V(r)$.*

Proof. There are the following three cases to consider.

Case 1: r shares exactly two common vertices with the other rectangles in Π . The sought-for decomposition is shown in Figure 7.1. First, we apply an exchange move that replaces r with a ‘thin’ rectangle, and then remove it by a half-wrinkle reduction move.

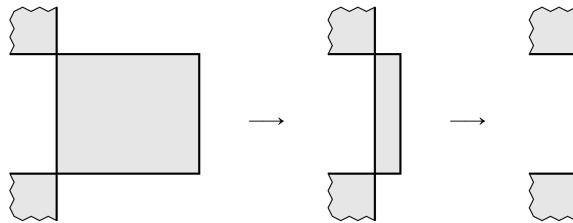


FIGURE 7.1. Removing a rectangle having two common vertices with other rectangles

Case 2: r shares a single common vertex with the other rectangles in Π . The decomposition is shown in Figure 7.2. First, we apply an exchange move, then a half-wrinkle creation move, and finally, a destabilization move.

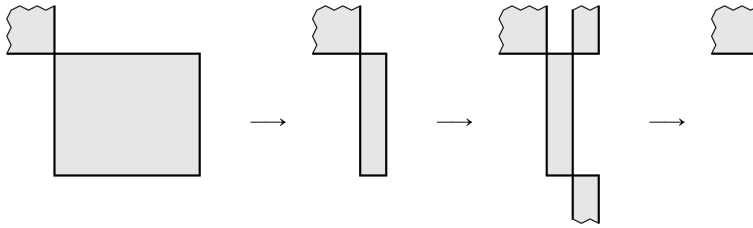


FIGURE 7.2. Removing a rectangle having a single common vertex with other rectangles

Case 3: r shares exactly three common vertices with the other rectangles in Π . We use the two previously resolved cases. First, we add, one by one, four new rectangles as shown in Figure 7.3, each sharing a single vertex with others at the moment of the addition. Then we apply a flype, and then remove five rectangles, each sharing either one or two vertices with others at the moment of the removal. \square

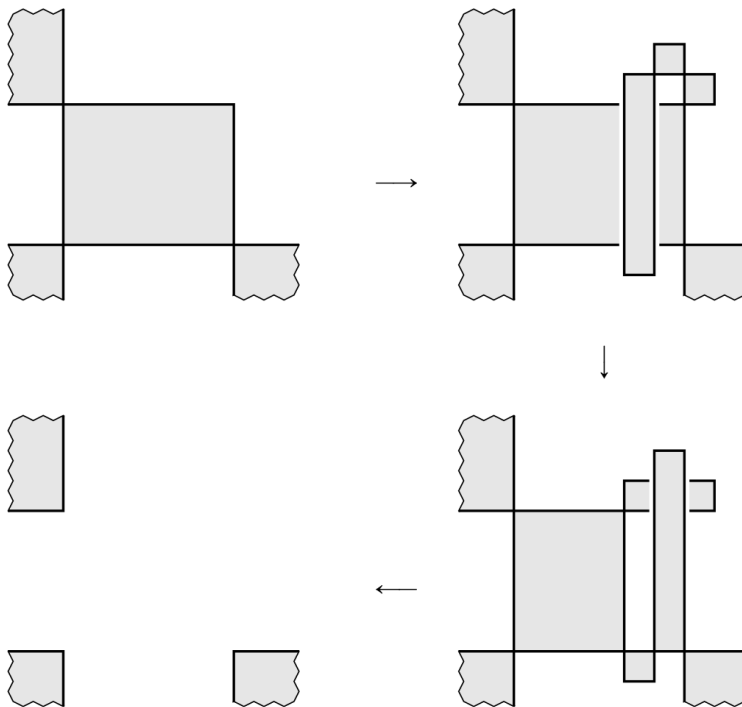


FIGURE 7.3. Removing a rectangle having three common vertices with other rectangles

Lemma 7.2. *Let Π and Π' be rectangular diagrams of surfaces such that $\Pi \subset \Pi'$ and the inclusion $\widehat{\Pi} \hookrightarrow \widehat{\Pi}'$ is a homotopy equivalence. Then there exists a sequence of basic moves*

$$\Pi = \Pi_0 \mapsto \Pi_1 \mapsto \Pi_2 \mapsto \dots \mapsto \Pi_N = \Pi' \quad (7.1)$$

such that:

- (1) each move $\Pi_i \mapsto \Pi_{i+1}$ is fixed on the common components of $\partial\Pi$ and $\partial\Pi'$;
- (2) the composition of moves (7.1) induces the morphism $\widehat{\Pi} \rightarrow \widehat{\Pi}'$ that is represented by any homeomorphism $\phi : (\mathbb{S}^3, \widehat{\Pi}) \rightarrow (\mathbb{S}^3, \widehat{\Pi}')$ isotopic to the identity in the class of maps $(\mathbb{S}^3, \widehat{\Pi}) \rightarrow (\mathbb{S}^3, \widehat{\Pi}')$.

Proof. The claim follows from Lemma 7.1 by induction in the number of rectangles in $\Pi' \setminus \Pi$. \square

Proof of Theorem 2.9. The implication (ii) \Rightarrow (i) is easy and left to the reader. We will prove the inverse one. So, suppose that (i) holds true.

It suffices to prove the theorem in the case when $\partial\widehat{\Pi} \cap \partial\widehat{\Pi}' = L$. Indeed, by a small perturbation of Π' we can always obtain a diagram Π'' such that $\partial\widehat{\Pi} \cap \partial\widehat{\Pi}'' = \partial\widehat{\Pi}' \cap \partial\widehat{\Pi}'' = L$. Then the passage $\Pi \mapsto \Pi'$ will decompose in the following two: $\Pi \mapsto \Pi'' \mapsto \Pi'$ with the former assigned a morphism $[\phi']$ with ϕ' close to ϕ , and the latter the morphism $[\phi \circ \phi'^{-1}]$. An application of the theorem to these transformations yields the general case.

So, from now on, we assume $\partial\widehat{\Pi} \cap \partial\widehat{\Pi}' = L$.

Let $\{\phi_t\}_{t \in [0;1]}$ be an isotopy from $\text{id}|_{\mathbb{S}^3}$ to ϕ fixed on L and preserving the tangent plane to $\widehat{\Pi}$ along L . For selected moments $t_0, t_1, \dots, t_N \in [0; 1]$ we denote $\phi_{t_i}(\widehat{\Pi})$ by F_i .

The isotopy $\{\phi_t\}_{t \in [0;1]}$ can be perturbed slightly, and moments

$$t_0 = 0 < t_1 < t_2 < \dots < t_{N-1} < t_N = 1$$

can be chosen so that, for all $i = 1, \dots, N$, one of the following holds:

- (1) we have $\partial F_{i-1} = \partial F_i$, and the tangent plane to F_{i-1} at any point $p \in \partial F_{i-1}$ is preserved during the isotopy $\{\phi_t\}_{t \in [t_{i-1}; t_i]}$,
- (2) we have $\phi_t(F_0) \subset F_i$ for all $t \in [t_{i-1}; t_i]$, and each connected component of ∂F_i is either contained in ∂F_{i-1} or disjoint from $\bigcup_{j=0}^{i-1} \partial F_j$;
- (3) we have $\phi_t(F_0) \subset F_{i-1}$ for all $t \in [t_{i-1}; t_i]$, and each connected component of ∂F_i is either contained in ∂F_{i-1} or disjoint from $\bigcup_{j=0}^{i-1} \partial F_j$.

Denote by L_* the link $\bigcup_{i=0}^N \partial F_i$. For each $i = 0, \dots, N$ denote also by L_i the union of connected components of L_* that are contained by whole in F_i .

By [3, Lemma 2] and a slight generalization of [3, Theorem 1] there exist a tubular neighborhood U of L , a homeomorphism $\psi : \mathbb{S}^3 \rightarrow \mathbb{S}^3$ identical on U , a rectangular diagram of a link R_* , and rectangular diagrams of surfaces $\Pi_0, \Pi_1, \dots, \Pi_{N-1}, \Pi_N$ such that the following holds:

- (1) $\psi(L_*) = \widehat{R}_*$;
- (2) for each $i = 0, 1, \dots, N$, there is an isotopy from $\psi(F_i)$ to $\widehat{\Pi}_i$ fixed on $\psi(L_i)$ and preserving the tangent plane to $\psi(F_i)$ at any $p \in \psi(L_i)$.

The generalization of [3, Theorem 1] needed here consists in requesting that a given link contained in the surface F also becomes ‘rectangular’ in a rectangular presentation of F . This does not require any essential change of the proof. Indeed, [3, Proposition 5], which is the key ingredient, treats even more general case which allows to make ‘rectangular’ any graph embedded in F .

Now each passage $\Pi_{i-1} \mapsto \Pi_i$, ($i = 1, \dots, N$), $\Pi \mapsto \Pi_0$, and $\Pi_N \mapsto \Pi'$ (assigned with the morphism obtained naturally from the construction) decompose into basic moves either by Proposition 6.1 or Lemma 7.2. This completes the proof of the theorem. \square

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Received: September 16, 2020, accepted: November 25, 2020.

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