

Floer-Novikov cohomology and symplectic fixed points, revisited

Hông Vân Lê, Kaoru Ono

Abstract. This note is mostly an exposition of a few versions of Floer-Novikov cohomology with a few new observations. For example, we state a lower bound for the number of symplectic fixed points of a non-degenerate symplectomorphism, which is symplectomorphic isotopic to the identity, on a compact symplectic manifold, more precisely than previous statements in [9, 12].

Анотація. В огляді наведено кілька версій теорії когомологій Флоєра-Новікова та доведено кілька нових фактів. Зокрема, отримано (більш точну ніж в [9, 12]) нижню межу для числа симплектичних нерухомих точок не виродженого симплектоморфізму, який є симплектоморфно ізоморфний до тотожного відображення на компактному симплектичному многовиді.

1. INTRODUCTION

Fixed points and periodic points are the simplest objects in dynamical systems. For time periodic flows, they are identified with periodic orbits. For a time dependent Hamiltonian system on a closed symplectic manifold (M, ω) , Arnold conjectured that the number of fixed points of the time-one map is at least the minimal number of critical points of smooth functions on the manifold M . In case all the fixed points are non-degenerate, he also conjectured that the number of fixed points is at least the minimal number of critical points of Morse functions on M . Motivated by these conjectures, Floer developed what is now called Floer theory [2]. In the non-degenerate

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case, it is now known that the number of fixed points is at least the sum of Betti numbers of M , see [3, 5, 7, 10, 11].

In [9] we considered a larger class of time dependent locally Hamiltonian systems on closed symplectic manifolds (M, ω) . Recall that a vector field V on (M, ω) is said to be *locally Hamiltonian*, if $i(V)\omega$ is a closed 1-form on M^{2n} . Here $i(X)\omega$ is the interior product by the vector field X to the symplectic form ω . Note that the interior product $i(\cdot)\omega : \mathfrak{X}(M) \rightarrow \Omega^1(M)$ induces an isomorphism $i(\bullet) : TM \rightarrow T^*M$. Let $\{X_t\}_{t \in \mathbb{R}/\mathbb{Z}}$ be a 1-periodic family of locally symplectic vector fields and integrating $\{X_t\}_{0 \leq t \leq 1}$, we obtain a symplectic isotopy $\{\varphi_t\}_{0 \leq t \leq 1}$ with $\varphi_0 = \text{id}$. Write $\eta_t = i(X_t)\omega$ and define the flux $\text{Flux}(\{\varphi_t\}_{0 \leq t \leq 1})$ of $\{\varphi_t\}_{0 \leq t \leq 1}$ by the de Rham cohomology class¹ of $\int_0^1 \eta_t dt$.

In [9], we extended the construction of Floer homology for non-degenerate Hamiltonian systems to Floer-Novikov homology for non-degenerate locally Hamiltonian systems. Since the former is related to Morse homology of the manifold, the latter must be related to Novikov homology for the flux instead of Morse homology. As a result, we obtained the following:

Theorem 1.1. (Main Theorem in [9]) *Let (M, ω) be a $2n$ -dimensional closed symplectic manifold which enjoys the following properties:*

$$c_1|_{\pi_2(M)} = \lambda\omega|_{\pi_2(M)}, \quad \lambda \neq 0,$$

and if $\lambda < 0$ then $N > n - 3$, where N is the minimal Chern number. Denote by φ_1 the time-one map of the symplectic flow $\{\varphi_t\}_{0 \leq t \leq 1}$. Suppose also that all fixed points of φ_1 are non-degenerate. Then the number of fixed points of φ_1 is bounded below by the sum of the $\mathbb{Z}/2\mathbb{Z}$ -Novikov-Betti numbers, i.e., the ranks of Novikov homology $\text{HN}^([\eta]; \mathbb{Z}/2\mathbb{Z})$ associated to the flux of $\{\varphi_t\}_{0 \leq t \leq 1}$.*

We assumed that $N > n - 3$ when $\lambda < 0$ due to the argument for transversality at that time. Under the same assumption, using the argument on orientation of the moduli space of connecting orbits, e.g. [3, 5], see also [4], we can use the dimension of Novikov homology with $F_p = \mathbb{Z}/p\mathbb{Z}$ -coefficients² for any prime number p . Using virtual technique in [5], we can show

Theorem 1.2. *Let (M, ω) be a closed symplectic manifold such that*

$$c_1|_{\pi_2(M)} = \lambda\omega|_{\pi_2(M)}, \quad \lambda \neq 0.$$

Suppose that all fixed points of the time-one map φ_1 of a symplectic isotopy are non-degenerate. Then the number of fixed points of φ_1 is bounded below

¹It was called the Calabi invariant in [9].

²For Floer-(Novikov) cohomology with \mathbb{Z} -coefficients, the torsion part in different degrees are mixed-up, if $N \neq 0$. See [6, 14].

by the sum of the Novikov-Betti number³ of $[\eta]$, the dimension of Novikov cohomology $\mathrm{HN}^*([\eta])$ as a vector space over the field $\Lambda_{[\eta]}^{\mathbb{Q}}$, which is the Novikov field associated with the flux $[\eta]$ of $\{\varphi_t\}$.

Here we set a restriction to the class of positively or negatively monotone symplectic manifolds because of the difficulty in computing Floer-Novikov cohomology, especially, the coefficient ring of Floer-Novikov cohomology depends on $\mathrm{Flux}(\{\varphi_t\}_{0 \leq t \leq 1}) \in \mathrm{H}^1(M^{2n}, \mathbb{R})$, see Remark 2.13 below for detailed discussion.

In [12], the second author constructed Floer-Novikov chain complexes defined over a variant of Novikov ring which is strictly contained in the one defined in [9] so that Floer-Novikov cohomology with fluxes close to each other can be compared. The following is a result in [12].

Theorem 1.3. ([12, Theorem 1.1]) *Let $\{\varphi_t\}_{0 \leq t \leq 1}$ be a smooth family of symplectic diffeomorphisms on a closed symplectic manifold (M, ω) with $\varphi_0 = \mathrm{id}$. Suppose that all fixed points of $\varphi = \varphi_1$ are non-degenerate. Then the cardinality of fixed points $\mathrm{Fix}(\varphi)$ of φ is bounded below by $\sum_p \mathrm{min}\text{-nov}^p(M)$.*

Here $\mathrm{min}\text{-nov}^p(M)$ is the minimum of the rank of Novikov cohomology over all de Rham cohomology classes of degree 1, see [12]. For an element $[\eta] \in \mathrm{H}^1(M, \mathbb{R})$ we denote by $\mathrm{HN}^*([\eta])$ the Novikov cohomology over \mathbb{Q} associated with $[\eta]$. For generic $[\eta] \in \mathrm{H}^1(M; \mathbb{R})$, $\mathrm{rank}_{\Lambda_{[\eta]}^{\mathbb{Q}}} \mathrm{HN}^p([\eta])$ is equal to the minimum $\mathrm{min}\text{-nov}^p(M)$ of $\mathrm{rank}_{\Lambda_{[\eta']}^{\mathbb{Q}}} \mathrm{HN}^p([\eta'])$ over $[\eta'] \in \mathrm{H}^1(M; \mathbb{R})$. We call $\mathrm{min}\text{-nov}^p(M)$ the p -th minimal Novikov number. In fact, we have a better statement.

Theorem 1.4. *Let $\{\varphi_t\}_{0 \leq t \leq 1}$ be a smooth family of symplectic diffeomorphisms on (M, ω) with $\varphi_0 = \mathrm{id}$. If all the fixed points of the time-one map $\varphi = \varphi_1$ are nondegenerate, then the number of fixed points of φ is bounded from below by the sum of the Novikov-Betti numbers of the flux $[\eta]$ of $\{\varphi_t\}_{0 \leq t \leq 1}$.*

Remark 1.5. Theorems 1.3 and 1.4 hold for any closed symplectic manifold. For weakly monotone closed symplectic manifolds, the conclusion holds using the dimension of Novikov cohomology with coefficient in any field. For general closed symplectic manifolds, the same follows using the argument in [6]. Since the details are not yet written, we state the results with \mathbb{Q} -coefficients for general symplectic manifolds.

³See section 1.5 in [1].

There are several variants of Floer-Novikov complexes [9, 12, 13]. We explain their difference although there are certainly similar properties enjoyed by them, see e.g. Proposition 3.10, and Remarks 3.11 and 3.12. We also explain the construction of Floer-Novikov complex for any component of the loop space of M . For the first step in the argument of the comparison of Floer-Novikov complexes with sufficiently close, but different, fluxes, we formulate Lemma 3.5, which is valid regardless of contractibility of periodic orbits. In [12], we used the maximal abelian covering of M in the construction of variants of Floer-Novikov complex to prove Theorem 1.3. Using a suitable intermediate abelian covering space of M , we improve it to Theorem 1.4.

The proof of Arnold's conjecture [5, 10] implies the existence of contractible 1-periodic orbits for any periodic Hamiltonian function. For a loop $\{\varphi_t^H\}$ of Hamiltonian diffeomorphisms based at the identity, the free homotopy class of a loop $t \in [0, 1] \mapsto \varphi_t^H(p)$ does not depend on $p \in M$, hence, contractible. In other words, the homomorphism

$$\pi_1(\text{Ham}(M, \omega), \text{id}) \rightarrow \pi_1(M, p_0)$$

by evaluating at $p_0 \in M$ is trivial. Although it is not the case for

$$\pi_1(\text{Symp}(M, \omega), \text{id}) \rightarrow \pi_1(M, p_0),$$

in general, we have

Proposition 1.6. *Let $\{\varphi_t\}_{0 \leq t \leq 1}$ be a symplectic isotopy with $\varphi_0 = \varphi_1 = \text{id}$. Denote by $[\eta] \in H^1(M; \mathbb{R})$ its flux. If $\text{rank HN}^*(M; [\eta]) \neq 0$, then the loop $t \mapsto \varphi_t(x)$, $x \in M$, is null homotopic.*

We observe a similar behavior for diffeomorphisms under the assumption that the Euler characteristic of M is not zero. It may be of independent interest.

Proposition 1.7. *Let $\{\varphi_t\}_{t=0}^1$ be an isotopy with $\varphi_0 = \text{id}$ on a closed manifold M . If the Euler characteristic $\chi(M)$ of M is non-zero, there is a fixed point p of φ_1 such that the loop $t \mapsto \varphi_t$ is null-homotopic. In particular, the homomorphism $ev_{x^*} : \pi_1(\text{Diff}(M), \text{id}) \rightarrow \pi_1(M, x)$ is trivial.*

As a corollary, we see that the flux group of a closed symplectically aspherical manifold is zero provided its Euler characteristic is non-zero (Corollary 4.2).

Our paper is organized as follows. In Section 2, we give a unified exposition of Floer-Novikov cochain complexes and their cohomology. We also collect some fundamental facts on Novikov rings in Section §2.2 for reader's convenience. For example, the Novikov ring $\Lambda_{\omega, \eta}(R)$ which is the ring of coefficients of Novikov-Floer chain complexes is an integral domain if R is an

integral domain R (Proposition 2.9). Then the rank of the Floer-Novikov cohomology groups is defined as the dimension after tensoring with the field of fractions of the Novikov ring⁴. In Section 3, we recall the construction of the Floer-Novikov cochain complex and their cohomology over smaller Novikov subrings and give a proof of Theorem 1.3. Namely, we compare the ranks of the cohomology of two “close” Floer-Novikov cochain complexes through “smaller Floer-Novikov cochain complexes over smaller subrings of $\Lambda_{\omega,\eta}$ ” as in [12], the argument of which uses results and ideas in our previous work [9]. In Section 4, we prove Proposition 1.6 and Proposition 1.7. In Appendix, a correction of [12, Lemma 5.1] is included.

In this note, M denotes a closed connected symplectic manifold. We shall use cohomological convention, i.e., Floer-Novikov cohomology as in [12] although in [9] we used homological convention.

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2. REVIEW ON FLOER-NOVIKOV COHOMOLOGY

We review the construction of Floer-Novikov cohomology for locally Hamiltonian systems, cf. [9, 12, 13]. Let $\{\eta_t\}$ be a one-parameter family of closed 1-forms on M . Denote by X_t the symplectic vector field defined by $i(X_t)\omega = \eta_t$ and by $\{\varphi_t\}$ the corresponding symplectic isotopy with $\varphi_0 = \text{id}$. Let φ be the time-one map φ_1 . Without changing the homotopy class of symplectic isotopies joining the identity and φ , we may assume that each η_t represents the same de Rham cohomology class, see [9, Lemma 2.1]. We may also assume that η_t is 1-periodic in t . Then we identify the set $\mathcal{P}(\omega, \{X_t\})$ of one-periodic solutions of the equation

$$\frac{d}{dt}x(t) = X_t(x(t)) \quad (2.1)$$

with the zero-set of the following closed 1-form $\alpha_{\{\varphi_t\}}$, in a formal sense, on the free loop space \mathcal{LM} over M :

$$\alpha_{\{\varphi_t\}}(x, \xi) = \int \omega(\xi, \dot{x}) + \eta_t(x(t))(\xi). \quad (2.2)$$

Denote by $[\eta]$ the flux of $\{\varphi_t\}$. What we call Floer-Novikov theory is a semi-infinite analogue of Morse-Novikov theory on the free loop space \mathcal{LM} .

⁴See page 560 in [12], where R is used in a different sense, and the case that R a (variant of) Novikov ring is discussed there.

Thus we take certain covering spaces of (a path connected component of) $\mathcal{L}M$ on which the lift of the closed 1-form $\alpha_{\{\eta_t\}}$ is exact. (There are some choices of covering spaces as we see in §2.1.)

§2.1. Floer-Novikov cochain complexes. In this section, we introduce a few versions of Floer-Novikov complex. For $a \in \pi_0(\mathcal{L}M)$ we denote by $\mathcal{L}^a M$ the corresponding connected component of $\mathcal{L}M$. Let $\gamma_0^a \in \mathcal{L}^a M$. Given de Rham cohomology classes $[\kappa]$ and $[\theta]$ of degree 2, and 1, respectively, we define the homomorphisms

$$\mathcal{I}_{[\kappa]}^{(2),a} : \pi_1(\mathcal{L}^a M, \gamma_0^a) \rightarrow \mathbb{R}, \quad \mathcal{I}_{[\theta]}^{(1),a} : \pi_1(\mathcal{L}^a M, \gamma_0^a) \rightarrow \mathbb{R}$$

as follows. Let $\{\gamma_\tau\}$ be a loop in $\mathcal{L}^a M$ with the base point γ_0^a and $C(\{\gamma_\tau\})$ the “torus” in M swept by $\{\gamma_\tau\}$. We set

$$\mathcal{I}_{[\kappa]}^{(2),a}(\{\gamma_\tau\}) = \int_{C(\{\gamma_\tau\})} \kappa, \quad \mathcal{I}_{[\theta]}^{(1),a}(\{\gamma_\tau\}) = \int_0^1 \theta\left(\frac{d\gamma_\tau}{d\tau}\right)|_{t=0} d\tau.$$

Note that

$$\mathcal{I}_{[\theta]}^{(1),a} = I_{[\theta]}^{(1)} \circ e_*,$$

where $I_{[\theta]}^{(1)}$ is the pairing between $[\theta] \in H^1(M; \mathbb{R})$ and loops in M and e is the evaluation at $t = 0$, i.e., $\gamma \in \mathcal{L}M \rightarrow \gamma(0) \in M$.

We consider covering spaces of $\mathcal{L}M$ such that the pull-back of $\alpha_{\{\varphi_t\}}$ becomes exact. Pick a closed 1-form η representing the flux of $\{\varphi_t\}$. The smallest one is the covering space of $\mathcal{L}^a M$, which is associated with

$$\ker(\mathcal{I}_{[\omega]}^{(2),a} + \mathcal{I}_{[\eta]}^{(1),a}) \subset \pi_1(\mathcal{L}^a M, \gamma_0^a).$$

Then its covering transformation group is isomorphic to

$$\pi_1(\mathcal{L}^a M, \gamma_0^a) / \{\ker(\mathcal{I}_{[\omega]}^{(2),a} + \mathcal{I}_{[\eta]}^{(1),a})\}.$$

Denote by N^a the minimal non-negative integer in $\text{Im } \mathcal{I}_{c_1}^{(2),a} \subset \mathbb{Z}$, (the number $N = N^0$ is often called the minimal Chern number.) To a zero of $\alpha_{\{\phi_t\}}$, i.e., a 1-periodic orbit γ , we assign

$$\text{Sign det}(\varphi_1|_p - \mathbf{1}).$$

In fact, this $\mathbb{Z}/2\mathbb{Z}$ -grading can be lifted to a $\mathbb{Z}/2N^a\mathbb{Z}$ -grading⁵. When $N^a \neq 0$, we can lift the $\mathbb{Z}/2\mathbb{Z}$ -grading to a \mathbb{Z} -grading after taking the covering space associated with

$$\ker(\mathcal{I}_{[\omega]}^{(2),a} + \mathcal{I}_{[\eta]}^{(1),a}) \cap \ker \mathcal{I}_{c_1}^{(2),a} \subset \pi_1(\mathcal{L}^a M, \gamma_0^a).$$

⁵Unless a is the contractible component, there is no canonical choice of $\mathbb{Z}/2N^a\mathbb{Z}$ -grading when $N^a \neq 1$.

We recall a description of this covering space. For $\gamma \in \mathcal{L}^a M$, there is a cylinder $v : [0, 1] \times \mathbb{R}/\mathbb{Z} \rightarrow M$ such that

$$v(0, t) = \gamma_0^a(t), \quad v(1, t) = \gamma(t).$$

Set $\ell_v = v(s, 0)$. For $(\gamma, v), (\gamma', v')$, we define the equivalence relation $(\gamma, v) \sim (\gamma', v')$ if and only if

$$\begin{aligned} \gamma &= \gamma', \\ \int_{[0,1] \times S^1} v^* \omega + \int_0^1 \ell_v^* \eta &= \int_{[0,1] \times S^1} (v')^* \omega + \int_0^1 \ell_{v'}^* \eta, \\ c_1(M)[v\#(-v')] &= 0. \end{aligned}$$

Here $v\#(-v')$ is the ‘‘torus’’ obtained by gluing v and v' with orientation reversed along the boundaries $\gamma^a \cup \gamma$. The space of equivalence classes of (γ, v) becomes a covering space $\tilde{\mathcal{L}}^a M$ of $\mathcal{L}^a M$ with the projection

$$[(\gamma, v)] \mapsto \gamma.$$

Then the covering space $\tilde{\mathcal{L}}^a M$ of $\mathcal{L}^a M$ is associated with

$$\ker(\mathcal{I}_{[\omega]}^{(2),a} + \mathcal{I}_{[\eta]}^{(1),a}) \cap \ker \mathcal{I}_{c_1}^{(2),a} \subset \pi_1(\mathcal{L}^a M, \gamma_0^a),$$

the covering transformation group of which is isomorphic to

$$\Gamma_{\omega, \eta}^a := \pi_1(\mathcal{L}^a M, \gamma) / \{\ker(\mathcal{I}_{[\omega]}^{(2),a} + \mathcal{I}_{[\eta]}^{(1),a}) \cap \ker \mathcal{I}_{c_1}^{(2),a}\}.$$

We choose a primitive function of the closed 1-form $\alpha_{\{\varphi_t\}}$ to be the action functional $\mathcal{A}_{\{\varphi_t\}}^a : \tilde{\mathcal{L}}^a M \rightarrow \mathbb{R}$ defined by

$$\mathcal{A}_{\{\varphi_t\}}^a([\gamma, v]) = \int_{[0,1] \times S^1} v^* \omega + \int_0^1 \ell_v^* \eta + \int_0^1 f_t(\gamma(t)) dt, \quad (2.3)$$

where $\{f_t\}$ is a 1-periodic family of smooth functions on M such that

$$\eta_t - \eta = df_t.$$

There is no canonical grading for $[\gamma, v]$, when $a \neq 0$. We pick and fix a 1-periodic orbit γ_* , a cylinder w joining γ and γ_* . Then the relative index $\mu(\gamma, \gamma_*; w)$ along w is the difference of the Conley-Zehnder indices of γ and γ_* with respect to a trivialization of w^*TM as a symplectic vector bundle. Set the grading of $[\gamma_*, v_0]$ to be $k \in \mathbb{Z}$ such that k is even (resp. odd) if

$$\det(d\varphi_1|_{\gamma_0(0)} - \mathbf{1})$$

is positive (resp. negative). Then the grading of $[\gamma, v_0\#w]$ is defined as $\mu(\gamma, \gamma_*; w) + k$. By the third condition in the definition of the equivalence relation \sim , we obtain a well-defined grading for each critical point $[\gamma, v]$ of $\mathcal{A}_{\{\varphi_t\}}^a$.

We define the Floer-Novikov cochain complex for $\{\varphi_t\}$ as follows. From now on, we assume that all 1-periodic orbits in $\mathcal{L}^a M$ are non-degenerate. Let R be a ground ring. We set

$$\text{CFN}^k(\{\varphi_t\}, R)^a = \left\{ \sum_i a_i [\gamma_i, v_i] \mid \text{where } a_i \in R \text{ and } [\gamma_i, v_i] \in \text{Crit} \mathcal{A}_{\{\varphi_t\}}^a \right. \\ \left. \text{satisfy conditions (2.4) and (2.5) below} \right\} :$$

$$\bullet \text{ the set } \{i \mid a_i \neq 0, \mathcal{A}_{\{\varphi_t\}}^a([\gamma_i, v_i]) < c\} \text{ is finite for any } c \in \mathbb{R}; \quad (2.4)$$

$$\bullet \text{ the degree of } [\gamma_i, v_i] \text{ equals } k. \quad (2.5)$$

Note that $\mathcal{I}_{[\omega]}^{(2),a} + \mathcal{I}_{[\eta]}^{(1),a}$ descends to $\Gamma_{\omega,\eta}^a \rightarrow \mathbb{R}$. We set

$$\Lambda_{\omega,\eta}^a(R) = \left\{ \sum a_i g_i \mid \text{where } a_i \in R \text{ and } g_i \in \Gamma_{\omega,\eta}^a \right. \\ \left. \text{satisfy the following condition (2.6) below} \right\} :$$

$$\bullet \text{ the set } \{i \mid a_i \neq 0, (\mathcal{I}_{[\omega]}^{(2),a} + \mathcal{I}_{[\eta]}^{(1),a})(g_i) < c\} \text{ is finite for any } c \in \mathbb{R}. \quad (2.6)$$

The grading is given by $g \mapsto \mathcal{I}_{c_1}^{(2),a}(g)$. The graded module $\text{CFN}^*(\{\varphi_t\}, R)^a$ is a finitely generated free module over $\Lambda_{\omega,\eta}^a(R)$, [13], see also [9, 12]. The coboundary operator δ is defined by counting isolated solutions⁶

$$u : \mathbb{R} \times S^1 \rightarrow M$$

of the following equation:

$$\frac{\partial}{\partial \tau} u(\tau, t) + J_t \left(\frac{\partial}{\partial t} u(\tau, t) - X_t(u(\tau, t)) \right) = 0, \quad (2.7)$$

$$\lim_{\tau \rightarrow \pm\infty} u(\tau, t) = \gamma^\pm(t) \quad (2.8)$$

for some 1-periodic solutions γ^\pm . Here J_t , $t \in \mathbb{R}/\mathbb{Z}$, is a 1-periodic family of almost complex structures compatible with ω . We call a solution u a connecting orbit joining $[\gamma^-, v^-]$ and $[\gamma^+, v^+]$, $[\gamma^+, v^+] \sim [\gamma^+, v^- \# u]$, where $v^- \# u$ is the concatenation of v^- and u along γ^- . For $[\gamma, v]$, we set

$$\delta[\gamma, v] := \sum \langle [\gamma, v], [\gamma', v'] \rangle [\gamma', v'], \quad (2.9)$$

where the sum runs over all $[\gamma', v']$ whose degree is bigger than that of $[\gamma, v]$ by 1 and $\langle [\gamma, v], [\gamma', v'] \rangle$ is the count of the connecting orbits. Then we can show that

$$\delta(\text{CFN}^*\{\varphi_t\}, R)^a \subset \text{CFN}^*(\{\varphi_t\}, R)^a.$$

⁶In general, we use the virtual count, as in [5], of the zero dimensional components of the moduli space of those solutions.

This is Floer-Novikov complex used in [13], which is suitable for the proof of the flux conjecture.

Let $\pi : \widetilde{M} \rightarrow M$ be a covering space of M on which $\pi^*\eta$ is an exact 1-form, i.e.,

$$[\eta] \in \ker(\pi^* : H^1(M; \mathbb{R}) \rightarrow H^1(\widetilde{M}; \mathbb{R})).$$

Denote by $\Gamma(\widetilde{M})$ its covering transformation group. Here we do not assume that \widetilde{M} is the minimal covering space enjoying this property⁷ and introduce Floer-Novikov complex for symplectic isotopies with flux $[\eta]$ associated with the covering \widetilde{M} .

Let $a \in \pi_1(\mathcal{L}M)$. Pick $p_0 \in \widetilde{M}$ such that $\pi(p_0) = \gamma_0^a(0)$. Consider triples (γ, v, p) , where $p \in \widetilde{M}$ such that $\pi(p) = \gamma(0)$ and the path ℓ_v lifts to a path from p_0 to p in \widetilde{M} . Note that, for a generic

$$[\eta'] \in \ker(\pi^* : H^1(M; \mathbb{R}) \rightarrow H^1(\widetilde{M}; \mathbb{R})),$$

we have

$$\ker I_{[\eta']}^{(1)} = \pi_*(\pi_1(\widetilde{M}, p_0)),$$

hence,

$$\ker \mathcal{I}_{[\eta']}^{(1),a} = e_*^{-1}(\pi_*(\pi_1(\widetilde{M}, p_0))).$$

We write $(\gamma, v, p) \approx (\gamma', v', p')$ if and only if $(\gamma, v) \sim (\gamma', v')$ and $p = p'$. Denote by $[[\gamma, v, p]]$ this equivalence class. Clearly, $[[\gamma, v, p]]$ determines $[\gamma, v]$. In particular, $\mathcal{A}_{\{\varphi_t\}}^a$ is well defined for $[[\gamma, v, p]]$.

The projection $[[\gamma, v, p]] \mapsto \gamma$ gives a covering space of $\mathcal{L}^a M$, which is denoted by $\widetilde{\mathcal{L}}^a \widetilde{M}$. Consider the covering space obtained by the pull-back of $\widetilde{M} \rightarrow M$ by the composition of the covering projection $\widetilde{\mathcal{L}}^a M \rightarrow \mathcal{L}^a M$ and $e : \mathcal{L}^a M \rightarrow M$. We take the connected component containing $[[\gamma_0^a, v_{\gamma_0^a}, p_0]]$, where $v_{\gamma_0^a}$ is the composition of the projection $[0, 1] \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ and $\gamma_0^a : \mathbb{R}/\mathbb{Z} \rightarrow M$. We find that this covering space is isomorphic to

$$\widetilde{\mathcal{L}}^a \widetilde{M} \rightarrow \mathcal{L}^a M.$$

In other words, it is the covering space of $\mathcal{L}^a M$ associated with

$$\ker \mathcal{I}_{[\omega]}^{(2),a} \cap \ker \mathcal{I}_{c_1}^{(2),a} \cap e_*^{-1}(\pi_*(\pi_1(\widetilde{M}, p_0))) \subset \pi_1(\mathcal{L}^a M).$$

We set

$$\text{CFN}^k(\{\varphi_t\} : \widetilde{M})^a = \left\{ \sum_i a_i [[\gamma_i, v_i, p_i]] \mid a_i \in \mathbb{R}, [[\gamma_i, v_i, p_i]] \in \text{Crit} \mathcal{A}_{\{\varphi_t\}}^a \right. \\ \left. \text{satisfying conditions (2.4) and (2.5)} \right\}.$$

⁷The construction in [9] is the case that $a = 0$ and $\widetilde{M} \rightarrow M$ is the minimal covering space on which $\pi^*\eta$ is exact.

Set

$$\bar{\Gamma}_{\omega, \tilde{M} \rightarrow M}^a = \pi_1(\mathcal{L}^a M) / \left(\ker \mathcal{I}_{[\omega]}^{(2),a} \cap \ker \mathcal{I}_{c_1}^{(2),a} \cap e_*^{-1}(\pi_*(\pi_1(\tilde{M}, p_0))) \right).$$

Then, for $[\eta] \in \ker(H^1(M; \mathbb{R}) \rightarrow H^1(\tilde{M}; \mathbb{R}))$, we define the corresponding Novikov ring by

$$\Lambda_{\omega, \eta, \tilde{M} \rightarrow M}^a(R) = \left\{ \sum a_i g_i \mid a_i \in R, g_i \in \bar{\Gamma}_{\omega, \tilde{M} \rightarrow M}^a \text{ satisfying (2.6)} \right\}. \quad (2.10)$$

Note that $\Lambda_{\omega, \eta, \tilde{M} \rightarrow M}^a$ depends on $[\eta]$ through condition (2.6). For the coboundary operator, we count connecting orbits u joining $[[\gamma^-, v^-, p^-]]$ and $[[\gamma^+, v^+, p^+]]$ in the sense that

$$[[\gamma^+, v^- \# u, p^+]] \approx [[\gamma^+, v^+, p^+]].$$

This Floer-Novikov complex for $\{\varphi_t\}$ with respect to $\tilde{M} \rightarrow M$ was introduced in [12], when $\mathcal{L}^a M$ is the space of contractible loops.

Remark 2.1. Denote by $\bar{M}^\eta \rightarrow M$ the covering space of M associated with $\ker I_{[\eta]}^{(1)} \subset \pi_1(M, x_0)$. Recall that $I_{[\eta]}^{(1)} : \pi_1(M, x_0) \rightarrow \mathbb{R}$ given by the pairing between $[\eta]$ and loops. Consider the case that $\tilde{M} = \bar{M}^\eta$, i.e., the minimal covering for η . Then the construction above is an extension of the one in [9] to the case of arbitrary connected component $\mathcal{L}^a M$ of the loop space $\mathcal{L}M$. Since $\tilde{\mathcal{L}}^0 \bar{M}^\eta \rightarrow \mathcal{L}^0 M$ is also regarded as the composition of

$$\tilde{\mathcal{L}}^0 \bar{M}^\eta \rightarrow \mathcal{L}^0 \bar{M}^\eta \quad \text{and} \quad \mathcal{L}^0 \bar{M}^\eta \rightarrow \mathcal{L}^0 M$$

and the latter is the pull back of $\bar{M}^\eta \rightarrow M$ by $e : \mathcal{L}^0 M \rightarrow M$. The group $\bar{\Gamma}_{\omega, \tilde{M} \rightarrow M}^0$ is isomorphic to $\Gamma_\omega \oplus \Gamma_\eta$. Here Γ_η is the covering transformation group of $\bar{M}^\eta \rightarrow M$.

When the covering space $\tilde{M} \rightarrow M$ is the minimal covering \bar{M}^η for $[\eta]$, i.e., the case that $\ker I_{[\eta]}^{(1)} = \pi_*(\pi_1(\bar{M}^\eta, p_0))$, we set

$$\widetilde{\text{CFN}}^* (\{\varphi_t\})^a = \text{CFN}^* (\{\varphi_t\} : \bar{M}^\eta)^a, \quad \tilde{\Lambda}_{\omega, \eta}^a = \Lambda_{\omega, \eta, \bar{M}^\eta \rightarrow M}^a.$$

Then we obtain the Floer-Novikov cochain complexes⁸

$$(\text{CFN}^* (\{\varphi_t\})^a, \delta), \quad (\widetilde{\text{CFN}}^* (\{\varphi_t\})^a, \delta), \quad (\text{CFN}^* (\{\varphi_t\}; \tilde{M})^a, \delta).$$

Their cohomology groups are denoted by $\text{HFN}^* (\{\varphi_t\})^a$, $\widetilde{\text{HFN}}^* (\{\varphi_t\})^a$ and $\text{HFN}^* (\{\varphi_t\}; \tilde{M})^a$, respectively.

⁸The construction works for ground ring R if (M, ω) is semi-positive (weakly monotone). Using the virtual technique in [5, 10], the construction works for any closed symplectic manifold (M, ω) with R containing \mathbb{Q} . A sketch for general construction with general R , in particular, \mathbb{Z} is given in [6].

Remark 2.2. In [9], we let $\gamma_0^{a=0}$ to be a constant path, $\gamma_0^{a=0} = c_{x_0}$ for a point $x_0 \in M$, and the path connected component $\mathcal{L}^0 M := \mathcal{L}^{[c_{x_0}]} M$ is the space of all contractible loops on M . In this case, $[\gamma, v]$ is an equivalence class of a pair of a loop γ equipped with a bounding disk v . The grading of $[\gamma, v]$ is defined as the Conley-Zehnder index of γ with respect to a trivialization of $v^* TM$. We take the covering space $\widetilde{\mathcal{L}}^0 \overline{M}^\eta$ of $\mathcal{L}^0 M$ defined in the following diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{L}}^0 \overline{M}^\eta & \xrightarrow{\tilde{j}} & \mathcal{L}^0 \overline{M}^\eta & \xrightarrow{\tilde{e}} & \overline{M}^\eta \\ \downarrow \tilde{\Pi} & & \downarrow \Pi & & \downarrow \pi \\ \widetilde{\mathcal{L}}^0 M & \xrightarrow{j} & \mathcal{L}^0 M & \xrightarrow{e} & M. \end{array} \quad (2.11)$$

Recall that

$$\mathcal{I}_{[\eta]}^{(1),0} = I_{[\eta]}^{(1)} \circ e_*.$$

Denote by $I_{c_1}^{(2)}, I_{[\omega]}^{(2)} : \pi_2(M) \rightarrow \mathbb{R}$ the homomorphisms given by the evaluation of $c_1(M)$, $[\omega]$, respectively. In the diagram (2.11), $\widetilde{\mathcal{L}}^0 M \rightarrow \mathcal{L}^0 M$ is the covering of $\mathcal{L}^0 M$ associated with

$$\ker I_{c_1}^{(2)} \cap \ker I_{[\omega]}^{(2)} \subset \pi_2(M).$$

This is the covering space used for Hamiltonian Floer theory. Its covering transformation group is

$$\Gamma_\omega = \pi_2(M) / \ker I_{c_1}^{(2)} \cap \ker I_{[\omega]}^{(2)}.$$

The covering space $\overline{M}^\eta \rightarrow M$ is associated with $\ker I_{[\eta]}^{(1)} \subset \pi_1(M, x_0)$. Its covering transformation group is $\Gamma_\eta = \pi_1(M, \mathbb{Z}) / \ker I_{[\eta]}^{(1)}$.

Note that, for $\mathcal{L}^0 M$, we have a section $M \rightarrow \mathcal{L}^0 M$ of e and hence

$$\pi_1(\mathcal{L}^0 M) \cong \pi_2(M) \rtimes \pi_1(M).$$

Note also that the $\pi_1(M)$ -action on $\pi_2(M)$ leaves $I_{[\omega]}^{(2)} : \pi_2(M) \rightarrow \mathbb{R}$ invariant. Thus $I_{[\omega]}^{(2)} : \pi_2(M) \rightarrow \mathbb{R}$ extends to $\mathcal{I}_{[\omega]}^{(2),0} : \pi_1(\mathcal{L}^0 M) \rightarrow \mathbb{R}$. Under this identification, we see that

$$\ker \mathcal{I}_{[\omega]}^{(2),0} \cap \ker \mathcal{I}_{[\eta]}^{(1),0} = (\ker I_{[\omega]}^{(2)} \rtimes \pi_1(M)) \cap (\pi_2(M) \rtimes \ker I_{[\eta]}^{(1)}).$$

Hence, the covering space $\widetilde{\mathcal{L}}^0 \overline{M}^\eta \rightarrow \mathcal{L}^0 M$ is the one associated with

$$\ker \mathcal{I}_{[\omega]}^{(2),0} \cap \ker \mathcal{I}_{[\eta]}^{(1),0} \cap \ker \mathcal{I}_{c_1}^{(2),0} \subset \pi_1(\mathcal{L}^0 M).$$

Since $\ker(\mathcal{I}_{[\omega]}^{(2),0} + \mathcal{I}_{[\eta]}^{(1),0}) \cap \ker(\mathcal{I}_{c_1}^{(2),0})$ contains

$$\ker(\mathcal{I}_{[\omega]}^{(2),0}) \cap \ker(\mathcal{I}_{[\eta]}^{(1),0}) \cap \ker(\mathcal{I}_{c_1}^{(2),0}),$$

the covering factors through $\tilde{\mathcal{L}}^0 M$, hence, the lift of the closed 1-form $\alpha_{\{\varphi_t\}}$ on $\tilde{\mathcal{L}}^0 \overline{M}^\eta$ is an exact 1-form. Clearly the covering transformation group of $\tilde{\mathcal{L}}^0 \overline{M}^\eta \rightarrow \mathcal{L}^0 M$ is $\Gamma_\omega \oplus \Gamma_\eta$.

Remark 2.3. Note that, for a generic $[\eta] \in \ker(H^1(M; \mathbb{R}) \rightarrow H^1(\tilde{M}; \mathbb{R}))$, $\text{Im } \mathcal{I}_{[\omega]}^{(2),a}$ and $\text{Im } \mathcal{I}_{[\eta]}^{(1),a} \subset \text{Im } I_{[\eta]}^{(1)}$ are independent over \mathbb{Z} . Thus we find that

$$\ker(\mathcal{I}_{[\omega]}^{(2),a} + \mathcal{I}_{[\eta]}^{(1),a}) = \ker \mathcal{I}_{[\omega]}^{(2),a} \cap \ker \mathcal{I}_{[\eta]}^{(1),a}.$$

If $[\eta]$ is generic in $\ker(H^1(M; \mathbb{R}) \rightarrow H^1(\tilde{M}; \mathbb{R}))$, we may also assume that $\tilde{M} \rightarrow M$ is the covering space associated with $\ker I_{[\eta]}^{(1)}$, i.e., $\tilde{M} = \overline{M}^\eta$. Hence, if the flux of $\{\varphi_t\}$ enjoys this property, the complexes

$$\text{CFN}^*(\{\varphi_t\}; \overline{M}^\eta)^a \quad \text{and} \quad \text{CFN}^*(\{\varphi_t\})^a$$

are the same.

§2.2. Novikov rings $\Lambda_{\omega,\eta}^a(R)$, $\tilde{\Lambda}_{\omega,\eta}^a(R)$. In this subsection we assume that R is a commutative unital ring. Given a group Γ and a homomorphism $\phi : \Gamma \rightarrow \mathbb{R}$, we denote by $R((\Gamma, \phi))$ the upward completion of the group ring $R[\Gamma]$ with respect to the weight homomorphism ϕ . More precisely, we define

$$R((\Gamma, \phi)) := \left\{ \sum \lambda_g \cdot g, g \in \Gamma, \lambda_g \in R \mid \text{for all } c \in \mathbb{R} \text{ there is only} \right. \\ \left. \text{finite number of } g \text{ such that } \lambda_g \neq 0 \text{ and } \phi(g) < c \right\}.$$

If Γ is a subgroup of \mathbb{R} and $\iota : \Gamma \rightarrow \mathbb{R}$ is the natural embedding, then we abbreviate $R((\Gamma, \iota))$ to $R((\Gamma))$. We recall the following Proposition 2.4 attributed to J.-C. Sikorav.

Proposition 2.4. *If ϕ is an injective homomorphism, R is a principal ideal domain, resp. a field, so is $R((\Gamma, \phi))$.*

Remark 2.5. This proposition is shown by looking at the leading terms, i.e., the lowest order terms with respect to the weight homomorphism ϕ . It also implies that if ϕ is injective and R is an integral domain, so is $R((\Gamma, \phi))$.

Example 2.6. (1) The Novikov rings $\Lambda_{\omega,\eta}^a(R)$ and $\Lambda_{\omega,\eta,\tilde{M} \rightarrow M}^a(R)$ defined in (2.6) and in (2.10) respectively can be written as $R((\Gamma_{\omega,\eta}^a, \mathcal{I}_{[\omega]}^a + \mathcal{I}_{[\eta]}^a))$ and $R((\tilde{\Gamma}_{\omega,\tilde{M} \rightarrow M}^a, \mathcal{I}_{[\omega]}^a + \mathcal{I}_{[\eta]}^a))$ respectively. Note that $\Gamma_{\omega,\eta}^a$ and $\tilde{\Gamma}_{\omega,\tilde{M} \rightarrow M}^a$ are

finitely generated free abelian groups. We may abbreviate $\Lambda_{\omega,\eta}^a(R), \tilde{\Lambda}_{\omega,\eta}^a(R), \Lambda_{\omega,\eta,\tilde{M}\rightarrow M}^a(R)$ by $\Lambda_{\omega,\eta}^a, \tilde{\Lambda}_{\omega,\eta}^a, \Lambda_{\omega,\eta,\tilde{M}\rightarrow M}^a$, respectively.

(2) Let η be a closed 1-form on M and $I_\eta : H_1(M, \mathbb{Z}) \rightarrow \mathbb{R}$ the period homomorphism. Assume that R be an integral domain. The Novikov ring $\Lambda_{[\eta]}(R) = R((\Gamma_\eta, I_\eta))$ is the Novikov ring associated to the cohomology class $[\eta] \in H^1(M, \mathbb{R})$. Novikov cochain complex $CN^*(\eta)$ is a complex over $\Lambda_{[\eta]}$ and Novikov cohomology $HN^*(\eta)$ is a module over $\Lambda_{[\eta]}$ ⁹.

In the remainder of this subsection, we assume further that R is an integral domain.

Lemma 2.7. *Let Γ be a finitely generated abelian group and $\phi : \Gamma \rightarrow \mathbb{R}$ a homomorphism. Write $\bar{\Gamma} = \phi(\Gamma)$. Then we have*

$$\Gamma \cong \ker \phi \oplus \bar{\Gamma}.$$

Proof. Since $\bar{\Gamma} = \phi(\Gamma)$ is a torsion free finitely generated abelian group, hence a free abelian group, we take a splitting of

$$0 \rightarrow \ker \phi \rightarrow \Gamma \rightarrow \bar{\Gamma} \rightarrow 0$$

to obtain $\Gamma = \ker \phi \oplus \bar{\Gamma}$. □

Denote by $\bar{\phi} : \bar{\Gamma} \rightarrow \mathbb{R}$ the homomorphism induced from ϕ . Note that $\bar{\phi}$ is injective.

Proposition 2.8. *Assume that Γ is a finitely generated abelian group and $\phi : \Gamma \rightarrow \mathbb{R}$ is a homomorphism. Then we have a ring isomorphism*

$$R((\Gamma, \phi)) = R[\ker \phi]((\bar{\Gamma}, \bar{\phi})).$$

Proof. By Lemma 2.7, we have $\Psi : \Gamma \cong \ker \phi \oplus \bar{\Gamma}$, which induces an isomorphism

$$R[\Gamma] \rightarrow R[\ker \phi] \otimes_R R[\bar{\Gamma}].$$

Taking the completion with respect to $\phi : \Gamma \rightarrow \mathbb{R}$, which factors through $\bar{\phi} : \bar{\Gamma} \rightarrow \mathbb{R}$, we obtain

$$R((\Gamma, \phi)) = R[\ker \phi]((\bar{\Gamma}, \bar{\phi})).$$
 □

Proposition 2.9. *Assume that Γ is a finitely generated free abelian group and R is an integral domain. Then the ring $R((\Gamma, \phi))$ is an integral domain.*

Proof. Let k be the rank of $\ker \phi$ as a finitely generated free abelian group. Then $R[\ker \phi]$ is isomorphic to the Laurent polynomial ring $R[t_1^\pm, \dots, t_k^\pm]$, in particular, an integral domain. Since $\bar{\phi}$ is injective, $R[\ker \phi]((\bar{\Gamma}, \bar{\phi}))$ is an integral domain (Proposition 2.4). The conclusion follows from Proposition 2.8. □

⁹For Novikov complex, see [1, 15].

Taking into account Example 2.6, we obtain immediately, from Proposition 2.9, the following

Corollary 2.10. *Let R be an integral domain. Then the Novikov ring $\Lambda_{\omega,\eta}^a(R)$ and its unital subrings are integral domains.*

§2.3. Computation of Floer-Novikov cohomology. As in Floer theory for Hamiltonian systems, Floer-Novikov cohomology is invariant under Hamiltonian deformations.

Theorem 2.11. (cf. [9, Theorem 4.3], [12, Theorem 3.1], [13, Theorem 3.4]) *Let $\{\varphi_t\}$ and $\{\varphi'_t\}$ be symplectic isotopies such that their flux are the same. Then we have*

$$\begin{aligned} \text{HFN}^*(\{\varphi_t\}; \mathbb{Q})^a &\cong \text{HFN}^*(\{\varphi'_t\}; \mathbb{Q})^a, \\ \text{HFN}^*(\{\varphi_t\} : \widetilde{M}; \mathbb{Q})^a &\cong \text{HFN}^*(\{\varphi'_t\} : \widetilde{M}; \mathbb{Q})^a, \\ \widetilde{\text{HFN}}^*(\{\varphi_t\}; \mathbb{Q})^a &\cong \widetilde{\text{HFN}}^*(\{\varphi'_t\}; \mathbb{Q})^a. \end{aligned}$$

When (M, ω) is semi-positive, we also have the isomorphisms with coefficients in any coefficient ring.

Here (M, ω) is said to be semi-positive, if any $A \in \pi_2(M)$ with

$$3 - n \leq \langle c_1(M), A \rangle < 0$$

satisfies $\langle [\omega], A \rangle \leq 0$. For the proof, we use a τ -dependent analogue of (2.7). Let $\{X_t\}$ and $\{X'_t\}$ be 1-periodic families of symplectic vector fields generating $\{\varphi_t\}$ and $\{\varphi'_t\}$, respectively. Take a two parameter family $\{X_{\tau,t}\}$ of symplectic vector fields of the form

$$X_{\tau,t} = (1 - \beta(\tau))X_t + \beta(\tau)X'_t.$$

Here β is a smooth function such that, for some $d > 0$, $\beta(\tau) = 0$ for $\tau < -d$, and $\beta(\tau) = 1$ for $\tau > d$. Roughly speaking, we count isolated solutions of the following equation to construct a chain homomorphism $\text{CFN}^*(\{\varphi_t\}) \rightarrow \text{CFN}^*(\{\varphi'_t\})$:

$$\frac{\partial}{\partial \tau} u(\tau, t) + J_{\tau,t} \left(\frac{\partial}{\partial t} u(\tau, t) - X_{\tau,t}(u(\tau, t)) \right) = 0 \quad (2.12)$$

with a similar asymptotic condition as (2.8). Note that the cohomology class $[i(X_{\tau,t})\omega]$ is independent of τ and t . This fact guarantees that the energy estimate for solutions of (2.12) holds as in Floer theory for Hamiltonian systems, see the proof of [9, Theorem 4.3].

When $c \in H^1(M; \mathbb{R})$ is sufficiently close to 0, we can pick a C^1 -small closed 1-form η representing the class c so that all 1-periodic orbits of the symplectic vector field X defined by $i(X)\omega = \eta$ are null-homotopic. Moreover, Floer-Novikov cochain complex can be described by Novikov

complex. For $(\text{CFN}(\{\phi_t\} : \widetilde{M}), \delta)$, we take $\widetilde{M} = \overline{M}^\eta$, which is used in the construction of Novikov cochain complex of η . As a consequence, we obtain the following:

Proposition 2.12. (cf. [9, Lemma 6.5], [12, Theorem 4.9], [13, Theorem 3.12]) *If the flux $[\eta]$ of the symplectic isotopy $\{\varphi_t\}$ is sufficiently small, we have*

$$\begin{aligned} \text{HFN}^*(\{\varphi_t\}; \mathbb{Q})^0 &\cong \text{HN}^*([\eta]; \mathbb{Q}) \otimes_{\Lambda_\eta(\mathbb{Q})} \Lambda_{\omega, \eta}(\mathbb{Q}), \\ \text{HFN}^*(\{\varphi_t\} : \overline{M}^\eta; \mathbb{Q})^0 &\cong \text{HN}^*([\eta]; \mathbb{Q}) \otimes_{\Lambda_\eta(\mathbb{Q})} \Lambda_{\omega, \eta, \overline{M}^\eta \rightarrow M}(\mathbb{Q}), \end{aligned}$$

i.e.,

$$\widetilde{\text{HFN}}^*(\{\varphi_t\}; \mathbb{Q})^0 \cong \text{HN}^*([\eta]; \mathbb{Q}) \otimes_{\Lambda_\eta(\mathbb{Q})} \widetilde{\Lambda}_{\omega, \eta}(\mathbb{Q}).$$

Here $\text{HN}^*(\eta; R)$ is Novikov cohomology for η with coefficients in R , and $\Lambda_\eta(R)$ is the Novikov ring over R associated to $[\eta]$. For component $a \neq 0$, Floer-Novikov cohomologies for $\{\varphi_t\}$ with a sufficiently small flux vanishes cf. Remark 3.11.

Remark 2.13. There are two issues in comparing Floer-Novikov homologies associated to symplectic isotopies whose flux are not the same. The first issue is that Novikov rings for them are not the same when $c \neq 0$. The second issue is that the derivation of an energy inequality to guarantee the weak-compactness of the moduli space used for the construction of chain homomorphisms, chain homotopies under Hamiltonian deformations does not work for deformations with varying flux. The strategy in [9] is as follows. To deal with the first issue, we restrict the class of \pm monotone symplectic manifolds. Then the degree 0-part of the Novikov ring $\widetilde{\Lambda}_{\omega, \eta}$ is identified with $\Lambda_{[\eta]}$, which depends on the positively proportional class of $[\eta]$. In general case, we compare Floer-Novikov cohomology with varying flux through the comparison of their *smaller* Floer-Novikov cohomology defined over *smaller* subring as in [12]. To deal with the second issue, we use a special deformation of $\{\varphi_t\}$ changing the flux based on Lemma 3.7 below, see Section 5 (see, the paragraph after Lemma 5.2) in [9]. This trick and related refining techniques, then work for general case. We shall discuss these technique in more details in the next section.

3. THE RANK OF THE FLOER-NOVIKOV COHOMOLOGY

In this section we first show that the ranks of Floer-Novikov cohomology associated to a symplectic isotopy $\{\varphi_t\}$ is well-defined. Then we shall prove the following

Theorem 3.1. *Let $\{\varphi_t\}_{0 \leq t \leq 1}$ be a smooth family of symplectic diffeomorphisms on (M, ω) such that $\varphi_0 = \text{id}$ and all the fixed points of φ_1 are non-degenerate. Then we have*

$$\text{rank}_{\tilde{\Lambda}_{[\omega], [\eta]}^{a=0}} \widetilde{\text{HFN}}^* (\{\varphi_t\}, \mathbb{Q})^0 = \text{rank}_{\Lambda_{[\eta]}} \text{HN}^* ([\eta], \mathbb{Q}).$$

As in Remark 1.5, the conclusion of Theorem 3.1 also holds for any field instead of \mathbb{Q} , if (M, ω) is weakly monotone.

Finally we derive Theorem 1.4 from Theorem 3.1 using a known relation between the rank of the Floer-Novikov cohomology and with the number of fixed points of φ .

§3.1. The ranks of (Floer-)Novikov cohomologies. Recall that the rank of a module L over an integral domain \mathfrak{R} is defined to be the dimension of the vector space $F(\mathfrak{R}) \otimes_{\mathfrak{R}} L$ over the field of fractions $F(\mathfrak{R})$ of \mathfrak{R} , cf. [12, page 560]. Now assume that \mathfrak{R} is an integral domain and C is a (co)chain complex with coefficients in \mathfrak{R} such that C is a finitely generated free module over \mathfrak{R} . In what follows, C is a cochain complex over \mathfrak{R} . Clearly the cohomology $H^*(C)$ is a module over \mathfrak{R} . Then we define the rank of $H^*(C)$ as follows

$$\text{rank} (H^*(C)) = \dim_{F(\mathfrak{R})} H^*(C) \otimes_{\mathfrak{R}} F(\mathfrak{R}). \quad (3.1)$$

Proposition 3.2. *Let $\mathbb{F} = F(\mathfrak{R})$ be the field of fractions of an integral domain \mathfrak{R} , Γ a finitely generated free abelian group and $\phi : \Gamma \rightarrow \mathbb{R}$ a homomorphism. Assume that C be a cochain complex over \mathfrak{R} . Then*

- (i) $F(\mathfrak{R}(\Gamma, \phi)) = F(\mathbb{F}(\Gamma, \phi))$,
- (ii) $\text{rank} H^*(C) \leq \text{rank} (C)$,
- (iii) $\text{rank} H^*(C) = \dim H^*(C \otimes_{\mathfrak{R}} \mathbb{F})$.

Proof. (i) Since $\mathfrak{R} \subset \mathbb{F}$ we have $F(\mathfrak{R}(\Gamma, \phi)) \subset F(\mathbb{F}(\Gamma, \phi))$. Next we note that $F(\mathfrak{R}(\Gamma, \phi)) \supset F(\mathbb{F}(\Gamma, \phi))$ since $F(\mathfrak{R}(\Gamma, \phi)) \supset \mathbb{F}$. This proves the first assertion of Proposition 3.2.

(ii) The second assertion of Proposition 3.2 follows from the universal coefficient theorem. Since the field of fractions $F(\mathfrak{R})$ is a flat \mathfrak{R} -module, we obtain

$$\text{rank} H^*(C) = \dim_{\mathbb{F}} H^*(C \otimes_{\mathfrak{R}} \mathbb{F}) \leq \text{rank} C. \quad (3.2)$$

(iii) The last assertion of Proposition 3.2 is (3.2). This completes the proof of Proposition 3.2. \square

§3.2. Floer-Novikov cohomology over subrings of $\Lambda_{\omega, \eta}(R)$. Let R be an integral domain. Firstly, we recall a variant of Floer-Novikov complex over a smaller Novikov ring introduced in [12].

Let U be a neighborhood of 0 in $V = \ker(\pi^* : H^1(M; \mathbb{R}) \rightarrow H^1(\widetilde{M}, \mathbb{R}))$. Then we define a variant of Novikov ring by

$$\Lambda_{\omega, \eta, U}^a(R) = \left\{ \sum a_i g_i \mid a_i \in R, g_i \in \overline{\Gamma}_{\omega, \widetilde{M} \rightarrow M}^a \right. \\ \left. \text{satisfying the following condition } (\#) \right\}$$

(#) $\{i \mid a_i \neq 0, (\mathcal{I}_{[\omega]}^{(2),a} + \mathcal{I}_{[\eta+\zeta]}^{(1),a})(g_i) < c\}$ is finite for any $c \in \mathbb{R}$ and for any $[\zeta] \in U$.

Namely, we have

$$\Lambda_{\omega, \eta, U}^a(R) = \bigcap_{[\zeta] \in U} \Lambda_{\omega, \eta+\zeta}^a(R). \tag{3.3}$$

Let η_t be a 1-periodic family of closed 1-forms in the same cohomology class. Denote by $\psi_t^{\{\eta_t\}}$ the symplectic flow on M that is generated by the time-dependent symplectic vector field X_t defined by $i(X_t) = \eta_t$ with $\psi_0^{\{\eta\}} = \text{id}$.

The definition of a smaller variant Floer-Novikov cochain complex in [12] extends to any connected component of \mathcal{LM} as follows.

$$\widehat{\text{CFN}}_U^k(\{\varphi_t\}, R)^a = \left\{ \sum_i a_i [[\gamma_i, v_i, p_i]] \mid a_i \in R, [[\gamma_i, v_i, p_i]] \in \text{Crit} \mathcal{A}_{\{\varphi_t\}}^a \right. \\ \left. \text{satisfying the following finiteness condition} \right. \\ \left. \text{for any } [\zeta] \in U \text{ and the degree of } [[\gamma_i, v_i, p_i]] \text{ equals } k \right\}.$$

The *finiteness condition* for $[\zeta] \in U$ is that the set

$$\left\{ i \mid a_i \neq 0, \mathcal{A}_{\{\psi_t^{\{\zeta\}} \circ \varphi_t\}}^a ([[\gamma_i, v_i, p_i]]) < c \right\}$$

is finite for any $c \in \mathbb{R}$.

In [12], we proved that, if $[\zeta]$ is sufficiently small, we can arrange a representative ζ such that $\mathcal{A}_{\{\psi_t^{\{\zeta\}} \circ \varphi_t\}}^a$ increases along solutions of (2.7). Namely,

Proposition 3.3. (cf. [12, page 558]) *Let δ be the boundary operator defined in (2.9). Then, for a sufficiently small U , the boundary homomorphism δ preserves the finiteness condition for each $[\zeta] \in U$. In particular,*

$$\delta(\widehat{\text{CFN}}_U^k(\{\varphi_t\}, R)^a) \subset \widehat{\text{CFN}}_U^{k+1}(\{\varphi_t\}, R)^a.$$

Remark 3.4. In [12] the second author stated the proposition in the case $a = 0 \in \pi_1(\mathcal{LM})$ and $R = \mathbb{Q}$, but the same argument works for any connected component of \mathcal{LM} . In [8] the first author considered another subring

of $\Lambda_{\omega,\eta}^0$ containing $\Lambda_{\omega,\eta,U}^0$ for U containing the segment $[\tau_1 - 1, \tau_2 - 1]$ with coefficients in $R = \mathbb{Q}$ as follows

$$\Lambda_{\omega,\eta(\tau_1,\tau_2)}^0(R) := \Lambda_{\omega,\tau_1\cdot\eta}^0(R) \cap \Lambda_{\omega,\tau_2\cdot\eta}^0(R). \quad (3.4)$$

Then she defined a subcomplex $\text{CFN}_*^{(\tau_1,\tau_2)}(\{\varphi_t\}, R)$, which is nothing but

$$\text{CFN}_{\omega,\eta,U}(\{\varphi_t\}; R) \otimes_{\Lambda_{\omega,\eta,U}^0(R)} \Lambda_{\omega,\eta(\tau_1,\tau_2)}^0(R).$$

She called this chain complex reduced Floer-Novikov chain complex.

In what follows we shall compare smaller variant of Floer-Novikov chain complexes associated to symplectic isotopies of different fluxes. Firstly, we prepare the following lemma. Pick and fix a norm on the finite dimensional vector space $H^1(M; \mathbb{R})$. Let $\{\gamma_1, \dots, \gamma_N\}$ be the set of all 1-periodic orbits of $\{X_t\}$ and $c \in H^1(M; \mathbb{R})$. For each t , pick contractible neighborhoods, say a smooth family of embeddings of geodesic balls $U_\rho(\gamma_j(t))$ around $\gamma_j(t)$ with a sufficiently small radius ρ for some Riemannian metric on M . Then we have the following¹⁰

Lemma 3.5. *For any $\epsilon > 0$, there exists $\delta > 0$ with the following property¹¹. Let $\|\cdot\|_{H^1}$ be a norm on the real vector space $H^1(M; \mathbb{R})$. If $\|c\|_{H^1} < \delta$, there exists a smooth 1-periodic family $\{\theta_t\}$ of closed 1-forms on M such that $[\theta_t] = c$, θ_t vanishes on $U_\rho(\gamma_j(t))$, $j = 1, \dots, N$, for each t and the C^1 -norm of θ_t is less than ϵ .*

Since all periodic orbits of $\{\varphi_t\}$ are non-degenerate, there is $\delta_0 > 0$ such that

$$\max_t d(\gamma(t), \gamma'(t)) > \delta_0 \text{ for distinct 1-periodic orbits } \gamma, \gamma'.$$

Note also that, for any $r > 0$, there exists $\delta_1 > 0$ such that

$$\|X_t(\sigma(t)) - \dot{\sigma}(t)\| > \delta_1,$$

if a loop σ satisfies $\max_t d(\sigma(t), \gamma(t)) > r$ for any 1-periodic orbit γ , see [9, Lemma 5.2].

Let X_{θ_t} be the vector field defined by $i(X_{\theta_t})\omega = \theta_t$ and $\psi^{\{\theta_t\}}$ the symplectic flow generated by $\{X_{\theta_t}\}$.

¹⁰When the dimension of M is bigger than 2, all 1-periodic solutions may be assumed to simple closed curves by perturbing symplectic vector fields $\{X_t\}$ keeping the time-one map and the cohomology classes $[i(X_t)\omega]$, hence for any closed 1-form is exact on tubular neighborhoods of null-homotopic 1-periodic orbits. When the dimension of M is 2, we cannot argue in a similar way. Lemma 3.5 here works in both cases and for all 1-periodic solutions, which are not necessarily null-homotopic.

¹¹Since $H^1(M; \mathbb{R})$ is finite dimensional, this property (after changing $\delta > 0$, if necessary) does not depend on the choice of the norm.

If $c \in H^1(M; \mathbb{R})$ is sufficiently close to 0, we can take θ_t in Lemma 3.5 so that $\|X_{\theta_t}\| < \delta_1/3$ and $\|X_t - \phi^{\{\theta_t\}}(X_t)\| < \delta_1/3$.

Then, in the same way as in Section 3.2 in [12], φ_t and $\psi^{\{\theta_t\}} \circ \varphi_t$ have the same 1-periodic orbits and all of them are non-degenerate.

Take $\{\theta_t\}$ representing c as in Lemma 3.5 and compare $\text{CFN}^*(\{\varphi_t\}; \widetilde{M})$ and $\text{CFN}^*(\{\psi^{\{\theta_t\}} \circ \varphi_t\}; \widetilde{M})$ as in [12, Proposition 4.7]. Here we do not assume c is proportional to $[\eta]$. The following lemma in [9, 12] was proved in the case of $a = 0$ (contractible loops), but the same argument works for any a .

Lemma 3.6. ([9, Lemma 5.4], [12, Lemma 3.2]) *If the closed 1-forms θ_t satisfy $\|\theta_t\| < \delta_1/3$, then we have the following energy estimate for solutions of (2.12):*

$$E(u) \leq 3 \left(\mathcal{A}_{\{\varphi_t\}}([\gamma^+, v^+, p^+]) - \mathcal{A}_{\{\varphi_t\}}([\gamma^-, v^-, p^-]) \right)$$

Roughly speaking, we proved that the functional $\mathcal{A}_{\{\varphi_t\}}$ increases along solutions of (2.12). The proof of [9, Lemma 5.4] yields the slightly stronger statement as follows.

Lemma 3.7. *Let $\{\zeta_t\}$ be a 1-periodic family of closed 1-forms such that ζ_t vanishes on $U_\rho(\gamma(t))$ for each 1-periodic solution γ of X_t and ζ_t represents a class $c \in \ker(\pi^* : H^1(M; \mathbb{R}) \rightarrow H^1(\widetilde{M}; \mathbb{R}))$. Suppose that $\|\zeta_t\| < \delta_1/6$. Suppose also that $\{\theta_t\}$ in Lemma 3.6 satisfies $\|\theta_t\| < \delta_1/6$. Then, for a solution u of (2.12), we have*

$$E(u) \leq 3 \left(\mathcal{A}_{\{\psi_t^{\{\zeta_t\}} \circ \varphi_t\}}([\gamma^+, v^+, p^+]) - \mathcal{A}_{\{\psi_t^{\{\zeta_t\}} \circ \varphi_t\}}([\gamma^-, v^-, p^-]) \right).$$

Remark 3.8. If $\theta_t = 0$, the estimate above holds for solutions of (2.7).

Note that, for any $\epsilon > 0$, if we can take U , a sufficiently small neighborhood of the origin in V , the following condition $(*_\epsilon)$ holds:

$(*_\epsilon)$ *For any $c \in U$, there exists $\{\theta_t\}$ as in Lemma 3.5 and $\|\theta_t\|_{C^1} \leq \epsilon$.*

Using Lemma 3.7 we obtained the following

Proposition 3.9. ([12, Theorem 4.6]) *Let ϵ be a sufficiently small positive real number. Pick a neighborhood U of 0 in V , which enjoys the property $(*_\epsilon)$. For $c \in U$, pick $\{\theta_t\}$ as in $(*_\epsilon)$. Then $\widehat{\text{HFN}}_U^*(\{\varphi_t\})$ is isomorphic to $\widehat{\text{HFN}}_{U'}^*(\{\psi^{\{\theta_t\}} \circ \varphi_t\})$ as $\Lambda_{\omega, \eta, U} = \Lambda_{\omega, \eta+c, U'}$ -modules, where*

$$U' = \{\alpha - c \mid \alpha \in U\}.$$

The ranks of $\text{HFN}^*(\{\varphi_t\}, \widetilde{M})^a$ and $\text{HFN}^*(\{\psi^{\{\theta_t\}} \circ \varphi_t\}; \widetilde{M})^a$ are compared using the isomorphism in Proposition 3.9, see [12, page 560]. Namely, we have

Proposition 3.10. [12, Proposition 4.7] *Under the situation of Theorem 3.9, we have*

$$\begin{aligned} \text{rank}_{\Lambda_{\omega, \eta; \widetilde{M} \rightarrow M}} \text{HFN}^*({\varphi}_t, \widetilde{M})^a &= \\ &= \text{rank}_{\Lambda_{\omega, \eta+c, \widetilde{M} \rightarrow M}} \text{HFN}^*({\psi}^{\{\theta_t\}} \circ {\varphi}_t; \widetilde{M})^a. \end{aligned}$$

Here the ground ring R is \mathbb{Q} , in general and an integral domain in the case of semi-positive symplectic manifolds.

Remark 3.11. For $a \neq 0$, i.e., a component of non-contractible loops, there are no non-contractible 1 periodic orbits for C^1 -small symplectic vector field X . Therefore $\text{HFN}^*({\varphi}_t, \widetilde{M})^a = 0$ if the flux $[\eta]$ of ${\varphi}_t$ is sufficiently small. Then Proposition 3.10 ensures that

$$\text{rank}_{\Lambda_{\omega, \eta, \widetilde{M} \rightarrow M}} \text{HFN}^*({\varphi}_t, \widetilde{M})^a = 0.$$

Taking \widetilde{M} as the covering associated with $\ker I_\eta^{(1)}$, we find that

$$\text{rank}_{\widetilde{\Lambda}_{\omega, \eta}} \widetilde{\text{HFN}}^*({\varphi}_t)^a = 0.$$

On the other hand,

$$\text{rank}_{\Lambda_{\omega, \eta}^a} \text{HFN}^*({\varphi}_t)^a$$

may not vanish¹² when $[\eta] \neq 0$ is not so small. Note also that, for a Hamiltonian isotopy ${\varphi}_t^H$,

$$\text{HFN}^*({\varphi}_t^H)^a = 0 \quad \text{for } a \neq 0.$$

Remark 3.12. When the flux of ${\varphi}_t$ varies in $H^1(M; \mathbb{R})$,

$$\text{rank}_{\widetilde{\Lambda}_{\omega, \eta}} \widetilde{\text{HFN}}^*({\varphi}_t)$$

may not be constant as in the jumping phenomenon of the Novikov-Betti numbers of Novikov cohomology [1, Theorem 1.41 (Sikorav)]. For an example, see Example 4.3.

§3.3. Proof of Theorems 3.1 and 1.4. From Proposition 3.10, we derived the following proposition.

Proposition 3.13. [12, Proposition 4.8] *Suppose that the flux $[\eta]$ of ${\varphi}_t$ belongs to V ,*

$$\text{rank}_{\Lambda_{\omega, \eta, \widetilde{M} \rightarrow M}} \text{HFN}^*({\varphi}_t; \widetilde{M})$$

is independent of the flux $[\eta] \in V$.

¹²For example, consider the composition of the full rotation on the two-torus along meridians and a Hamiltonian isotopies.

Remark 3.14. When \widetilde{M} is the covering space of M associated with

$$\ker \eta \subset \pi_1(M),$$

$\text{HFN}^*(\{\varphi_t\}; \widetilde{M})$ is equal to $\widetilde{\text{HFN}}^*(\{\varphi_t\})$.

Proof of Theorem 3.1. Theorem 3.1 follows immediately from Proposition 2.12, Remark 3.14 and Proposition 3.13. \square

Proof of Theorem 1.4. Theorem 1.4 follows from Theorem 3.1, Proposition 3.2, since the cochain module $\widetilde{\text{CFN}}(\{\varphi_t\}; \mathbb{Q})^0$ is freely generated over $\widetilde{\Lambda}_{\omega, \eta}(\mathbb{Q})$ by the set of contractible 1-periodic orbits of $\{\varphi_t\}$ which is regarded as a subset of $\text{Fix}(\varphi_1)$. \square

4. SYMPLECTIC FIXED POINTS AND CONTRACTIBILITY OF ORBITS OF LOOPS OF SYMPLECTOMORPHISMS

Let $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M)$ be a 1-periodic Hamiltonian function and $\{\varphi_H^t\}$ the Hamiltonian isotopy generated by H with $\varphi_H^0 = \text{id}$. The proof of Arnold's conjecture for fixed points of Hamiltonian diffeomorphisms shows the existence of null-homotopic 1-periodic orbits of $\{X_{H_t}\}$, i.e., a fixed point x of φ_H^1 such that $t \mapsto \varphi_H^t(x)$ is a null-homotopic loop. In particular, for any $p \in M$, the homomorphism

$$ev_{x*} : \pi_1(\text{Ham}(M, \omega), \text{id}) \rightarrow \pi_1(M, x)$$

obtained by evaluating Hamiltonian loops at x is trivial. On the other hand,

$$ev_{x*} : \pi_1(\text{Symp}(M, \omega), \text{id}) \rightarrow \pi_1(M, x)$$

is not necessarily trivial, e.g., the S^1 -action on a symplectic torus by rotations. Note that the existence of a locally free S^1 -action implies that the Euler characteristic $\chi(M)$ of M is zero.

If the Euler characteristic $\chi(M)$ of M is non-zero, and hence any diffeomorphism isotopic to the identity has a fixed point, we have

Proposition 4.1. (Proposition 1.7 in §1) *Let $\{\varphi_t\}_{t=0}^1$ be an isotopy with $\varphi_0 = \text{id}$ on a closed manifold M . If the Euler characteristic $\chi(M)$ of M is non-zero, there is a fixed point p of φ_1 such that the loop $t \mapsto \varphi_t$ is null-homotopic. In particular, the homomorphism*

$$ev_{x*} : \pi_1(\text{Diff}(M), \text{id}) \rightarrow \pi_1(M, x)$$

is trivial.

Proof. Suppose that all fixed points of φ_1 are non-degenerate. Consider the subset

$$\Gamma = \{(t, x) \in [0, 1] \times M \mid \varphi_t(x) = x\}$$

of $[0, 1] \times M$. By perturbing the isotopy φ_t such that

- $\{\varphi_t\}_{t=0}^\epsilon$ is generated by a vector field X for some small ϵ . Moreover, all zeros of X are non-degenerate.
- $\Gamma' = \Gamma \cap [\epsilon, 1] \times M$ is an oriented embedded 1-dimensional submanifold with boundary.

We may assume that ϵ is so small that

$$t \in [0, 1] \mapsto \varphi_{\epsilon t}(p) \in M$$

is constant, especially, null-homotopic for any $(\epsilon, p) \in \Gamma$.

Since M is a closed manifold, Γ' is a compact manifold with boundary $\Gamma' \cap \{\epsilon\} \times M$ and $\Gamma' \cap \{1\} \times M$. Thus Γ' is diffeomorphic to the union of circles and arcs with boundary on $\Gamma' \cap \{\epsilon, 1\} \times M$.

Taking the orientation into account and the assumption that $\chi(M) \neq 0$, there must be at least one arc joining $\Gamma' \cap \{\epsilon\} \times M$ and $\Gamma' \cap \{1\} \times M$. Namely, there is a continuous $(\tau, g) : [0, 1] \rightarrow \Gamma'$ such that $\tau(0) = \epsilon$ and $\tau(1) = 1$. In particular, $\varphi_{\tau(s)}(g(s)) = g(s)$.

For $s \in [0, 1]$, $t \in [0, 1] \mapsto \varphi_{\tau(s)t}(g(s))$ is a loop in M . Hence

$$t \mapsto \varphi_{\epsilon t}(g(\epsilon)) \quad \text{and} \quad t \mapsto \varphi_t(g(1))$$

are homotopic as loops. Since $t \mapsto \varphi_{\epsilon t}(g(\epsilon))$ is null-homotopic, we find that $t \mapsto \varphi_t(g(1))$ is null-homotopic.

When φ_1 may have degenerate fixed points, we take a sequence of isotopies $\{\varphi_t^{(i)}\}_{t=0}^1$ isotopic to the identity such that $\{\varphi_t^{(i)}\}$ converges to $\{\varphi_t\}$ and all fixed points of $\varphi_1^{(i)}$ are non-degenerate for each i . As we saw, there exist $p^{(i)} \in M$ such that $s \mapsto \varphi_s^{(i)}(p^{(i)})$ is null-homotopic. Take a convergent subsequence, we assume that $p^{(i)}$ converges to p in M . Then we find a fixed point p of φ_1 such that $s \mapsto \varphi_s(p)$ is null-homotopic. \square

Here is an easy corollary of Proposition 1.7. The flux group of (M, ω) is defined to be the image of $\pi_1(\text{Symp}(M, \omega))$ under the flux homomorphism, i.e., the subgroup of $H^1(M; \mathbb{R})$ consisting of the fluxes for all loops of symplectomorphisms with $\varphi_0 = \varphi_1 = \text{id}$.

Corollary 4.2. *Let (M, ω) be a closed symplectically aspherical manifold, i.e., $I_{[\omega]}^{(2)}(\pi_2(M)) = 0$. If the Euler characteristic of M is not zero, the flux group is $\{0\}$.*

Proof of Proposition 1.6. We observe that $\{\varphi_t\}$ has a contractible 1-periodic orbit, if the rank of Novikov cohomology of the flux of $\{\varphi_t\}$ is non-zero.

Let $[\eta]$, (resp. $[\eta']$) $\in H^1(M; \mathbb{R})$ be the flux of $\{\varphi_t\}$, (resp. $\{\varphi'_t\}$). Let $\overline{M}^\eta \rightarrow M$ be the covering space associated with $\ker I_{[\eta]}^{(1)}$. Proposition 3.13

implies that

$$\text{rank}_{\Lambda_{\omega, \eta, \overline{M}^\eta \rightarrow M}} \text{HFN}(\{\varphi_t\} : \overline{M}^\eta)^0 = \text{rank}_{\Lambda_{\omega, \eta', \overline{M}^\eta \rightarrow M}} \text{HFN}(\{\varphi'_t\} : \overline{M}^\eta)^0,$$

whenever $[\eta], [\eta'] \in V$. We take $\{\varphi'_t\}$ so that $[\eta'] = \epsilon[\eta]$ for a sufficiently small $\epsilon > 0$. Then, by Proposition 2.12, we see that if

$$\text{HN}^*([\eta]; \mathbb{Q}) \cong \text{HN}^*([\eta']; \mathbb{Q}) \neq 0,$$

then

$$\text{rank}_{\Lambda_{\omega, \eta', \overline{M}^\eta \rightarrow M}} \text{HFN}^*(\{\varphi'_t\}; \overline{M}^\eta)^0 = \text{rank}_{\widetilde{\Lambda}_{\omega, \eta''}} \widetilde{\text{HFN}}^*(\{\varphi'_t\})^0 \neq 0.$$

Then Proposition 3.13 yields that the rank of $\widetilde{\text{HFN}}^*(\{\varphi'_t\})^0$ is not zero, hence the existence of contractible 1-periodic orbits of $\{\varphi_t\}$. Then the rest goes as in the case of Hamiltonian diffeomorphisms. \square

Example 4.3. Let Σ_g , (resp. T) be an oriented surface of genus $g \geq 2$, (resp. 1) equipped with an area form. Denote by $p : \Sigma_g \times T \rightarrow \Sigma_g$ the projection to the first factor. If the flux $[\eta]$ of a symplectic isotopy $\{\varphi_t\}$ with $\varphi_0 = \text{id}$ belongs to

$$\text{Im } p^* \subset H^1(\Sigma_g \times T; \mathbb{R}),$$

i.e., $[\eta] = p^*[\theta]$ for $[\theta] \in H^1(\Sigma_g; \mathbb{R})$, then we have

$$\text{HN}^*([\eta]) \cong \text{HN}^*([\theta]) \otimes_{\mathbb{Q}} H^*(T).$$

Since the alternating sum of ranks of $\text{HN}^*([\theta])$ is the usual Euler characteristic of Σ_g , the rank of $\text{HN}^*([\eta])$ is non-zero. Thus, Proposition 1.6 implies that such a $\{\varphi_t\}$ has contractible 1-periodic orbits. For $[\eta] \notin \text{Im } p^*$, there is $\{\varphi_t\}$ with the flux $[\eta]$ without contractible 1-periodic orbits. For example, for $[\eta] \notin \text{Im } p^*$, there is a symplectic isotopy of the form

$$\{\varphi_t = \varphi_t^{\Sigma_g} \times \varphi_t^T\}$$

with the flux $[\eta]$. Here $\{\varphi_t^{\Sigma_g}\}$, (resp. $\{\varphi_t^T\}$), is a symplectic isotopy on Σ_g , (resp. T). We may take $\{\varphi_t^T\}$ as a family of rotations on T , which has no contractible orbits as long as its flux is non zero. Hence there are no contractible orbits of such $\{\varphi_t\}$. In particular, $\widetilde{\text{HFN}}^*(\{\varphi_t\}) = 0$. By the invariance under Hamiltonian isotopy, we find that

$$\widetilde{\text{HFN}}^*(\{\varphi_t\}) = 0$$

whenever $\{\varphi_t\} \notin \text{Im } p^*$.

When the flux of $\{\varphi_t\}$ is $[\eta] = p^*[\theta]$, we take a symplectic isotopy $\{\psi_t\}$ on Σ_g with the flux $[\theta]$. We find that

$$\widetilde{\text{HFN}}^*(\{\varphi_t\}) \cong \widetilde{\text{HFN}}^*(\{\psi_t\}) \otimes H^*(T).$$

Since the alternating sum of ranks of $\widetilde{\text{HFN}}^*(\{\psi_t\})$ is equal to the Euler characteristic of Σ_g , $\widetilde{\text{HFN}}^*(\{\psi_t\})$ is non-zero, when $g > 1$. Thus the rank of $\widetilde{\text{HFN}}^*(\{\psi_t\})$ jumps along the space of symplectic isotopies with flux in $\text{Im } p^*$.

Appendix by Kaoru Ono. Erratum on Lemma 5.1 in [12]. The second author take this opportunity to correct an inaccurate point in [12]. Lemma 5.1 in [12] states that the degree zero part $\Lambda^0_{\omega,\eta,U}$ of $\Lambda_{\omega,\eta,U}$ is a principal ideal domain¹³, but it is not correct, although it is an integral domain. ($\Lambda_{\omega,\eta,U}$ is a subring of $\tilde{\Lambda}_{\omega,\eta+\zeta}$ for $\zeta \in U$. When $\zeta \in U$ is generic, $\tilde{\Lambda}_{\omega,\eta+\zeta}$ coincides with $\Lambda_{\omega,\eta+\zeta}$, the degree zero part of which is a principal ideal domain whenever the ground ring is so.) Since this lemma was used in the proof of Theorem 5.2 therein, we give an argument independent of it. Theorem 5.2, hence also Theorem 5.3¹⁴, holds when the flux of $\{\varphi_t\}$ is generic in the sense of [12, Remark 4.2]. For general $\{\varphi_t\}$ such that all fixed points of φ_1 are non-degenerate, we replace the conclusion in Theorem 5.3 by that the number of fixed points of φ_1 is at least the sum of Novikov-Betti numbers. This statement reduces to the one in the case of generic flux as follows.

Let $\overline{M} = \overline{M}^\eta \rightarrow M$ be the covering space associated with

$$\ker I_{[\eta]}^{(1)} \subset \pi_1(M).$$

For a generic real number $c \neq 0$, we have $\text{im } I_{[\omega]}^{(2)} \cap \text{im } I_{c[\eta]}^{(1)} = \{0\}$. Then $c[\eta]$ is generic in the sense of [12, Remark 4.2], i.e.,

$$\ker \mathcal{I}_{[\omega]}^{(2)} + \mathcal{I}_{c[\eta]}^{(1)} = \ker \mathcal{I}_{[\omega]}^{(2)} \cap \ker \mathcal{I}_{c[\eta]}^{(1)}.$$

We may assume that $c = 1 + \epsilon$ is arbitrary close to 1. Let $\{\varphi_t\}$ be a symplectic isotopy with $\varphi_0 = \text{id}$ and the flux $[\eta]$. We may take a representative η as a closed 1-form. We take a symplectic isotopy $\{\psi_t^{\epsilon\eta}\}$ with flux $\epsilon\eta$ for a sufficiently small $\epsilon \neq 0$, e.g., the symplectic isotopy generated by the vector field X such that $i(X) = \epsilon\eta$. If ϵ is sufficiently small, the number of fixed points of $\psi_1^{\epsilon\eta} \circ \varphi_1$ and the one of φ_1 are the same, since all fixed points of φ_1 are non-degenerate. We apply Theorem 5.2 valid for generic flux to $\{\psi_t^{\epsilon\eta} \circ \varphi_t\}$ for a sufficiently small generic $\epsilon \neq 0$. Note that the degree zero part of $\tilde{\Lambda}_{[\omega],c[\eta]}(\mathbb{Q})$ is a field, when $c = 1 + \epsilon$ is generic in the sense. Then the rank of Floer-Novikov cohomology gives a lower bound for the number

¹³In [12], $\Lambda_{\omega,\eta,U}$ and $\Lambda^0_{\omega,\eta,U}$ are denoted by $\widehat{\Lambda}_{\omega,\eta,U}$ and $\widehat{\Lambda}^0_{\omega,\eta,U}$, respectively.

¹⁴As for the estimate of the number of contractible 1-periodic orbits, we can also argue as in the proof of Theorem 1.4.

of contractible 1-periodic orbits of $\{\psi_t^{\epsilon\eta} \circ \varphi_t\}$. Thus we find the desired lower bound for the number of fixed points.

Next, we show Theorem 5.2 for generic flux in the sense of [12, Remark 4.2]. It is actually a consequence of [12, Proposition 4.7, Proposition 4.8, Theorem 4.9], which are cited as Proposition 3.10, Proposition 3.13 and Theorem 2.12 in this note, respectively, and Sikorav's theorem for finite dimensional Novikov theory. (In Appendix, we refer these results in [12] as Proposition 3.10, Proposition 3.13 and Theorem 2.12.) The argument goes as follows. Let $\{\varphi_t\}$ be a symplectic isotopy with $\varphi_0 = \text{id}$ and the flux $[\eta]$ and $\overline{M} \rightarrow M$ the covering space associated with $\ker I_{[\eta]}^{(1)} \subset \pi_1(M)$ as above. We pick a representative η of the flux of $\{\varphi_t\}$ as a Morse 1-form. By Proposition 3.13, we find that

$$\text{rank}_{\Lambda_{\omega,\eta,\overline{M} \rightarrow M}} \text{HFN}^*(\{\varphi_t\}; \overline{M}) = \text{rank}_{\Lambda_{\omega,\eta,\overline{M} \rightarrow M}} \text{HFN}^*(\{\psi_t^{\epsilon\eta}\}; \overline{M}).$$

For a sufficiently small ϵ , Theorem 2.12 implies that

$$\text{rank}_{\Lambda_{\omega,\eta,\overline{M} \rightarrow M}} \text{HFN}^*(\{\psi_t^{\epsilon\eta}\}; \overline{M}) = \sum_p \text{rank}_{\Lambda_\eta} \text{HN}^p(M, [\eta]).$$

Combining them, we have

$$\text{rank}_{\Lambda_{\omega,\eta,\overline{M} \rightarrow M}} \text{HFN}^*(\{\varphi_t\}; \overline{M}) = \sum_p \text{rank}_{\Lambda_\eta} \text{HN}^p(M, [\eta]).$$

Then as noted in the proof of [12, Theorem 4.9] (see also Remark 2.3 in this note), $(\text{CFN}^*(\{\varphi_t\}; \overline{M}), \delta)$ (resp. $\Lambda_{\omega,\eta,\overline{M} \rightarrow M}$) coincides with $\text{CFN}^*(\{\varphi_t\}, \delta)$ (resp. $\Lambda_{\omega,\eta}$), when the flux $[\eta]$ is generic in the sense above. Hence, we have

$$\text{rank}_{\Lambda_{\omega,\eta}} \text{HFN}^*(\{\varphi_t\}) = \sum_p \text{rank}_{\Lambda_{[\eta]}} \text{HN}^p(M; [\eta]).$$

Let $pr : \widetilde{M} \rightarrow M$ be a covering space such that $pr^*\eta = 0$. Pick a closed 1-form η' on M such that $pr_*\pi_1(\widetilde{M}) = \ker \eta'$. Then Proposition 3.13 and Theorem 2.12 imply that

$$\text{rank HFN}^*(\{\varphi\}; \widetilde{M}) = \sum_p \text{rank HN}^p(M; [\eta']).$$

For closed Morse 1-forms η and η' such that $\ker I_{[\eta']}^{(1)} \subset \ker I_{[\eta]}^{(1)}$, [1, Theorem 1.41] (Sikorav) implies that

$$\text{rank}_{\Lambda_{\eta'}} \text{HN}^p(M : [\eta']) \leq \text{rank}_{\Lambda_\eta} \text{HN}^p(M : [\eta]).$$

Therefore we obtain

$$\text{rank}_{\Lambda_{\omega,\eta}} \text{HFN}^*(\{\varphi_t\}) \geq \text{rank}_{\Lambda_{\omega,\eta,\widetilde{M} \rightarrow M}} \text{HFN}^*(\{\varphi_t\}; \widetilde{M}).$$

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Hồng Vân Lê

INSTITUTE OF MATHEMATICS OF CZECH ACADEMY OF SCIENCE, ŽITNA 25, 11567 PRAHA 1, CZECH REPUBLIC.

Email: hvle@math.cas.cz

Kaoru Ono

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

Email: ono@kurims.kyoto-u.ac.jp