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On generalized Inoue manifolds

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Abstract. This paper is about a generalization of celebrated Inoue's surfaces. To each matrix M in $SL(2n+1,\mathbb{Z})$ we associate a complex non-Kähler manifold T_M of complex dimension $n+1$. This manifold fibers over S^1 with the fiber $\mathbb{T}^{2n+1}$ and monodromy $M^\top$. Our construction is elementary and does not use algebraic number theory. We show that some of the Oeljeklaus-Toma manifolds are biholomorphic to the manifolds of type T_M. We prove that if M is not diagonalizable, then T_M does not admit a Kähler structure and is not homeomorphic to any of Oeljeklaus-Toma manifolds.

Анотація. Стаття присвячена узагальненню поверхонь Інуе. Кожній матриці M в $SL(2n+1,\mathbb{Z})$ ми ставимо у відповідність некелеровий комплексний многовид T_M комплексної розмірності $n+1$. Це многовид розшаровується над колом S^1 з шаром $\mathbb{T}^{2n+1}$ і монодромією $M^\top$. Запропонована нами конструкція є елементарною і не використовує алгебраїчну теорію чисел. Ми показуємо, що деякі многовиди Олеклауса-Тома є біголоморфними до многовидів типу T_M. Ми також доводимо, що якщо M неможливо діагоналізувати, то T_M не допускає келерової структури і не є гомеоморфним жодному з многовидів Олеклауса-Тома.

1. Introduction

1.1. **Background.** In 1972 M. Inoue [5] constructed complex surfaces having remarkable properties: they have second Betti number equal to zero and contain no complex curves. Inoue surfaces attracted a lot of attention. It was proved by F. Bogomolov [2] (see also the works of J. Li, S.-T. Yau, and F. Zheng [9] and [10], and A. Teleman [15]) that each complex surface of class VII_0 with $b_2(X) = 0$ and containing no complex curves is isomorphic to an Inoue surface. Inoue surfaces are not algebraic, and moreover they do not admit Kähler metric (since their first Betti number is odd).

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Let us say that a matrix $M \in \mathrm{SL}(2n+1,\mathbb{Z})$ is of type $\mathcal{I}$, if it has only one real eigenvalue which is irrational and simple. Inoue's construction associates to every such matrix $M \in \mathrm{SL}(3,\mathbb{Z})$ a complex surface T_M obtained as a quotient of $\mathbb{H} \times \mathbb{C}$ by action of a discrete group (here $\mathbb{H}$ is the upper half-plane). This manifold fibers over S^1 with fiber $\mathbb{T}^3$ and the monodromy of this fibration equals the diffeomorphism of $\mathbb{T}^3$ determined by $M^\top$.

Inoue's construction was generalized to higher dimensions in several papers in particular in a celebrated paper of K. Oeljeklaus and M. Toma [12]. The construction of Oeljeklaus and Toma uses algebraic number theory. It starts with an algebraic number field K. Denote by s the number of embeddings of K to $\mathbb{R}$ and by $2t$ number of non-real embeddings of K to $\mathbb{C}$, so that

$$(K : \mathbb{Q}) = s + 2t.$$

K. Oeljeklaus and M. Toma constructed an action of a certain semi-direct product $\mathbb{Z}^s \ltimes \mathbb{Z}^{2t+s}$ on $\mathbb{H}^s \times \mathbb{C}^t$, such that the quotient is a compact complex manifold of complex dimension $s+t$. The original Inoue surface corresponds to the algebraic number field generated by the eigenvalues of the matrix M. The Oeljeklaus-Toma manifolds (*OT-manifolds* for short) have very interesting geometric properties, studied in [12]; in particular, they do not admit Kähler metric. These manifolds were recently studied by many authors. In the work of L. Ornea, M. Verbitsky, and V. Vuletescu [13] it is shown that in many cases the OT-manifolds do not contain proper analytic subvarieties. In the article [6] of N. Istrati and A. Otiman the De Rham cohomology of OT-manifolds is computed. The paper of D. Angella, M. Parton, and V. Vuletescu [1] is devoted to the proof of the rigidity of the complex structure of the OT-manifolds. The non-existence of complex curves in OT-manifolds is proved in the paper [17] of S. Verbitsky.

1.2. **Outline of the paper.** In the present paper we introduce another generalization of Inoue's construction. Our method does not use algebraic number theory, it generalizes the original Inoue's approach.

Let $M \in \mathrm{SL}(2n+1,\mathbb{Z})$ be a matrix of type $\mathcal{I}$. We construct an action of a certain semi-direct product $\mathbb{Z} \ltimes \mathbb{Z}^{2n+1}$ on $\mathbb{H} \times \mathbb{C}^n$, the quotient is a complex non-Kähler manifold T_M. It fibers over S^1 with fiber $\mathbb{T}^{2n+1}$ and the monodromy of this fibration equals the diffeomorphism of $\mathbb{T}^{2n+1}$ determined by $M^\top$. The construction of the manifolds is done in Section 2 and their properties are studied in Sections 3 and 4.

The basic difference of our construction from the preceding generalizations of Inoue's work is that the matrix M can be non-diagonalizable. In Section 4.3 we show that if M is non-diagonalizable, then the manifold T_M and its cartesian powers do not admit a structure of Kähler manifold. The proof is based on a theorem from [14] asserting that the monodromy of a fibration of a Kähler manifold over a circle is diagonalizable[1].

In Section 5 we show that some of the Oeljeklaus-Toma manifolds are biholomorphic to manifolds T_M for some special choices of the matrix M. Then we show that if M is non-diagonalizable then the manifold T_M is not homeomorphic to any of Oeljeklaus-Toma manifolds (see Subsection 5.4).

2. Manifold T_M: the construction

Let n be a positive integer $\geqslant 1$ and $M = (m_{ij})$ an element of the group $\mathrm{SL}(2n+1, \mathbb{Z})$. Suppose that M has exactly one real eigenvalue α, and assume moreover that $\alpha > 0$, $\alpha \neq 1$m and the multiplicity of α equals 1.

Remark 2.1. Observe that these conditions imply that α is irrational. Indeed, α is a root of the characteristic polynomial of M, which has integer coefficients and its principal coefficient equals 1. Therefore α is an algebraic integer, and if it were rational, it would be a natural number $\neq 1$, which is impossible since the free term of the characteristic polynomial of M equals -1.

Denote the remaining eigenvalues by $\beta_1, \ldots, \beta_k, \overline{\beta}_1, \ldots, \overline{\beta}_k$, $1 \leqslant k \leqslant n$. We can assume that $\mathfrak{Im}(\beta_j) > 0$ for every $j \in \{1, \ldots, k\}$.

The eigenspace V of M corresponding to α has dimension one. Denote the generalized eigenspace of M corresponding to an eigenvalue β by $W(\beta)$, namely

$$W(\beta) := \{x \in \mathbb{C}^{2n+1} \mid (M - \beta I)^N x = 0 \text{ for some positive integer } N\}.$$

We then obtain a direct sum decomposition of $\mathbb{C}^{2n+1}$ into complex M-invariant subspaces

$$\mathbb{C}^{2n+1} = V \oplus W \oplus \overline{W}, \qquad W := \bigoplus_{j=1}^{k} W(\beta_j), \qquad \overline{W} := \bigoplus_{j=1}^{k} W(\overline{\beta}_j).$$

Let a be a real eigenvector of M corresponding to α and $b_1, \ldots, b_n$ a basis of W. Then $\overline{b}_1, \ldots, \overline{b}_n$ is a basis of $\overline{W}$, and $a, b_1, \ldots, b_n, \overline{b}_1, \ldots, \overline{b}_n$ is a basis of $\mathbb{C}^{2n+1}$. Let $f_M : W \to W$ be the restriction of M to W and $R = (r_{ij})$ the matrix of f_M in the basis $b_1, \ldots, b_n$, namely

$$Mb_j = \sum_{\ell=1}^{n} r_{\ell j} b_\ell, \qquad (r_{\ell j} \in \mathbb{C}). \tag{2.1}$$

[1] A brief account of the proof of this theorem is included in Section 4.3.

Write

$$a = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(2n+1)} \end{pmatrix}, \quad b_1 = \begin{pmatrix} b_1^{(1)} \\ \vdots \\ b_1^{(2n+1)} \end{pmatrix}, \quad \ldots, \quad b_n = \begin{pmatrix} b_n^{(1)} \\ \vdots \\ b_n^{(2n+1)} \end{pmatrix},$$

where $a^{(1)}, \ldots, a^{(2n+1)}$ are real numbers. We also consider the vectors

$$v_i := (a^{(i)}, b_1^{(i)}, \ldots, b_n^{(i)}) \in \mathbb{R} \times \mathbb{C}_n = \mathbb{R}_{2n+1},$$
$$u_i := {v_i}^\top \in \mathbb{R} \times \mathbb{C}^n = \mathbb{R}^{2n+1}, \quad (i \in \{1, \ldots, 2n+1\}).$$

The following lemma is easy to prove.

Lemma 2.2. *The vectors $v_1, \ldots, v_{2n+1}$ are linearly independent over $\mathbb{R}$.* □

Consider the following matrices and vectors:

$$B := (b_1, \ldots, b_n) = \begin{pmatrix} b_1^{(1)} & \cdots & b_n^{(1)} \\ \vdots & & \vdots \\ b_1^{(2n+1)} & \cdots & b_n^{(2n+1)} \end{pmatrix},$$
$$b^{(i)} := \left(b_1^{(i)}, \ldots, b_n^{(i)}\right) \in \mathbb{C}_n \quad (i \in \{1, \ldots, 2n+1\}).$$

A direct computation proves the following lemma.

Lemma 2.3. *The equality $MB = BR$ holds. In particular*

$$b^{(i)} R = \sum_{j=1}^{2n+1} m_{ij} b^{(j)} \quad \textit{for every} \quad i \in \{1, \ldots, 2n+1\}.$$

Let $\mathbb{H}$ be the upper half of the complex plane, namely

$$\mathbb{H} = \{w \in \mathbb{C} \mid \mathfrak{Im}(w) > 0\}.$$

Consider complex-analytic automorphisms

$$g_0, g_1, \ldots, g_{2n+1} : \mathbb{H} \times \mathbb{C}^n \to \mathbb{H} \times \mathbb{C}^n$$

defined by

$$g_0(w, z) := (\alpha w,\ R^\top z), \qquad g_i(w, z) := (w, z) + u_i,$$

for every $(w, z) \in \mathbb{H} \times \mathbb{C}^n$ and $i \in \{1, \ldots, 2n+1\}$. Let G_M be the subgroup of $\mathrm{Aut}(\mathbb{H} \times \mathbb{C}^n)$ generated by $g_0, g_1, \ldots, g_{2n+1}$, H_M be the subgroup of $\mathrm{Aut}(\mathbb{H} \times \mathbb{C}^n)$ generated by $g_1, \ldots, g_{2n+1}$, and $\langle g_0 \rangle$ be the infinite cyclic group generated by g_0. Then Lemma 2.2 implies that H_M is a free abelian group of rank $2n + 1$.

Lemma 2.4. *For every $i \geqslant 1$ we have*

$$g_0 g_i g_0^{-1} = g_1^{m_{i1}} \cdots g_{2n+1}^{m_{i,2n+1}}.$$

In particular, H_M is a normal subgroup of G_M.

Proof. Let $(w, z) \in \mathbb{H} \times \mathbb{C}$. By Lemma 2.3 we have

$$\begin{aligned} g_0(g_i(w,z)) = g_0((w,z)+u_i) &= \Big(\alpha(w + a^{(i)}), R^\top(z + (b^{(i)})^\top)\Big) \\ &= \left(\alpha w + \sum_{j=1}^{2n+1} m_{ij} a^{(j)}, \ \ R^\top z + \sum_{j=1}^{2n+1} m_{ij} (b^{(j)})^\top \right); \end{aligned}$$

the last term is by definition $(g_1^{m_{i1}} \cdots g_{2n+1}^{m_{i,2n+1}})(g_0(w,z))$. □

Observe that the group G_M/H_M is generated by one element g_0. For $(w,z) \in \mathbb{H} \times \mathbb{C}$ denote $\mathfrak{Im}(w)$ by $p_1(w,z)$. Then $p_1(g_i(w,z)) = p_1(w,z)$ for $i > 0$ and $p_1(g_0(w,z)) = \alpha \cdot p_1(w,z)$. Therefore the element g_0^n is not in H_M for any $n \in \mathbb{Z}$.

Proposition 2.5. *The group G_M is isomorphic to a semi-direct product of $\mathbb{Z}$ and $\mathbb{Z}^{2n+1}$ associated to the action of $\mathbb{Z}$ on $\mathbb{Z}^{2n+1}$ given by the formula $t \cdot v = M^\top v$, where t is a generator of $\mathbb{Z}$ and $v \in \mathbb{Z}^{2n+1}$.*

Proof. It follows from the observation above that the group G_M/H_M is infinite cyclic. Therefore the exact sequence

$$1 \longrightarrow H_M \hookrightarrow G_M \longrightarrow G_M/H_M \longrightarrow 1$$

is isomorphic to

$$1 \longrightarrow \mathbb{Z}^{2n+1} \longrightarrow G_M \longrightarrow \mathbb{Z} \longrightarrow 1.$$

The action of the group $\mathbb{Z}$ on $\mathbb{Z}^{2n+1}$ is easily deduced from Lemma 2.4. □

Corollary 2.6. *The group G_M admits a finite presentation with generators $g_0, g_1, \ldots, g_{2n+1}$ and defining relations*

$$\begin{gathered} g_i g_j = g_j g_i, \quad (i, j \in \{1, \ldots, 2n+1\}), \\ g_0 g_i g_0^{-1} = g_1^{m_{i1}} \cdots g_{2n+1}^{m_{i,2n+1}}, \quad (i \in \{1, \ldots, 2n+1\}). \end{gathered}$$

Proof. It follows from Lemma 2.4 and Proposition 2.5 (see [7, Section 5.4]). □

Corollary 2.7. *The group H_M includes the commutator subgroup $[G_M, G_M]$ of G_M, and the quotient $H_M/[G_M, G_M]$ is finite.*

Proof. The first part of the Lemma follows from Corollary 2.6. We already observed that the group $H_M/[G_M, G_M]$ is isomorphic to the abelian group generated by $g_1, \dots, g_{2n+1}$ with relations

$$g_i = m_{i1}g_1 + \cdots + m_{i,2n+1}g_{2n+1}, \quad (i \in \{1, \dots, 2n+1\}).$$

Since M does not have eigenvalue 1, we see $\det(M - I) \neq 0$. Thus the group $H_M/[G_M, G_M]$ is finite. □

Proposition 2.8. *The action of G_M on $\mathbb{H} \times \mathbb{C}^n$ is free and properly discontinuous.*

Proof. We will prove that the action is free, the proof of the discontinuity is similar. Let $(w, z) \in \mathbb{H} \times \mathbb{C}^n$, and $g \in G_M$. Assume that $g(w, z) = (w, z)$ for some $(w, z) \in \mathbb{H} \times \mathbb{C}^n$. Write $g = g_0^{m_0} \cdot h$, where $h \in H_M$. Observe that $p_1(g(w, z)) = \alpha^{m_0} \cdot \mathfrak{Im}(w)$; therefore $m_0 = 0$, and $g \in H_M$. The action of H_M leaves invariant the $(2n+1)$-dimensional real affine subspace

$$V = \{(w', z') \mid \mathfrak{Im}(w') = \mathfrak{Im}(w)\}.$$

On this space H_M acts as a full lattice generated by vectors $v_1, \dots, v_{2n+1}$. This action is free, therefore $g = 1$. □

Consider the map $g_M : \mathbb{R} \times \mathbb{C}^n \to \mathbb{R} \times \mathbb{C}^n$ defined by

$$g_M(x, z) := (\alpha x,\ R^\top z), \quad (x \in \mathbb{R},\ z \in \mathbb{C}^n).$$

A direct computation using Lemma 2.3 proves the following Lemma.

Lemma 2.9. *The matrix of the linear transformation g_M with respect to the basis $(u_1, \dots, u_{2n+1})$ is equal to $M^\top$.* □

By Proposition 2.8, the quotient $T_M := (\mathbb{H} \times \mathbb{C}^n)/G_M$ is a complex manifold of complex dimension $n+1$. If $n = 1$, the manifold T_M is called *Inoue surface* (see [5]). Since the action of H_M on $\mathbb{H} \times \mathbb{C}^n$ is also free and properly discontinuous, $C_M := (\mathbb{H} \times \mathbb{C}^n)/H_M$ is also a complex manifold of dimension $n+1$. If we regard $\mathbb{H}$ as $\sqrt{-1}\mathbb{R}_+^* \times \mathbb{R}$, then the group H_M acts on $\sqrt{-1}\mathbb{R}_+^*$ trivially. The quotient $(\mathbb{R} \times \mathbb{C}^n)/H_M$ is a $(2n+1)$-dimensional torus $\mathbb{T}^{2n+1}$. The map g_M descends to a self-diffeomorphism of $\mathbb{T}^{2n+1}$. Thus we have $C_M = \sqrt{-1}\mathbb{R}_+^* \times \mathbb{T}^{2n+1}$. Observe that the matrix $M^\top$ determines a self-diffeomorphism of $\mathbb{T}^{2n+1}$, this diffeomorphism will be denoted by the same symbol $M^\top$.

Proposition 2.10. *The manifold T_M is diffeomorphic to the mapping torus of*

$$M^\top : \mathbb{T}^{2n+1} \to \mathbb{T}^{2n+1}.$$

In particular, T_M is compact.

Proof. From Proposition 2.5, we have the equality

$$T_M = (\mathbb{H} \times \mathbb{C}^n)/G_M = C_M/\langle g_0 \rangle = (\sqrt{-1}\mathbb{R}_+^* \times \mathbb{T}^{2n+1})/\langle g_0 \rangle.$$

The latter manifold is diffeomorphic to the manifold obtained from the product $[1, \alpha] \times \mathbb{T}^{2n+1}$ by gluing $\{1\} \times \mathbb{T}^{2n+1}$ with $\{\alpha\} \times \mathbb{T}^{2n+1}$ by g_M. The conclusion now follows from Lemma 2.9. □

3. Topological properties of T_M

We begin by computation of the first Betti number of T_M. Then we show that the homeomorphism type of T_M determines the matrix M up to conjugacy in $\mathrm{SL}(2n+1, \mathbb{Z})$ and inverting M (see Theorem 3.2). This result implies in particular (Subsection 5.4) that if M is not diagonalizable, then the manifold T_M is not homeomorphic to any of the manifolds constructed in [12].

The first Betti number.

Lemma 3.1. *The first Betti number $b_1(T_M)$ of T_M is equal to* 1.

Proof. The fundamental group $\pi_1(T_M)$ of T_M is isomorphic to G_M, which has the finite presentation given in Corollary 2.6. Hence the first homology group $H_1(T_M; \mathbb{Z})$ is isomorphic to the abelian group generated by $g_0, g_1, \ldots, g_{2n+1}$ with relations

$$g_i = m_{i1} g_1 + \cdots + m_{i,2n+1} g_{2n+1}, \quad (i \in \{1, \ldots, 2n+1\}).$$

Since 1 is not an eigenvalue of M, we have $\det(M - I) \neq 0$. Thus the first homology group $H_1(T_M; \mathbb{Q})$ with rational coefficient is isomorphic to $\mathbb{Q}$. □

On fundamental groups of mapping tori. Let k be a natural number. Then any matrix $A \in \mathrm{SL}(k, \mathbb{Z})$ yields a homeomorphism $\phi_A : \mathbb{T}^k \to \mathbb{T}^k$. Denote by $\mathscr{T}_A$ the mapping torus of this map. Then we get a fibration $p_A : \mathscr{T}_A \to S^1$ with fiber $\mathbb{T}^k$.

Theorem 3.2. *Let $A, B \in \mathrm{SL}(k, \mathbb{Z})$. Assume that* 1 *is not an eigenvalue of A neither of B. Assume also that $\pi_1(\mathscr{T}_A) \approx \pi_1(\mathscr{T}_B)$. Then A is conjugate to B or to B^{-1} in* $\mathrm{SL}(k, \mathbb{Z})$.

Proof. Consider the infinite cyclic covering $\overline{\mathscr{T}_A} \to \mathscr{T}_A$ induced from the universal covering $\mathbb{R} \to S^1$ by p_A. The space $\overline{\mathscr{T}_A}$ is homotopy equivalent to the fiber of p_A, that is, to $\mathbb{T}^k$. Therefore the Milnor exact sequence [11] of the covering $\overline{\mathscr{T}_A} \to \mathscr{T}_A$ is isomorphic to the following sequence

$$H_1(\mathbb{T}^k) \xrightarrow{A-1} H_1(\mathbb{T}^k) \longrightarrow H_1(\mathscr{T}_A) \longrightarrow H_0(\mathbb{T}^k) \xrightarrow{0} H_0(\mathbb{T}^k).$$

Since $A - 1$ is injective, the group $H_1(\mathscr{T}_A)$ is isomorphic to $\mathbb{Z} \oplus F$ where F is a finite abelian group. Therefore there are exactly two epimorphisms $\pi_1(\mathscr{T}_A)$ onto $\mathbb{Z}$, and they are obtained from one another via multiplication by (-1). Consider the exact sequence of the fibration p_A:

$$0 \longrightarrow \pi_1(\mathbb{T}^k) \longrightarrow \pi_1(\mathscr{T}_A) \xrightarrow{(p_A)_*} \pi_1(S^1) \longrightarrow 0.$$

It follows from this sequence that $\pi_1(\mathscr{T}_A)$ is isomorphic to the semi-direct product $\mathbb{Z} \ltimes \mathbb{Z}^k$ where the action of the generator t of $\pi_1(S^1)$ equals A. Denote by ι the canonical generator of $\pi_1(S^1)$ and choose an element $\theta_A \in \pi_1(\mathscr{T}_A)$ such that $(p_A)_*(\theta_A) = \iota$. Let $f : \pi_1(\mathscr{T}_A) \to \pi_1(\mathscr{T}_B)$ be an isomorphism. It follows from the remark above that the following diagram is commutative

$$\begin{array}{ccccc} \pi_1(\mathscr{T}_A) & \xrightarrow{(p_A)_*} & \pi_1(S^1) & \longrightarrow & 0 \\ \Big\downarrow f & & \Big\downarrow \varepsilon & & \\ \pi_1(\mathscr{T}_B) & \xrightarrow{(p_B)_*} & \pi_1(S^1) & \longrightarrow & 0 \end{array}$$

where ε equals 1 or -1. Therefore the element $f(\theta_A)$ equals $\theta_B \cdot g$ or $(\theta_B)^{-1} \cdot g$ with some $g \in \pi_1(\mathbb{T}^k)$. Thus the homomorphism A is conjugate to B or to B^{-1} in $\mathrm{SL}(k, \mathbb{Z})$. □

Corollary 3.3. *If $\pi_1(\mathscr{T}_A) \approx \pi_1(\mathscr{T}_B)$ and A is diagonalizable, then B is also diagonalizable.* □

Theorem 3.2 above can be reformulated in terms of semi-direct products of groups. Let $A \in \mathrm{SL}(k, \mathbb{Z})$. Consider the action $\circ$ of $\mathbb{Z}$ on $\mathbb{Z}^k$ defined by $m \circ x = A^m \cdot x$; denote by S_A the corresponding semi-direct product $\mathbb{Z} \ltimes \mathbb{Z}^k$.

Corollary 3.4. *Let $A, B \in \mathrm{SL}(k, \mathbb{Z})$, assume that 1 is not an eigenvalue of A neither of B. Assume that $S_A \approx S_B$. Then A is conjugate to B or to B^{-1} in* $\mathrm{SL}(k, \mathbb{Z})$.

Proof. Define an action $\circ$ of S_A on $\mathbb{R} \times \mathbb{R}^k$ as follows:

$$(m, h) \circ (t, v) = (t + m, A^m v + h)$$

It is easy to see[2] that the quotient space is the mapping torus of the map $\phi_A : \mathbb{T}^k \to \mathbb{T}^k$.

Thus $S_A \approx S_B$ implies $\pi_1(\mathscr{T}_A) \approx \pi_1(\mathscr{T}_B)$, and applying the preceding theorem we deduce the Corollary. □

[2]Although we do not use it in the proofs, let us observe that this space is $K(S_A, 1)$, that is, it has only one non-zero homotopy group, namely the fundamental group, which is isomorphic to S_A.

Definition 3.5. We say that the semi-direct product $S_A = \mathbb{Z} \ltimes \mathbb{Z}^k$ is *of diagonal type*, if A is diagonalizable over $\mathbb{C}$ and its eigenvalues are distinct from 1.

We say that the semi-direct product $S_A = \mathbb{Z} \ltimes \mathbb{Z}^k$ is *of non-diagonal type*, if A is non-diagonalizable over $\mathbb{C}$ and its eigenvalues are distinct from 1.

Corollary 3.6. *A semi-direct product of diagonal type is not isomorphic to a semi-direct product of non-diagonal type.* □

Proposition 3.7. *Let $A, B \in \mathrm{SL}(2n+1, \mathbb{Z})$. The manifolds $\mathscr{T}_A$ and $\mathscr{T}_B$ have then natural orientations. Assume that A is conjugate to B or to B^{-1} in $\mathrm{SL}(2n+1, \mathbb{Z})$. Then there is an orientation preserving diffeomorphism $\mathscr{T}_A \approx \mathscr{T}_B$.*

Proof. 1) If $A = C^{-1}BC$ with $C \in \mathrm{SL}(\mathbb{Z}, 2n+1)$ then the required diffeomorphism is given by the formula $(x, t) \mapsto (Cx, t)$.

2) If $A = C^{-1}B^{-1}C$ with $C \in \mathrm{SL}(\mathbb{Z}, 2n+1)$ then the required diffeomorphism is defined as the composition $\chi \circ \phi \circ \psi$ with

$$\begin{aligned}
&\psi : \mathscr{T}_A \to \mathscr{T}_A, && \psi(x,t) = (-x, t),\\
&\chi : \mathscr{T}_{B^{-1}} \to \mathscr{T}_B, && \chi(x,t) = (x, 1-t),\\
&\phi : \mathscr{T}_A \to \mathscr{T}_{B^{-1}}, && \phi(x,t) = (Cx, t).
\end{aligned}$$

Observe that ψ and χ reverse orientation, and ϕ is orientation preserving. The proposition is proved. □

4. Geometric properties of T_M

This section is about the properties of the manifolds T_M related to its complex structure. These properties are mostly similar to the properties of OT-manifolds. The first section is about the holomorphic bundles over T_M and their sections.

In the last two subsections we investigate the questions of existence of Kähler and locally conformally Kähler structures on manifolds T_M. Here we concentrate ourselves on the case when the matrix M is not diagonalizable.

Holomorphic bundles on T_M and their sections.

Proposition 4.1. *Any H_M-invariant holomorphic function on $\mathbb{H} \times \mathbb{C}^n$ is constant.*

Proof. Let $f : \mathbb{H} \times \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function. Suppose that $f(g(w, z)) = f(w, z)$ for every $g \in H_M$ and $(w, z) \in \mathbb{H} \times \mathbb{C}^n$. In particular, for $i \geqslant 1$ we have

$$f(w,z) = f(g_i(w,z)) = f((w,z)+u_i). \tag{4.1}$$

For $w_0 \in \mathbb{H}$ let

$$A_w = \{(w_0, z) \mid z \in \mathbb{C}^n\}, \quad B_w = \{(w,z) \mid z \in \mathbb{C}^n, \mathfrak{Im}(w) = \mathfrak{Im}(w_o)\}.$$

Then A_w is an n-dimensional complex space, and B_w is a real vector space of dimension $2n+1$. We have $A_w \subset B_w$. The abelian group generated by the vectors $u_1, \ldots, v_{2n+1}$ is a full lattice in B_w, therefore $f \mid B_w$ is bounded, and so is $f \mid A_w$. The function $f \mid A_w$ is holomorphic and bounded, therefore it is constant. Thus $f(w,0) = f(w,z)$ for every $(w,z) \in \mathbb{H} \times \mathbb{C}^n$. Consider a subset

$$A := \left\{ \sum_{i=1}^{2n+1} s_i a^{(i)} \;\middle|\; s_1, \ldots, s_{2n+1} \in \mathbb{Z} \right\} \subset \mathbb{R}.$$

Using (4.1) repeatedly, we deduce that $f(w,0) = f(w+\xi, 0)$ for every $\xi \in A$. Since a is an eigenvector of M corresponding to α, we have $\alpha a^{(i)} \in A$ for every i. The set $A_0 := \{(n_1 + n_2\alpha)a^{(i)} \mid n_1, n_2 \in \mathbb{Z}\}$ is included in A, and it is dense in $\mathbb{R}$ by Kronecker's density theorem. Therefore A is also dense in $\mathbb{R}$, and $f(w,0)$ does not depend on w. □

Our next proposition is similar to [12, Prop. 2.5].

Proposition 4.2. *The following statements hold.*

1) *There are no non-trivial holomorphic 1-forms on T_M.*
2) *Let $K = K_{T_M}$ be the canonical bundle on T_M, and $k \in \mathbb{N}$, $k > 0$. Then the bundle $K^{\otimes k}$ admits no non-trivial global sections. The Kodaira dimension of T_M is therefore equal to $-\infty$.*

Proof. 1) Let λ be a holomorphic 1-form on T_M and $u : \mathbb{H} \times \mathbb{C}^n \to T_M$ the universal covering of T_M. Then

$$u^*\lambda = f_0(w,z)dw + f_1(w,z)dz_1 + \ldots + f_1(w,z)dz_n,$$

where f_i are holomorphic functions on $\mathbb{H} \times \mathbb{C}^n$. They are invariant with respect to H_M, and therefore constant by Proposition 4.1. The form $u^*\lambda$ is also g_0-invariant. Since $g_0^*(dw) = \alpha dw$, we have $f_0 = 0$. Similarly, since 1 is not an eigenvalue of M, we deduce that $f_i(w,z) = 0$ for every $i \geqslant 1$.

2) Let ρ be a section of $K^{\otimes k}$. Then

$$u^*\rho = f(w,z)\Big(dw \wedge dz_1 \wedge \ldots \wedge dz_n\Big).$$

Similarly to the item 1) we deduce that $f(z,w)$ is a constant function. Since $u^*\rho$ is also g_0-invariant, we have $(\alpha \cdot \beta_1 \cdots \beta_n)^k = 1$. The condition

$\det M = 1$ implies then that $(\bar{\beta}_1 \cdots \bar{\beta}_n)^k = 1$, and finally $\alpha^k = 1$, which is impossible. □

4.3. **Kähler structures.** Proceeding to the non-existence of Kähler structures on T_M and its cartesian powers let us begin with a brief overview of the proof of a theorem from [14].

Theorem 4.4. *Let X be a Kähler manifold, and $p : X \to S^1$ a C^∞ fibration with fiber F. Then the homological monodromy $H_*(F) \to H_*(F)$ of the fibration is diagonalizable.*

Overview of the proof. Let us begin with a fibration $p : Y \to S^1$ where Y is any C^∞ compact manifold; denote by U its fiber. Let $\xi \in H^1(Y, \mathbb{C})$ be the p^*-image of the fundamental class of the circle. Denote by m_k the maximal length of a non-zero higher Massey product of the form $\langle \xi, \ldots, \xi, y \rangle$ where $y \in H^k(Y, \mathbb{C})$. It is proved in [14] that the maximal size of a Jordan block with eigenvalue 1 of the monodromy $H_k(U, \mathbb{C}) \to H_k(U, \mathbb{C})$ equals m_k.

Therefore if Y is a Kähler manifold, the maximal size of Jordan block with eigenvalue 1 equals 1, since all higher Massey products vanish in $H^*(Y, \mathbb{C})$ (see [4], and [8]).

A slightly more complicated argument, using cohomology with local coefficients and the corresponding Massey products, proves that the Jordan blocks with all eigenvalues are of size 1 when Y is a Kähler manifold. □

It is clear that the manifold T_M is not Kähler, since $b_1(T_M) = 1$. The next proposition asserts a much stronger property

Proposition 4.5. *Assume that M is non-diagonalizable. Let X be a C^∞ manifold diffeomorphic to $X = (T_M)^l$ where $l \in \mathbb{N}$. Then X does not admit the structure of a Kähler manifold.*

Proof. Consider the composition $\pi' : (T_M)^l \xrightarrow{p_1} T_M \xrightarrow{\pi} S^1$ where π is the fibration induced by the mapping torus structure on T_M. The map π' is a fibration with fiber $(T_M)^{l-1} \times \mathbb{T}^{2n+1}$. The monodromy homomorphism of this fibration equals $\mathrm{Id} \times M^\top$. This matrix is not diagonalizable,and the main theorem of [14] implies that $(T_M)^l$ does not admit a Kähler structure. □

Locally conformally Kähler structures. In 1982 F. Tricerri [16] proved that Inoue manifold admits an LCK-structure. The case of OT-manifolds is different, it is proved in [12] that the OT-manifolds $X(K, U)$ do not admit an LCK-structure for $s = 1$.

Proposition 4.6. *Assume that M is not diagonalizable. Then T_M does not admit an LCK-structure.*

Proof. The proof follows the lines of the corresponding theorem of Oeljeklaus-Toma [12, Prop. 2.9]. In our case the argument is somewhat simpler. Assume that there exists an LCK-structure on T_M. Let J be the corresponding complex structure, $\langle\cdot,\cdot\rangle$ the Hermitian metric on M, and Ω the 2-form associated with g and J, so that $\Omega(\xi,\eta) = \langle\xi, J\eta\rangle$. Let also be the corresponding 1-form on M, so that $d\Omega = \omega \wedge \Omega$. Consider the infinite cyclic covering $p : \overline{T_M} \to T_M$ corresponding to the mapping torus structure of T_M. The universal covering $\mathbb{H} \times \mathbb{C}^n \to T_M$ factors as follows:

$$\mathbb{H} \times \mathbb{C}^n \xrightarrow{\ q\ } \overline{T_M} \xrightarrow{\ p\ } T_M.$$

We have a diffeomorphism $\overline{T_M} \approx \mathbb{T}^{2n+1} \times \mathbb{R}$, and replacing the form $(q\circ p)^*\Omega$ by its average with respect to the action of $\mathbb{T}^{2n+1}$ we can assume that the form $(q\circ p)^*\Omega$ on $\mathbb{H}\times\mathbb{C}^n$ does not depend on the coordinates $z = (z_1, \ldots, z_n)$ on every subspace $\{h\} \times \mathbb{C}^n$. Let $(q \circ p)^*\omega = df$, with $f : \mathbb{H} \times \mathbb{C}^n \to \mathbb{R}$. Since $(q \circ p)^*\Omega$ is a symplectic form on $\{h\} \times \mathbb{C}^n$, this implies $df = 0$ on $\{h\} \times \mathbb{C}^n$. Put $\tau = e^{-f} \cdot (q \circ p)^*\Omega$. Then

$$\tau = \sum_{0 \leqslant i < j \leqslant n} g_{ij}(z) dz_i \wedge d\bar{z}_j,$$

(where we have denoted the first coordinate of $\{h\} \times \mathbb{C}^n$ by z_0). Here $g_{ij}(z_0, z_1, \ldots, z_n)$ does not depend on $(z_1, \ldots, z_n)$. Moreover, $d\tau = 0$, and this implies easily that $g_{ij}(z)$ does not depend on z_0 either. We can assume that $f(\sqrt{-1}, 0, \ldots, 0) = 0$. Let $\xi = f(\sqrt{-1}, 0, \ldots, 0) = 0 \in \mathbb{R}$, and $\mu = e^{-\xi}$. Denote by τ_0 the restriction of τ to $i \times \mathbb{C}^n$. Then we have $(M^\top)^*\tau_0 = \mu \cdot \tau_0$, which implies that the linear map $M^\top/\mu$ preserves the non-degenerate 2-form $\tau_0 \in \Lambda^2(\mathbb{C}^n)$. The symmetric form $\sigma(x, y) = \tau_0(x, iy)$ on $\mathbb{C}^n$ is a scalar product (since Ω is the imaginary part of a Hermitian form), therefore $M^\top/\mu$ preserves a scalar product, which is impossible since $M^\top$ is non-diagonalizable. □

5. Relations with the Oeljeklaus-Toma construction

In this section we study the relation between the manifold T_M constructed in Section 2 and the manifolds constructed by K. Oeljeklaus and M. Toma in [12] (*OT-manifolds* for short). In Subsection 5.2 we show that some of OT-manifolds appear as T_M-manifolds. In Subsection 5.4 we show that the manifold T_M with M non-diagonalizable is not homeomorphic to any of OT-manifolds.

5.1. **Construction of OT-manifolds.** Let us first recall the construction from [12] (in a slightly modified terminology). Let K be an algebraic number field. An embedding $K \hookrightarrow \mathbb{C}$ is called *real* if its image is in $\mathbb{R}$. An embedding which is not real is called *complex.* Denote by s the number of real embeddings and by t the number of complex embeddings. Then $(K : \mathbb{Q}) = s+2t$. Let $\sigma_1, \dots, \sigma_s$ be the real embeddings and $\sigma_{s+1}, \dots, \sigma_{s+2t}$ be the complex embeddings. One can assume that $\sigma_i = \overline{\sigma_{t+i}}$ for $i \geqslant s+1$. Then the map

$$\sigma : K \to \mathbb{R}^s \times \mathbb{C}^t, \qquad \sigma(x) = \big(\sigma_1(x), \dots, \sigma_{s+t}(x)\big),$$

is an embedding (known as *geometric representation* of the field K, see [3, Ch. II, § 3]). Let $\mathscr{O}$ be any order in K, then $\sigma(\mathscr{O})$ is a full lattice in $\mathbb{R}^s \times \mathbb{C}^t$. Denote by $\mathscr{O}^*$ the group of all units of $\mathscr{O}$. The Dirichlet Unit Theorem (see [3, Ch. II, § 4, Th. 5]) says that the group $\mathscr{O}^*/Tors$ is a free abelian group of rank $s+t-1$. Assume that $t \geqslant 1$. Choose any elements $u_1, \dots, u_s$ of $\mathscr{O}^*$ generating in $\mathscr{O}^*/Tors$ a free abelian subgroup of rank s. A unit $\lambda \in \mathscr{O}$ will be called *positive* if $\sigma_i(\lambda) > 0$ for every $i \leqslant s$. Replacing u_i by u_i^2 if necessary we can assume that every u_i is positive. The subgroup U of $\mathscr{O}^*$ generated by $u_1, \dots, u_s$ acts on $\mathscr{O}$ and we can form the semi-direct product $\mathscr{P} = U \ltimes \mathscr{O}$. The group $\mathscr{P}$ acts on $\mathbb{C}^r = \mathbb{C}^s \times \mathbb{C}^t$ as follows:

- any element $\xi \in \mathscr{O}$ acts by translation by vector $\sigma(\xi) \in \mathbb{R}^s \times \mathbb{C}^r$.
- any element $\lambda \in U$ acts as follows:

$$\lambda \cdot (z_1, \dots, z_{s+t}) = (\sigma_1(\lambda) z_1, \dots, \sigma_{s+t}(\lambda) z_{s+t}).$$

For $i \leqslant s$ the numbers $\sigma_i(\lambda)$ are real and positive, so the subset $\mathbb{H}^s \times \mathbb{C}^t$ is invariant under the action of $\mathscr{P}$. This action is properly discontinuous and the quotient is a complex analytic manifold of dimension $s+t$ which will be denoted by $X(K, \mathscr{O}, U)$. The notation $X(K, U)$ used in the article [12] pertains to the case when the order $\mathscr{O}$ is the maximal order of K.

5.2. **OT-manifolds as manifolds of type T_M.** Consider the case $s = 1$. In this subsection we will denote the number of complex embeddings of K by n, in order to fit to the terminology of the previous sections. Then $(K : \mathbb{Q}) = 2n+1$. We assume that $n \geqslant 1$. Assume that there is a Dirichlet unit ξ in K such that $\mathbb{Q}(\xi) = K$. This assumption holds for example when there are no proper subfields $\mathbb{Q} \subsetneq K' \subsetneq K$; this is always the case if $2n+1$ is a prime number. Replacing ξ by ξ^2 if necessary we can assume that ξ is positive. Denote by $\mathscr{O}$ the order $\mathbb{Z}[\xi]$, and let U be the group of units, generated by ξ. Let also P be the minimal polynomial of ξ, C_P the companion matrix of P, and $D_P = C_P^\top$.

Proposition 5.3. *We have a biholomorphism*

$$T_{D_P} \approx X(K, \mathscr{O}, U).$$

Proof. Let us give explicit descriptions of both these manifolds.

1) The manifold $X(K, \mathscr{O}, U)$.

The lattice $\sigma(\mathscr{O})$ is a free $\mathbb{Z}$-module generated by $e_i = \sigma(\xi^i)$. Denote by $\alpha, \beta_1, \ldots, \beta_n, \bar{\beta}_1, \ldots, \bar{\beta}_n$, the roots of P (here $\alpha \in \mathbb{R}, \beta_i \notin \mathbb{R}$). Then $\sigma(\xi^k) = (\alpha^k, \beta_1^k, \ldots, \beta_n^k)$, and the action of ξ on $\mathbb{H} \times \mathbb{C}$ is given by the following formula:

$$\xi \cdot (w, z_1, \ldots, z_n) = (\alpha w, \beta_1 z_1, \ldots, \beta_n z_n).$$

2) The manifold T_{D_P}.

The eigenvalues of the matrix D_P are the same as of the matrix C_P, that is, $\alpha, \beta_1, \ldots, \beta_n, \bar{\beta}_1, \ldots, \bar{\beta}_n,$. The corresponding eigenvectors of D_P are:

$$a = (1, \alpha, \ldots, \alpha^{2n}), \qquad b_i = (1, \beta_i, \ldots, \beta_i^{2n}), \qquad 1 \leqslant i \leqslant n.$$

The vectors u_i generating the group H_{D_P} of translations (see Section 2, page 26) are given by the formula

$$u_1 = (1, \ldots, 1), \quad u_2 = (\alpha, \beta_1, \ldots, \beta_n), \quad \ldots, \quad u_{2n+1} = (\alpha^{2n}, \beta_1^{2n}, \ldots, \beta_n^{2n}).$$

The element $g_0 \in G_{D_P}$ acts as follows

$$g_0 \cdot (w, z_1, \ldots, z_n) = (\alpha w, \beta_1 z_1, \ldots, \beta_n z_n).$$

The proposition follows. □

5.4. **The case of non-diagonalizable matrix M.**

Lemma 5.5. *Let K be an algebraic number field with $s = 1$. Denote $(K : \mathbb{Q}) = 2n + 1$. Let $\mathscr{O}$ be an order in K, and ξ a positive unit of $\mathscr{O}$. Let also $X = X(K, \mathscr{O}, \xi)$ be the corresponding OT-manifold. Then the group $\pi_1(X)$ is a semi-direct product $\mathbb{Z} \ltimes \mathbb{Z}^{2n+1}$ of diagonal type.*

Proof. The $\pi_1(X)$ is a semi-direct product S_A where A is the matrix of the action of the unit ξ on $\mathscr{O}$. Let P be the minimal polynomial of ξ. The roots of P are simple and different from 1. Since $P(A) = 0$, the minimal polynomial of A has the same properties. Therefore A is diagonal and Lemma is proved. □

Proposition 5.6. *Let $M \in \mathrm{SL}(2n + 1, \mathbb{Z})$ be a matrix of type $\mathcal{I}$ and non-diagonalizable over $\mathbb{C}$. Then the group $\pi_1(T_M)$ is not isomorphic to the fundamental group of any of manifolds $X(K, U)$ constructed in [12]. Therefore T_M is not homeomorphic to any of manifolds $X(K, U)$.*

Proof. Assume that we have an isomorphism $\pi_1(X(K,U)) \approx \pi_1(T_M)$. Since $b_1(\pi_1(X(K,U))) = s$, and $b_1(\pi_1(T_M)) = 1$, we have $s = 1$. By Lemma 5.5 $\pi_1(X(K,U))$ is isomorphic to a semi-direct product $\mathbb{Z} \ltimes \mathbb{Z}^{2n+1}$ of diagonal type. Recall that $\pi_1(T_M)$ is a a semi-direct product of a *non-diagonal type*. Apply Corollary 3.6 and the proof is over. □

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