

Incidence coefficients in the Novikov Complex for Morse forms: rationality and exponential growth properties

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Dedicated to the memory of V. V. Sharko

Abstract. Let $f : M \rightarrow S^1$ be a Morse map, v a transverse f -gradient. The construction of the Novikov complex associates to these data a free chain complex $C_*(f, v)$ over the ring $\mathbb{Z}[[t]][t^{-1}]$, generated by the critical points of f . This complex computes the completed homology module of the corresponding infinite cyclic covering of M . *Novikov's Exponential Growth Conjecture* says that the boundary operators in this complex are power series of non-zero convergence radius (see [10]).

In [14] the author announced the proof of the Novikov conjecture for the case of C^0 -generic gradients together with several generalizations. The proofs of the first part of this work were published in [15], see also [16].

The present article contains the proofs of the second part.

There is a refined version of the Novikov complex, defined over a suitable completion of the group ring of the fundamental group. We prove that for a C^0 -generic f -gradient the corresponding incidence coefficients belong to the image in the Novikov ring of a (non commutative) localization of the fundamental group ring.

The Novikov construction generalizes also to the case of Morse 1-forms. In this case the corresponding incidence coefficients belong to a certain completion of the ring of integral Laurent polynomials of several variables. We prove that for a given Morse form ω and a C^0 -generic ω -gradient these incidence coefficients are rational functions.

The incidence coefficients in the Novikov complex are obtained by counting the algebraic number of the trajectories of the gradient, joining the zeros of the Morse form. There is V. I. Arnold's version of the exponential growth conjecture, which concerns the total number of trajectories. We confirm this

Keywords: Novikov complex, non-commutative localization, exponential growth conjecture

Ключові слова: Комплекс Новікова, некомутативна локалізація, гіпотеза експоненційного росту

DOI: <http://dx.doi.org/10.15673/tmgc.v13i4.1747>

stronger form of the conjecture for any given Morse form and a C^0 -dense set of its gradients.

We give an example of explicit computation of the Novikov complex.

Анотація. Нехай $f : M \rightarrow S^1$ – відображення Морса і v – f -градієнтне векторне поле. Комплекс Новікова $C_*(f, v)$ – це вільний ланцюговий комплекс над кільцем $\mathbb{Z}[[t]][t^{-1}]$, породжений критичними токами f . Він дозволяє обчислити поповнений модуль гомологій відповідного нескінченного циклічного накриття M . *Гіпотеза Новікова про експоненційний зріст* стверджує що граничні оператори цього комплексу є степеневими рядами з ненульовими радіусами збіжності (див. [10]).

В роботі [14] анонсовано доведення гіпотези Новікова для випадку C^0 -типових градієнтних векторних полів та деякі узагальнення цієї гіпотези. Доведення першої частини вказаних результатів опубліковано в [15], див. також [16]. Представлена робота містить доведення другої частини [14].

Існує спрощена версія комплексу Новікова, визначена над певним поповненням групового кільця фундаментальної групи. Ми показуємо, що для C^0 -типових f -градієнтних векторних полів відповідні коефіцієнти інцидентності належать образу кільця Новікова (некомутативної) локалізації фундаментального групового кільця.

Конструкція Новікова також узагальнюється на випадок морсівських 1-форм. В цій ситуації відповідні коефіцієнти інцидентності належать поповненню кільця поліномів Лорана від кількох змінних з цілими коефіцієнтами. Ми доводимо, що для заданої морсівської форми ω і C^0 -типового ω -градієнтного векторного поля ці коефіцієнти інцидентності є раціональними функціями.

Коефіцієнти інцидентності в комплексі Новікова можна отримати обчислюючи алгебраїчне число траєкторій градієнтного векторного поля, що з'єднують нулі даної форми. Існує версія В. І. Арнольда гіпотези експоненційного росту про число таких траєкторій. В роботі доведено сильну форму цієї гіпотези для кожної морсівської форми та C^0 -щільної множини градієнтних векторних полів.

Ми також наводимо приклади конкретних обчислень комплексу Новікова.

1. INTRODUCTION

Let $f : M \rightarrow S^1$ be a Morse map, v a transverse f -gradient. Denote by $\overline{M} \rightarrow M$ the infinite cyclic covering corresponding to f . The classical construction of the Morse chain complex of a real-valued Morse function can be generalized to this case; one obtains the *Novikov complex* $C_*(f, v)$ defined over the ring $\mathbb{Z}[[t]][t^{-1}]$ of Laurent power series with integer coefficients and finite negative part. The homology of this chain complex is isomorphic to a certain completion of the homology module $H_*(\overline{M})$. *Novikov's Exponential Growth Conjecture* says that the boundary operators in $C_*(f, v)$ are power

series of non-zero convergence radius. In 1995 the author proved that these boundary operators are rational functions of t for the case of C^0 -generic gradient v . That implies the Novikov Exponential Growth Conjecture for C^0 -generic gradients. This result and its generalizations were announced in [14], and the proofs of first part of these results were published in [15]. The proofs of the second part were available as an e-print [13]. Since then the results of this e-print were used in several papers, see e.g. [5]. Recently this domain attracted again the interest of researchers, see the article [6], where a family of examples of Novikov complex with infinite power series as incidence coefficients was constructed.

The present paper is a revised version of the eprint [13]. Several references were updated, and references to more recent articles were added. Thus this article completes the publication of the results announced in [14].

The paper falls naturally into three parts. The first part (Section 2) contains a recollection of the material of the author's paper [15] in a compressed and rearranged form, suitable for our present purposes. The second part (Section 3, 4) is the hard core of the paper. We prove here that the rationality property of the boundary operators is C^0 -generic¹. In Section 3 we deal with non-abelian Novikov rings and the rationality property is expressed in terms of a certain non-abelian localization of the group ring of the fundamental group of the manifold. In Section 4 we consider the case of arbitrary Morse forms and free abelian coverings.

In the third part we study the exponential growth properties. We begin by an example of a Novikov complex where the incidence coefficients grow exponentially (Section 5). Then we apply the results of the previous sections to show that the exponential growth property of the incidence coefficients of the gradient of a Morse form is C^0 -generic.

In the rest of the Introduction we give the statement of our main results preceded by short background subsections 1.1 and 1.2.

1.1. Morse-Novikov theory for circle-valued Morse functions. The classical Morse-Thom-Smale construction associates to a Morse function $g : M \rightarrow \mathbb{R}$ on a closed manifold a free chain complex $C_*(g)$ where the number $m(C_p(g))$ of free generators of $C_p(g)$ equals the number of the critical points of g of index p for each p . The boundary operator in this complex is defined in a geometric way, counting the trajectories of a gradient of g , joining critical points of g (see [8, 12, 17–19]).

In the early 80s S. P. Novikov generalized this construction to the case of maps $f : M \rightarrow S^1$ (see [9]). The corresponding analogue of Morse complex is a free chain complex $C_*(f)$ over $\mathbb{Z}[[t]][t^{-1}]$. Its number of free generators

¹ For the precise meaning of C^0 -genericity in our context see the statements of the Theorems A, B, C

equals the number of critical points of f of index p , and the homology of $C_*(f)$ equals to the completed homology of the cyclic covering.

Fix some k . The boundary operator $\partial : C_k(f) \rightarrow C_{k-1}(f)$ is represented by a matrix, which entries are in the ring of Laurent power series. That is $\partial_{ij} = \sum_{n=-N}^{\infty} a_n t^n$, where $a_n \in \mathbb{Z}$, and N is some natural number.

Since the beginning S. P. Novikov conjectured that the power series ∂_{ij} have some nice analytic properties. In particular he conjectured that:

- *generically the coefficients a_n of $\partial_{ij} = \sum_{n=-N}^{\infty} a_n t^n$ grow at most exponentially with n .*

In [15] we have proved that for a C^0 generic f -gradient the incidence coefficients above are actually rational functions. To recall the statement of the Main Theorem of [15] let M be a closed connected manifold and $f : M \rightarrow S^1$ a Morse map, non-homotopic to zero. Denote the set of critical points of f by $S(f)$. The set of f -gradients of the class C^∞ , satisfying the transversality assumption (see §1 for terminology), will be denoted by $\mathcal{G}t(f)$. By Kupka-Smale theorem it is residual in the set of all the C^∞ gradients. Choose $v \in \mathcal{G}t(f)$. Denote by $\overline{M} \xrightarrow{\mathcal{P}} M$ the connected infinite cyclic covering for which $f \circ \mathcal{P}$ is homotopic to zero. Choose a lift $F : \overline{M} \rightarrow \mathbb{R}$ of $f \circ \mathcal{P}$ and let t be the generator of the structure group of \mathcal{P} such that $F(xt) < F(x)$. The t -invariant lift of v to \overline{M} will be denoted by the same letter v . For every critical point x of f choose a lift \bar{x} of x to \overline{M} . Choose orientations of stable manifolds of critical points. Then for every $x, y \in S(f)$ with $\text{ind } x = \text{ind } y + 1$ and every $k \in \mathbb{Z}$ the incidence coefficient $n_k(x, y; v)$ is defined (as the algebraic number of $(-v)$ -trajectories joining \bar{x} to $\bar{y}t^k$).

Theorem. ([15, p.971]) *In the set $\mathcal{G}t(f)$ there is a subset $\mathcal{G}t_0(f)$ with the following properties.*

- (1) $\mathcal{G}t_0(f)$ is open and dense in $\mathcal{G}t(f)$ with respect to C^0 topology.
- (2) If $v \in \mathcal{G}t_0(f)$, $x, y \in S(f)$ and $\text{ind } x = \text{ind } y + 1$, then

$$\sum_{k \in \mathbb{Z}} n_k(x, y; v) t^k$$

is a rational function of t of the form $\frac{P(t)}{t^m Q(t)}$, where $P(t)$ and $Q(t)$ are polynomials with integral coefficients, $m \in \mathbb{N}$, and $Q(0) = 1$.

- (3) Let $v \in \mathcal{G}t_0(f)$ and U be a neighborhood of $S(f)$. Then for every $w \in \mathcal{G}t_0(f)$ such that $w = v$ in U and w is sufficiently close to v in C^0 topology we have: $n_k(x, y; v) = n_k(x, y; w)$ for every $x, y \in S(f)$, $k \in \mathbb{Z}$.

The present paper uses the terminology and definitions of the paper [15], and should be read as a follow up of [15].

1.2. Morse forms and Novikov rings. To state our results, we recall some algebraic and Morse-theoretic definitions. Let G be a group and

$$\xi : G \rightarrow \mathbb{R}$$

be a group homomorphism. We denote by $(\mathbb{Z}G)^{\wedge\wedge}$ the abelian group of all formal linear combinations $\sum_{g \in G} n_g g$ (infinite in general). Recall that the

Novikov ring $\mathbb{Z}G_{\xi}^{-}$ is the subring of all $\lambda \in (\mathbb{Z}G)^{\wedge\wedge}$ such that for every $c \in \mathbb{R}$ the set $\text{supp } \lambda \cap \xi^{-1}([c, \infty))$ is finite.

Let ω be a closed 1-form on a manifold M . The de Rham cohomology class of ω will be denoted by $[\omega]$ and the corresponding homomorphism $\pi_1 M \rightarrow \mathbb{R}$ will be denoted by $\{\omega\}$. We say that ω is a *Morse form*, if locally it is the differential of a Morse function. If $f : M \rightarrow S^1$ is a Morse map, then its differential is a Morse form, which cohomology class is in $H^1(M, \mathbb{Z})$. A Morse form ω is proportional to a differential of a Morse map $M \rightarrow S^1$ if and only if there exists $\lambda \in \mathbb{R}$ such that $\lambda[\omega] \in H^1(M, \mathbb{Z})$.

The terminology of [15, §1.1] (which concerns Morse functions) is extended in an obvious way to the case of Morse forms, and we shall make free use of it. In particular we shall assume the notion of ω -*gradient*. (See Subsection 2.1 for a detailed recollection.) The set of all ω -gradients, satisfying the transversality assumption will be denoted by $\mathcal{G}t(\omega)$.

Let ω be a Morse form on a closed connected manifold M , $v \in \mathcal{G}t(\omega)$, and $x, y \in S(\omega)$ with $\text{ind } x = \text{ind } y + 1$. Choose some lifts \tilde{x}, \tilde{y} of x, y to \tilde{M} and some orientations of the stable manifolds of x and of y . In Subsection 4.2 we define the incidence coefficient $\tilde{n}(\tilde{x}, \tilde{y}; v) \in \mathbb{Z}(\pi_1 M)_{\xi}^{-}$ (Definition 4.2.5).

1.3. Statement of the results.

1.3.1. Morse maps $M \rightarrow S^1$. Let $\xi : G \rightarrow \mathbb{Z}$ be a group epimorphism. Denote $\text{Ker } \xi$ by H . For $n \in \mathbb{Z}$ denote

- $\xi^{-1}(n)$ by $G_{(n)}$;
- $\{x \in \mathbb{Z}G \mid \text{supp } x \subset G_{(n)}\}$ by $\mathbb{Z}G_{(n)}$;
- $\xi^{-1}((-\infty, -1])$ by G_- ;
- and $\{x \in \mathbb{Z}G \mid \text{supp } x \subset G_-\}$ by $\mathbb{Z}G_-$.

Choose $\theta \in \mathbb{Z}G_{(-1)}$. It is easy to see that $\mathbb{Z}G_{\xi}^{-}$ is identified with the ring of power series of the form $\{a_{-n}\theta^{-n} + \dots + a_1\theta + \dots \mid a_i \in \mathbb{Z}H\}$. Put

$$\Sigma_n = \{\mathbf{1} + A \mid A \in \text{Mat}_{n \times n}(\mathbb{Z}G_{(-1)})\}, \quad \Sigma = \bigcup_{n \geq 1} \Sigma_n.$$

There is the corresponding localization ring $\mathbb{Z}G_{\Sigma}$ (see [4, p. 291]). Every matrix in Σ_n is invertible in $\text{Mat}_{n \times n}(\mathbb{Z}G_{\xi}^{-})$, the inverse of $\mathbf{1} + A$ being given

by $\sum_{n=0}^{\infty} (-1)^n A^n$, therefore the localization map $\lambda : \mathbb{Z}G \rightarrow \mathbb{Z}G_{\Sigma}$ is injective and the inclusion $i : \mathbb{Z}G \hookrightarrow \mathbb{Z}G_{\xi}^{-}$ factors through a ring homomorphism $\ell : \mathbb{Z}G_{\Sigma} \rightarrow \mathbb{Z}G_{\xi}^{-}$.

Let M be a connected closed manifold and $f : M \rightarrow S^1$ be a Morse map, non-homotopic to zero. Let $G = \pi_1(M)$ and denote by ξ the induced homomorphism $G \rightarrow \mathbb{Z}$. Denote by $p : \widetilde{M} \rightarrow M$ the universal covering of M .

Theorem A. *In the set $\mathcal{G}t(f)$ there is a subset $\mathcal{G}t_1(f)$ with the following properties:*

- (1) $\mathcal{G}t_1(f)$ is open and dense in $\mathcal{G}t(f)$ with respect to C^0 topology.
- (2) If $v \in \mathcal{G}t_1(f)$ then for every $x, y \in S(f)$ with $\text{ind } x = \text{ind } y + 1$ we have $\tilde{n}(\tilde{x}, \tilde{y}; v) \in \text{Im } \ell$.
- (3) Let $v \in \mathcal{G}t_1(f)$. Let U be a neighborhood of $S(f)$. Then for every $w \in \mathcal{G}t_1(f)$ such that $w = v$ in U and w is sufficiently close to v in C^0 topology we have: $\tilde{n}(\tilde{x}, \tilde{y}; v) = \tilde{n}(\tilde{x}, \tilde{y}; w)$ for every $x, y \in S(f)$.

1.3.2. *Morse forms within arbitrary cohomology classes.* At present we can prove the analogue of Theorem A in the case of arbitrary Morse forms only for the incidence coefficients associated with free abelian coverings.

Let ω be a Morse form on a closed connected manifold M . If $\phi : \widehat{M} \rightarrow M$ is any regular covering with the structure group G , such that $\phi^*([\omega]) = 0$, then the homomorphism $\{\omega\} : \pi_1 M \rightarrow \mathbb{R}$ factors as $\pi_1 M \rightarrow G \rightarrow \mathbb{R}$ and it is not difficult to see that the incidence coefficients $\widehat{n}(\widehat{x}, \widehat{y}; v)$ are defined for every $v \in \mathcal{G}t(\omega)$ (here we suppose that $\text{ind } x = \text{ind } y + 1$, and that for every $p \in S(\omega)$ a lift \widehat{p} of p to \widehat{M} and an orientation of the stable manifold of p are chosen). In particular, it is the case for the maximal free abelian covering $\overline{M} \xrightarrow{\mathcal{P}} M$ with the structure group $H_1(M, \mathbb{Z})/\text{Tors} \approx \mathbb{Z}^m$. By abuse of notation we shall denote the corresponding homomorphism $\mathbb{Z}^m \rightarrow \mathbb{R}$ by the same symbol as the de Rham cohomology class $[\omega]$ of ω . Assume that $[\omega] \neq 0$. Put

$$\mathfrak{s}_{[\omega]} = \{P \in \mathbb{Z}[\mathbb{Z}^m] \mid P = \mathbf{1} + Q : \text{supp } Q \subset [\omega]^{-1}((-\infty, 0))\}.$$

Then $\mathfrak{s}_{[\omega]}$ is a multiplicative subset of $\mathbb{Z}[\mathbb{Z}^m]$.

Theorem B. *There is a subset $\mathcal{G}t_1(\omega) \subset \mathcal{G}t(\omega)$ with the following properties:*

- (1) $\mathcal{G}t_1(\omega)$ is open and dense in $\mathcal{G}t(\omega)$ with respect to C^0 topology.
- (2) For every $v \in \mathcal{G}t_1(\omega)$ and every $x, y \in S(\omega)$ with $\text{ind } x = \text{ind } y + 1$ we have: $\overline{n}(\overline{x}, \overline{y}; v) \in \mathfrak{s}_{[\omega]}^{-1} \mathbb{Z}[\mathbb{Z}^m]$.

(3) Let $v \in \mathcal{G}t_1(\omega)$. Let U be a neighborhood of $S(\omega)$. Then for every $w \in \mathcal{G}t_1(\omega)$ such that $w = v$ in U and w is sufficiently close to v in C^0 topology we have: $\bar{n}(\bar{x}, \bar{y}; v) = \bar{n}(\bar{x}, \bar{y}; w)$ for every $x, y \in S(\omega)$, $k \in \mathbb{Z}$.

1.3.3. *An example.* In Section 5 we construct a three-manifold M , a Morse map $f : M \rightarrow S^1$ and an f -gradient v such that $n_0(\bar{x}, \bar{y}; v) = 0$ and for $k \geq 0$ we have

$$n_{k+1}(\bar{x}, \bar{y}; v) = -\frac{4}{\sqrt{5}} \cdot \left(\left(\frac{3 + \sqrt{5}}{2} \right)^k - \left(\frac{3 - \sqrt{5}}{2} \right)^k \right).$$

1.3.4. *Exponential growth estimates.* The Novikov Exponential Growth conjecture [10] has a natural generalization to the case of Morse forms of arbitrary irrationality degree. Let ω be a Morse form and $\mathcal{P} : \overline{\overline{M}} \rightarrow M$ be the maximal free abelian covering. Let $F : \overline{\overline{M}} \rightarrow \mathbb{R}$ be a function such that $dF = \mathcal{P}^*([\omega])$. For critical point x, y of F with $\text{ind } x = \text{ind } y + 1$ and $h \in \mathbb{Z}^m$ let $n(x, y; h)$ be the algebraic number of $(-v)$ -trajectories joining x with yh . Then the generalized exponential growth conjecture says that there are constants $A, B > 0$ such that for every x, y with $\text{ind } x = \text{ind } y + 1$ and every $h \in \mathbb{Z}^m$ we have

$$|n(x, y; h)| \leq A \cdot e^{B[\omega](h)}. \quad (1.1)$$

Theorem C below asserts a much stronger property, pertaining to the universal covering of M and counting for every $g \in G = \pi_1(M)$ the absolute value of the algebraic number of trajectories joining x with $y \cdot g$. To state this theorem we need one more definition. Let G be a group. For an element $a = \sum n_g g \in \mathbb{Z}G$ we denote by $\|a\|$ the sum $\sum |n_g|$. Let $\xi : G \rightarrow \mathbb{R}$ be a homomorphism. For $\lambda = \sum n_g g \in \mathbb{Z}G_\xi^-$ and $c \in \mathbb{R}$ we denote by $\lambda[c]$ the element $\sum_{\xi(g) \geq c} n_g g$ of $\mathbb{Z}G$ and we set $N_c(\lambda) = \|\lambda[c]\|$. We shall say that λ is of *exponential growth* if there are $A, B \geq 0$ such that for every $c < 0$ we have $N_c(\lambda) \leq Ae^{-cB}$. It is easy to prove that the elements of exponential growth form a subring of $\mathbb{Z}G_\xi^-$ containing $\mathbb{Z}G$.

Theorem C. *Let v be an ω -gradient, belonging to $\mathcal{G}t_1(\omega)$, and $x, y \in S(\omega)$ with $\text{ind } x = \text{ind } y + 1$. Then $\tilde{n}(\tilde{x}, \tilde{y}; v) \in \mathbb{Z}[\pi_1 M]_{\{\omega\}}^-$ is of exponential growth.*

In the paper [3, Theorem 2.21] D. Burghelea and S. Haller also prove that the property (1.1) holds for C^0 -generic gradients².

² The author thanks the anonymous referee for this information.

1.3.5. *Exponential estimates of absolute number of trajectories: Morse maps* $M \rightarrow S^1$. The first published version of the Exponential Growth conjecture appeared in the paper of V. I. Arnold “Dynamics of intersections”, 1989. This version is also the strongest one. Arnold writes in [1, p. 83]:

“The author is indebted to S. P. Novikov who has communicated the following conjecture, which was the starting point of the present paper. Let $p : \widetilde{M} \rightarrow M$ be a covering of a compact manifold M with fiber \mathbb{Z}^n , and let α be a closed 1-form on M such that $p^*\alpha = df$, where $f : M \rightarrow \mathbb{R}$ is a Morse function. The Novikov conjecture states that, “generically” the number of the trajectories of the vector field $-\text{grad } f$ on M starting at a critical point x of the function f of index k and connecting it with the critical points y having index $k-1$ and satisfying $f(y) \geq f(x) - n$, grows in n more slowly than some exponential, e^{an} .”

In the present paper we prove this conjecture for the C^0 -dense subset in the set of all gradients (Theorems C and D below). We assume here the terminology of Subsection 1.3.1. The set of all f -gradients of class C^∞ will be denoted by $\mathcal{G}(f)$. An f -gradient v is called *good* if for every $p, q \in S(f)$ we have

$$(\text{ind } p \leq \text{ind } q + 1) \Rightarrow (D(p, v) \pitchfork D(q, -v))$$

The set of all good f -gradients will be denoted by $\mathcal{G}d(f)$. For $v \in \mathcal{G}(f)$ we denote by the same letter v the t -invariant lift of v to \overline{M} . Choose a lift

$$F : \overline{M} \rightarrow \mathbb{R}$$

of f to \overline{M} . It is easy to prove that for $p, q \in S(F)$, $\text{ind } p = \text{ind } q + 1$ and for for $v \in \mathcal{G}d(f)$ the set of $(-v)$ -trajectories, joining p to q is finite. The lifts of critical points of f to \overline{M} being chosen, denote by $N_k(x, y; v)$ the number of $(-v)$ -trajectories joining \bar{x} to $\bar{y}t^k$ (where $\text{ind } x = \text{ind } y + 1$).

Theorem D. *In the set $\mathcal{G}(f)$ there is a subset $\mathcal{G}_0(f)$ with the following properties:*

- (1) $\mathcal{G}_0(f)$ is C^0 -dense in $\mathcal{G}(f)$ and $\mathcal{G}_0(f) \subset \mathcal{G}d(f)$.
- (2) Let $v \in \mathcal{G}_0(f)$. Then there are constants $C, D > 0$ such that for every $x, y \in S(f)$ with $\text{ind } x = \text{ind } y + 1$ and for every $k \in \mathbb{Z}$ we have $N_k(x, y; v) \leq C \cdot D^k$.

1.3.6. *Exponential estimates of absolute number of trajectories: Morse forms.* Let ω be a Morse form on a closed connected manifold M . Let $\mathcal{P} : \overline{\overline{M}} \rightarrow M$ be the maximal free abelian covering of M with structure group \mathbb{Z}^m ; we identify the cohomology class $[\omega]$ of ω with the corresponding homomorphism $\mathbb{Z}^m \rightarrow \mathbb{R}$. We denote by $\mathcal{G}(\omega)$ the set of all ω -gradients of class C^∞ and by $\mathcal{G}d(\omega)$ the set of all good ω -gradients of class C^∞ . Let

$v \in \mathcal{G}d(\omega)$. For every zero x of ω choose a lift \bar{x} of x to \overline{M} and an orientation of the stable manifold of x . Then for every $g \in \mathbb{Z}^m$ and every $x, y \in S(\omega)$ with $\text{ind } x = \text{ind } y + 1$ the set of $(-v)$ -trajectories joining \bar{x} to $\bar{y}g$ is finite and we denote its cardinality by $N(\bar{x}, \bar{y}, g; v)$. For $c \in \mathbb{R}$ we denote by $N_{\geq c}(\bar{x}, \bar{y}; v)$ the sum $\sum_{g: [\omega](g) \geq c} N(\bar{x}, \bar{y}, g; v)$.

Theorem E. *In the set $\mathcal{G}(\omega)$ there is a subset $\mathcal{G}_0(\omega)$ with the following properties:*

- (1) $\mathcal{G}_0(\omega)$ is dense in $\mathcal{G}(\omega)$ with respect to C^0 topology; $\mathcal{G}_0(\omega) \subset \mathcal{G}d(\omega)$.
- (2) Let $v \in \mathcal{G}_0(\omega)$. There exist constants $C, D > 0$ such that for every $x, y \in S(\omega)$ with $\text{ind } x = \text{ind } y + 1$ and every $\lambda < 0$ we have

$$N_{\geq \lambda}(x, y; v) \leq C \cdot D^{-\lambda}.$$

Remark 1.3.7. Observe that Theorem D follows from Theorem E. We still keep both statements of Theorem D since the essential of the proof of Theorem E is based on the proof of Theorem D.

2. RANGING SYSTEMS AND RANGING FLOWS (AN OVERVIEW OF RESULTS OF [15])

The results of the present paper are based on the author's paper [15]. For the convenience of the reader we recall in this section some of the basic definitions and theorems of [15]. The contents of this section falls into two parts. In the first part (Subsections 2.2-2.3) we present the main technical tool for the rationality theorem, namely *ranging systems*. We show that if an f -gradient admit a ranging system adapted to a pair of critical points p, q , then the incidence coefficient of these two points is a rational function of the type described in the statement of the Main Theorem of [15]. The final result of this part is Theorem 2.3.4.

In the second part (Subsections 2.4-2.6) we show that the set of f -gradients that admit a ranging system adapted to p, q , is open and dense in the set of all transverse gradients (with respect to C^0 -topology). The basic instruments here are the notions of ranging flow and ranging pair.

The exposition is preceded by a short introductory Subsection 2.1 where we collected the necessary definitions.

2.1. Terminology: functions and gradients. In the present subsection M is a closed manifold.

Definition 2.1.1. Let $f : M \rightarrow \mathbb{R}$ be a Morse function on a closed manifold M . Denote $\dim M$ by n . The set of critical points of f will be denoted by $S(f)$. A chart $\Phi_p : U_p \rightarrow B^n(0, r_p)$ (where $p \in S(f)$, U_p is a neighborhood

of p , $r_p > 0$) is called *standard chart for f around p of radius r_p* (or simply *f -chart*) if there is an extension of Φ_p to a chart $\tilde{\Phi}_p : V_p \rightarrow B^n(0, r'_p)$, (where $\overline{U_p} \subset V_p$ and $r'_p > r_p$), such that

$$(f \circ \tilde{\Phi}_p^{-1})(x_1, \dots, x_n) = f(p) + \sum_{i=1}^n \alpha_i x_i^2,$$

where $\alpha_i < 0$ for $i \leq \text{ind } p$ and $\alpha_i > 0$ for $i > \text{ind } p$. The domain U_p is called *standard coordinate neighborhood*. Any such extension $\tilde{\Phi}_p$ of Φ_p will be called *standard extension of Φ_p* .

The set $\Phi_p^{-1}(\mathbb{R}^k \times \{0\})$, resp. $\Phi_p^{-1}(\{0\} \times \mathbb{R}^{n-k})$, where k stands for $\text{ind } p$, is called *negative disc*, resp. *positive disc*. If for every i we have $\alpha_i = \pm 1$, we shall say that the coordinate system $\{\Phi_p\}$ is *strongly standard*.

A family

$$\mathcal{U} = \{\Phi_p : U_p \rightarrow B^n(0, r_p)\}_{p \in S(f)}$$

of f -charts is called *f -chart-system*, if the family $\{\overline{U_p}\}$ is disjoint. We denote $\min_p r_p$ by $d(\mathcal{U})$, and $\max_p r_p$ by $D(\mathcal{U})$. If all the r_p are equal to r , we shall say that \mathcal{U} is of *radius r* .

The set $\Phi_p^{-1}(B^n(0, \lambda))$, where $\lambda \leq r_p$ will be denoted by $U_p(\lambda)$. For $\lambda \leq d(\mathcal{U})$ we denote $\bigcup_{p \in S(f)} U_p(\lambda)$ by $\mathcal{U}(\lambda)$. Let

$$\mathcal{U} = \{\Phi_p : U_p \rightarrow B^n(0, r_p)\}_{p \in S(f)}, \quad \mathcal{U}' = \{\Phi'_p : U'_p \rightarrow B^n(0, r'_p)\}_{p \in S(f)}$$

be two f -chart-systems. We say, that \mathcal{U}' is a *restriction of \mathcal{U}* , if for every $p \in S(f)$ we have: $r'_p \leq r_p$, $U'_p \subset U_p$, $\Phi'_p = \Phi_p | U'_p$.

Given an f -chart system

$$\mathcal{U} = \{\Phi_p : U_p \rightarrow B^n(0, r_p)\}_{p \in S(f)}$$

we say, that a vector field v on M is an *f -gradient with respect to \mathcal{U}* , if

- 1) For every $x \in M \setminus S(f)$ we have $df(v)(x) > 0$;
- 2) For every $p \in S(f)$ we have

$$(\tilde{\Phi}_p)_*(v) = (-x_1, \dots, -x_k, x_{k+1}, \dots, x_n),$$

where $k = \text{ind } p$, and $\tilde{\Phi}_p$ is some standard extension of Φ_p .

We say that a vector field v is an *f -gradient* if there is an f -chart system \mathcal{U} , such that v is an f -gradient with respect to \mathcal{U} .

Definition 2.1.2. Assume that M is Riemannian and denote by $D_r(x)$, resp. $B_r(x)$ the closed, resp. open ball of radius r centered in x . Let also $f : M \rightarrow \mathbb{R}$ be a Morse function, v be an f -gradient, $p \in S(f)$. Set:

$$B_\delta(p, v) = \{x \in M \mid \exists t \geq 0 : \gamma(x, t; v) \subset B_\delta(p)\},$$

$$D_\delta(p, v) = \{x \in M \mid \exists t \geq 0 : \gamma(x, t; v) \subset D_\delta(p)\},$$

$$D(p, v) = \{x \in M \mid \lim_{t \rightarrow \infty} \gamma(x, t; v) = p\}.$$

We denote by $K(v)$ the union of all subsets $D(p, v)$, and by $B_\delta(v)$ the union of all subsets $B_\delta(p, v)$ where p ranges over critical points of f .

We also denote by $B_\delta(\text{ind} \leq s; v)$ the union of subsets $B_\delta(p, v)$ (where p ranges over critical points of f of index $\leq s$). Similar notations like $D_\delta(\text{ind} \leq s; v)$ or $K(\text{ind} = s; v)$ etc. are now clear without special definition.

Remark 2.1.3. Definitions 2.1.1 and 2.1.2 generalize in an obvious way to the case of Morse functions on cobordisms and to the case of circle-valued Morse functions.

Definition 2.1.4. Let $f : W \rightarrow [a, b]$ be a Morse function on a compact Riemannian cobordism, $f^{-1}(b) = V_1, f^{-1}(a) = V_0$, v be an f -gradient. For $x \in W$ we denote by $\gamma(x, t; v)$ the value at t of the integral curve of v satisfying $\gamma(x, 0; v) = x$. If $x \in V_1 \setminus K(-v)$, then the trajectory $\gamma(x, t; v)$ reaches V_0 at some moment t_0 ; put

$$\overset{\rightsquigarrow}{v}(x) = \gamma(x, t_0; v).$$

We obtain thus a diffeomorphism

$$\overset{\rightsquigarrow}{v} : V_1 \setminus K(-v) \xrightarrow{\approx} V_0 \setminus K(v).$$

Sometimes we denote it by $v_{[b,a]}^{\rightsquigarrow}$.

In the next section we take a closer look at this diffeomorphism.

2.2. Ranging systems. We proceed to the definition of ranging system (see [15, Definition 4.6]). Let $f : W \rightarrow [a, b]$ be a Morse function on a compact Riemannian cobordism, $f^{-1}(b) = V_1, f^{-1}(a) = V_0$, v be an f -gradient.

Definition 2.2.1. Let $\Lambda = \{\lambda_0, \dots, \lambda_k\}$ be a finite set of regular values of f , such that $\lambda_0 = a, \lambda_k = b$, and for each $0 \leq i \leq k-1$ we have $\lambda_i < \lambda_{i+1}$ and there is exactly one critical value of f in $[\lambda_i, \lambda_{i+1}]$. The values λ_i, λ_{i+1} will be called *adjacent*. The set of pairs $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ is called *ranging system for (f, v)* if

(RS1) For every $\lambda \in \Lambda$ the sets A_λ and B_λ are disjoint compacts in $f^{-1}(\lambda)$.

(RS2) Let $\lambda, \mu \in \Lambda$ be adjacent. Then for every $p \in S(f) \cap f^{-1}([\lambda, \mu])$ one of the two following properties i), ii) holds:

- i) $D(p, v) \cap f^{-1}(\lambda) \subset \text{Int}A_\lambda$,
- ii) $D(p, -v) \cap f^{-1}(\mu) \subset \text{Int}B_\mu$.

(RS3) Let $\lambda, \mu \in \Lambda$ be adjacent. Then

$$v_{[\mu, \lambda]}^{\rightsquigarrow}(A_\mu) \subset \text{Int}A_\lambda, \quad (-v)_{[\lambda, \mu]}^{\rightsquigarrow}(B_\lambda) \subset \text{Int}B_\mu.$$

The existence of a ranging system allows us to endow the gradient flow with a structure that resembles to some extent a continuous map between compact spaces. The next proposition is proved in [15, Lemma 4.8 and Propositions 4.9, 4.10].

Proposition 2.2.2. *Let $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ be a ranging system for (f, v) . Let also N be an oriented submanifold of $V_1 \setminus B_b$ such that $N \setminus \text{Int}A_b$ is compact. Then the following statements hold.*

(1) *There exists a homomorphism*

$$H(v) : H_*(V_1 \setminus B_b, A_b) \rightarrow H_*(V_0 \setminus B_a, A_a),$$

such that

$$H(v)([N]) = [v_{[b, a]}^{\rightsquigarrow}(N)],$$

where $[N]$ is the fundamental class of N in $H_(V_1 \setminus B_b, A_b)$ and $[v_{[b, a]}^{\rightsquigarrow}(N)]$ is the fundamental class of $v_{[b, a]}^{\rightsquigarrow}(N)$ in $H_*(V_0 \setminus B_a, A_a)$.*

(2) *There is $\epsilon > 0$ such that for any f -gradient w with $\|v - w\| < \epsilon$ the ranging system $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ is also a ranging system for (f, w) , and $H(v) = H(w)$.*

Remark 2.2.3. Let $\lambda < \mu$ be elements of Λ . Similarly to Proposition 2.2.2 one defines a homomorphism

$$H(v)_{[\mu, \lambda]} : H_*(f^{-1}(\mu) \setminus B_\mu, A_\mu) \rightarrow H_*(f^{-1}(\lambda) \setminus B_\lambda, A_\lambda)$$

which is C^0 -stable with respect to small C^0 -perturbations of v , as in property (2) of Proposition 2.2.2.

Ranging systems provide a convenient tool for computation of incidence coefficients in homology terms.

Definition 2.2.4. Let $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ be a ranging system for

$$f : W \rightarrow [a, b].$$

Let also $p \in S(f)$ and $\mu(p)$ and $\sigma(p)$ be the (uniquely determined) adjacent elements of Λ such that $f(p) \in (\mu(p), \sigma(p))$. We will use the following notations:

- $S_+(p, v)$ denotes the sphere $D(p, -v) \cap f^{-1}(\sigma(p))$, and
- $S_-(p, v)$ denotes the sphere $D(p, v) \cap f^{-1}(\mu(p))$.

If this does not lead to confusion, we denote these spheres simply by $S_-(p)$ and $S_+(p)$.

We say that a ranging system $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ is *adapted to a pair* p, q of critical points in $S(f)$, where $\text{ind } p = \text{ind } q + 1$, if

$$S_-(p) \cap B_{\mu(p)} = \emptyset \quad \text{and} \quad S_+(q) \cap A_{\sigma(q)} = \emptyset.$$

Let $l + 1 = \text{ind } p$. Then $\text{ind } q = l$.

If $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ is adapted to p, q , we denote by $[S_-(p)]$ also the image of this fundamental class in

$$H_l(f^{-1}(\mu(p)) \setminus B_{\mu(p)}, A_{\mu(p)}).$$

Consider the inclusion

$$j(v) : (f^{-1}(\sigma(q)) \setminus B_{\sigma(q)}, A_{\sigma(q)}) \hookrightarrow (f^{-1}(\sigma(q)), f^{-1}(\sigma(q)) \setminus S_+(q))$$

and let $]]S_+(q, v)[[$ be the $(j(v))^*$ -image of the class dual to $S_+(q)$ in the group $H^l(f^{-1}(\sigma(q)) \setminus B_{\sigma(q)}, A_{\sigma(q)})$.

See the proof of the following two propositions in [15, Proposition 4.12].

Proposition 2.2.5. *Let $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ be a ranging system adapted to p, q . If v is an f -gradient satisfying the transversality condition then*

$$n(p, q; v) = \langle]]S_+(q)[[, H_{[\mu(p), \sigma(q)]}(v)([S_-(p)]) \rangle.$$

This formula imply immediately a C^0 -stability property for the incidence coefficients.

Proposition 2.2.6. (1) *There is $\epsilon > 0$ such that for every f -gradient v with $\|w - v\| < \epsilon$ the system $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ is a ranging system for (f, w) adapted to p, q .*

(2) *Let $U(p)$ be a neighborhood of p , and $U(q)$ be a neighborhood of q . There is $\epsilon > 0$ such that for every f -gradient w with $w|_{U(p)} = v|_{U(p)}$ and $w|_{U(q)} = v|_{U(q)}$ and $\|w - v\| < \epsilon$ we have*

$$[S_-(p, w)] = [S_-(p, v)], \quad]]S_-(q, w)[[=]]S_-(q, v)[[$$

and

$$n(p, q; w) = n(p, q; v) \tag{2.1}$$

In our applications the cobordism W will be obtained as the result of cutting a closed manifold M along a regular level surface of a Morse function $f : M \rightarrow S^1$. Such cobordism is endowed naturally with a diffeomorphism of its lower boundary on its upper boundary.

Definition 2.2.7. A Riemannian cobordism W with $\partial W = V_0 \sqcup V_1$ is said to be *cyclic* if there is an isometry $\Phi : V_0 \rightarrow V_1$ (such an isometry will be regarded as part of the structure of a cyclic cobordism).

Definition 2.2.8. A ranging system $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ for (f, v) is said to be *cyclic* if $\Phi(A_a) = A_b, \Phi(B_a) = B_b$. A cyclic ranging system is said to be of *finite type* if $H_*(V_0 \setminus B_a, A_a)$ is finitely generated.

These techniques will be applied to the case of circle-valued Morse functions.

2.3. Equivariant ranging systems. Now we will apply the techniques of the previous section to investigation of the incidence coefficients in the Novikov complex. Let $f : M \rightarrow S^1$ be a Morse function, and v be an f -gradient. We will assume here that the class $[f] \in H^1(M, \mathbb{Z})$ is indivisible, so that for all $z \in \overline{M}$ we have that

$$F(z) - F(zt) = 1.$$

We assume also $S(f) \neq \emptyset$, and that 0 is a regular value of F . Choose a Riemannian metric on M . Then \overline{M} obtains a t -invariant Riemannian metric.

Definition 2.3.1. Let u be any f -gradient. Let Σ be a non empty set of regular values of F satisfying the following collection (\mathcal{S}) of conditions:

- (\mathcal{S}) : • the regular value 0 is in Σ ;
 • for every $A, B \in \mathbb{R}$ the set $\Sigma \cap F^{-1}([A, B])$ is finite;
 • if $\sigma \in \Sigma$ then $\sigma + n \in \Sigma$ for all $n \in \mathbb{Z}$;
 • if $\lambda, \mu \in \Sigma$ are adjacent, then there is only one critical value of F between λ and μ .

A set $\{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma}$ is called *t-equivariant ranging system for (F, u)* , if

- (1) For every $\mu, \nu \in \Sigma, \mu < \nu$ we have: $\{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma, \mu \leq \sigma \leq \nu}$ is a ranging system for $(F|_{F^{-1}([\mu, \nu]), u})$.
 (2) $A_{\sigma-n} = A_\sigma \cdot t^n$ and $B_{\sigma-n} = B_\sigma \cdot t^n$ for every $n \in \mathbb{Z}$.

Any cyclic ranging system on the cobordism $W = F^{-1}([0, 1])$ determines in an obvious way a t -equivariant ranging system. Let p and q be critical points of f , such that $\text{ind } p = \text{ind } q + 1$. Let $l = \text{ind } q$. Let

$$V_1 = F^{-1}(1), \quad V_0 = F^{-1}(0), \quad W = F^{-1}([0, 1]).$$

Assume that the ranging system is adapted to the pair (p, q) . We lift p and q to \overline{M} in such a way that $\bar{q} \in tW$ and $\bar{p} \in W$. Put

$$[[p]] = H_{[\mu(p), 0]}([S_-(p)]) \in H_l(V_0 \setminus B_0, A_0).$$

The cohomology class

$$]]S_+(\bar{q}, v)[[\in H^l(V_0 \setminus B_0, A_0)$$

will be denoted by $]]q[[$. Put

$$h = t^{-1} \circ H_{[0, -1]} : H^l(V_0 \setminus B_0, A_0) \rightarrow H^l(V_0 \setminus B_0, A_0).$$

The next three assertions form the first basic ingredient of the proof of the main theorem of [15] (see [15, Proposition 4.18] for the proof).

Proposition 2.3.2. *We have*

$$n_k(\bar{p}, \bar{q}; v) = \begin{cases} \langle]]q[[, (h(v))^k [[p]] \rangle & \text{if } k \geq 0, \\ 0 & \text{if } k < 0. \end{cases}$$

Lemma 2.3.3. *Let G be a finitely generated abelian group, A an endomorphism of G , and $\lambda : G \rightarrow \mathbb{Z}$ a homomorphism. Then for every $p \in G$ the series*

$$\sum_{k \geq 0} \lambda(A^k p) t^k \in \mathbb{Z}[[t]]$$

is a rational function of t of the form $\frac{P(t)}{Q(t)}$, where P, Q are polynomials and $Q(0) = 1$.

Theorem 2.3.4. *Let $f : M \rightarrow S^1$ be a Morse function, and v be a transverse f -gradient. Let also $F : \bar{M} \rightarrow \mathbb{R}$ be a lift of f and p, q be critical points of f such that $\text{ind } p = \text{ind } q + 1$.*

(1) *Assume that there exists a cyclic ranging system on the cobordism*

$$W = F^{-1}([0, 1]),$$

adapted to p, q . Then the incidence coefficient $N(p, q, v)$ is a rational function of t of the form $\frac{P(t)}{Q(t)}$, where P, Q are polynomials and $Q(0) = 1$.

(2) *The set of transverse f -gradients admitting a cyclic ranging system adapted to p, q is C^0 -open in the set of all transverse f -gradients.*

2.4. Almost transverse gradients. In this subsection we gathered some preliminaries necessary for construction of ranging systems for C^0 -generic flows; this construction will be recalled in the following subsections.

Definition 2.4.1. Let $f : W \rightarrow [a, b]$ be a Morse function and v an f -gradient. We say that a Morse function $\phi : W \rightarrow \mathbb{R}$ is *adjusted to the pair (f, v)* if

- (1) $S(\phi) = S(f)$,
- (2) v is also a ϕ -gradient,

- (3) the function $\phi - f$ is constant in a neighborhood of V_1 , in a neighborhood of V_0 , and in a neighborhood of each critical point of f .

Observe that it is not clear whether there is a ranging system for every gradient. It turns out that ranging systems exist for C^0 -generic gradients. We outline the proof in the next subsections.

Definition 2.4.2. Let $f : W \rightarrow [a, b]$ be a Morse function on a compact cobordism W , and v be an f -gradient. We say that v satisfies *the almost transversality assumption* if for every pair x, y of critical points of f we have

$$(\text{ind } x \leq \text{ind } y) \Rightarrow (D(x, v) \pitchfork D(y, -v)).$$

or equivalently

$$(\text{ind } x \leq \text{ind } y) \Rightarrow (D(x, v) \cap D(y, -v) = \emptyset).$$

We say also that v is *almost good* or *almost transverse*.

Lemma 2.4.3. *If V is almost good, then $K(\text{ind} \leq s; v)$ is compact for every s .*

Definition 2.4.4. We say that a f -gradient is δ -separated if there is an ordered Morse function $\phi : W \rightarrow [a, b]$, adjusted to (f, v) , with an ordering sequence a_0, \dots, a_{n+1} , such that

- (1) for every $p \in S(f)$ we have $D_\delta(p) \subset \phi^{-1}((a_k, a_{k+1}))$, where $k = \text{ind } p$.
- (2) for every $l : 0 \leq l \leq n$ and every $q \in S(f)$ with $\text{ind } q = l$ there is a Morse function

$$\psi : W_l \rightarrow [a_l, a_{l+1}],$$

adjusted to $(\phi|_{W_l}, v|_{W_l})$, where $W_l = \phi^{-1}([a_l, a_{l+1}])$, and a regular value $\mu \in (a_l, a_{l+1})$ of ψ , such that

- $D_\delta(q) \subset \phi^{-1}((a_l, \mu))$,
- and for every $r \in S(f)$, $\text{ind } r = l$, $r \neq q$ we have

$$D_\delta(r) \subset \phi^{-1}((\mu, a_{l+1})).$$

Lemma 2.4.5. *If v is almost good, then v is δ -separated for some $\delta > 0$.*

Proposition 2.4.6. *If v is δ_0 -separated, then for every $\delta \in (0, \delta_0)$ and for every s the set $D_\delta(\text{ind} \leq s; v)$ is compact.*

Definition 2.4.7. A pair (ϕ, u) where ϕ is a Morse function on a cobordism and u is a δ -separated ϕ -gradient will be called \mathcal{AM} -flow³.

³ Usually the number δ is clear from the context, so we omit it from the notation.

For an \mathcal{AM} -flow (ϕ, u) the sets $D_\delta(\text{ind} \leq s, u)$ form an increasing filtration of W . This is just one of versions of *handle decomposition of W* .

2.5. Stratified submanifolds and their thickenings. In this subsection we recall briefly the notions of s -submanifold and ts -submanifold introduced in [15]. These techniques will be used in the following subsection. We omit all proofs referring the reader to [15, §2].

2.5.A. Stratified submanifolds. Let $\mathbb{A} = \{A_0, \dots, A_k\}$ be a finite sequence of subsets of a topological space X . For $0 \leq s \leq k$, we denote A_s also by $\mathbb{A}_{(s)}$; the set $A_0 \cup \dots \cup A_s$ is denoted by $A_{\leq s}$ and also by $\mathbb{A}_{(\leq s)}$. We say that \mathbb{A} is a *compact family* if $\mathbb{A}_{(\leq s)}$ is compact for every s with $0 \leq s \leq k$.

Definition 2.5.1. (Stratified submanifolds) Let M be a manifold without boundary. A finite sequence $\mathbb{X} = \{X_0, \dots, X_k\}$ of subsets of M is called an *s -submanifold of M* (s for “stratified”) if

- (1) $X_i \cap X_j = \emptyset$ for $i \neq j$, and each X_i is a submanifold of M of dimension i with trivial normal bundle;
- (2) \mathbb{X} is a compact family.

For an s -submanifold $\mathbb{X} = \{X_1, \dots, X_s\}$, the largest k such that $X_k \neq \emptyset$ is called the *dimension* of \mathbb{X} and is denoted by $\dim \mathbb{X}$. For a diffeomorphism $\Phi : M \rightarrow N$ and an s -submanifold \mathbb{X} of M , we denote by $\Phi(\mathbb{X})$ the s -submanifold of N defined by $\Phi(\mathbb{X})_{(i)} = \Phi(\mathbb{X}_{(i)})$.

If V is a submanifold of M and \mathbb{X} is an s -submanifold of M , then we say that V is *transversal* to \mathbb{X} (in symbols: $V \pitchfork \mathbb{X}$) if $V \pitchfork \mathbb{X}_{(i)}$ for each i . If V is a compact submanifold of M transversal to an s -submanifold \mathbb{X} , then the family $\{\mathbb{X}_{(i)} \cap V\}$ is an s -submanifold of V ; it is denoted by $\mathbb{X} \cap V$.

Let \mathbb{X}, \mathbb{Y} be two s -submanifolds of M . We say that \mathbb{X} is *transversal* to \mathbb{Y} (in symbols: $\mathbb{X} \pitchfork \mathbb{Y}$) if $\mathbb{X}_{(i)} \pitchfork \mathbb{Y}_{(j)}$ for every i, j . We say that \mathbb{X} is *almost transversal* to \mathbb{Y} (in symbols: $\mathbb{X} \dagger \mathbb{Y}$) if $\mathbb{X}_{(i)} \pitchfork \mathbb{Y}_{(j)}$ for any i, j with $i + j < \dim M$. Note that $\mathbb{X} \dagger \mathbb{Y}$ if and only if $\mathbb{X}_{(\leq i)} \cap \mathbb{Y}_{(\leq j)} = \emptyset$ whenever $i + j < \dim M$.

Definition 2.5.2. Let $f : M \rightarrow \mathbb{R}$ be a Morse function, where M is a closed manifold or a compact cobordism, and let v be an f -gradient. Put

$$\mathbb{D}(v) = \{D(\text{ind} \leq i; v)\}_{0 \leq i \leq \dim M},$$

$$\mathbb{D}_\lambda(v) = \{D(\text{ind} \leq i; v) \cap f^{-1}(\lambda)\}_{0 \leq i \leq \dim M}.$$

Lemma 2.5.3. *Assume that v satisfies the almost transversality condition. Then:*

- (1) $\mathbb{D}(v)$ is a compact family;

- (2) if M is a closed manifold, then $\mathbb{D}(v)$ is an s -submanifold transversal to $f^{-1}(\lambda)$ for every regular value λ of f ;
- (3) if M is a cobordism, then for every regular value λ of f the family $\mathbb{D}_\lambda(v)$ is an s -submanifold of $f^{-1}(\lambda)$.

2.5.B. Good fundamental systems of neighborhoods, and ts -submanifolds.

Definition 2.5.4. Let X be a topological space, $\mathbb{A} = \{A_0, \dots, A_k\}$ a compact family of subsets of X , I an open interval $(0, \delta_0)$. A *good fundamental system of neighborhoods of \mathbb{A}* (briefly: a *gfn*-system for \mathbb{A}) is a family $\mathbf{A} = \{A_s(\delta)\}_{\delta \in I, 0 \leq s \leq k}$ of open subsets of X such that for any positive integer $s \leq k$ the following conditions are fulfilled:

- (FS1) for every $\delta \in I$ we have $A_s \subset A_s(\delta)$;
- (FS2) $\delta_1 < \delta_2 \Rightarrow A_s(\delta_1) \subset A_s(\delta_2)$;
- (FS3) $\overline{A_{\leq s}(\delta)} = \bigcap_{\theta > \delta} (A_{\leq s}(\theta))$ for every $\delta \in I$.
- (FS4) we have $A_{\leq s} = \bigcap_{\theta > 0} (A_{\leq s}(\theta))$.

The interval I is called the *interval of subdefinition* of the *gfn*-system and \mathbb{A} is called the *core* of \mathbf{A} . We shall denote $A_s(\delta)$ also by $\mathbf{A}_{(s)}(\delta)$ and $A_{\leq i}(\delta)$ also by $\mathbf{A}_{(\leq i)}(\delta)$.

Let M be a manifold without boundary, \mathbb{X} an s -submanifold of M , and \mathbf{X} a *gfn*-system for \mathbb{X} . We say that \mathbf{X} is a *ts-submanifold of M with core \mathbb{X}* . For a *ts*-submanifold $\mathbf{X} = \{X_s(\delta)\}_{\delta \in I, 0 \leq s \leq k}$, we shall also denote $X_i(\delta)$ by $\mathbf{X}_{(i)}(\delta)$, and $X_{\leq i}(\delta)$ by $\mathbf{X}_{(\leq i)}(\delta)$. The basic example of a *gfn*-system and a *ts*-manifold is given by the next lemma.

Lemma 2.5.5. *Let M be a closed Riemannian manifold or a Riemannian cobordism of dimension n . Let $f : M \rightarrow \mathbb{R}$ be a Morse function and v a δ -separated f -gradient. Then the family*

$$\mathbf{D}(v) = \{B_\delta(\text{ind}=s; v)\}_{\delta \in]0, \epsilon[, 0 \leq s \leq n}$$

*is a *gfn*-system for $\mathbb{D}(v)$. If M is a closed manifold, this family is a *ts*-submanifold with core $\mathbb{D}(v)$.*

Observe that there is no canonical way for choosing an interval of definition for this system. We shall say that $\lambda > 0$ is in an *interval of definition of $\mathbf{D}(v)$* if there is $\epsilon > \lambda$ such that $\{B_\delta(\text{ind}=s; v)\}_{\delta \in]0, \epsilon[, 0 \leq s \leq n}$ is a *gfn*-system. If v is λ -separated, then λ is in an interval of definition of $\mathbf{D}(v)$.

2.5.C. Tracks of subsets, s -submanifolds, and ts -submanifolds.

Definition 2.5.6. Let $f : W \rightarrow [a, b]$ be a Morse function on a compact Riemannian cobordism W , and v be an f -gradient. Let $X \subset V_1$. The set

$\{\gamma(x, t; -v) \mid t \geq 0, x \in X\}$ is called the *track* of X (with respect to v) and is denoted by $T(X, v)$.

Lemma 2.5.7. (1) *If X is compact, then $T(X, v) \cup D(v)$ is compact.*

(2) *If X is compact, and every $(-v)$ -trajectory starting at a point of X reaches V_0 , then $T(X, v)$ is compact.*

(3) *For any X we have $\overline{T(X, v) \cup D(v)} = T(\overline{X}, v) \cup D(v)$.*

(4) *For any X and any $\delta > 0$ we have $\overline{T(X, v) \cup B_\delta(v)} = T(\overline{X}, v) \cup D_\delta(v)$.*

Definition 2.5.8. Let v be an f -gradient satisfying the almost transversality condition. Let \mathbb{A} be an s -submanifold $\{A_0, \dots, A_k\}$ of V_1 such that $\mathbb{A} \uparrow \mathbb{D}_b(-v)$. By convention we write $A_{-1} = \emptyset$. Put

$$TA_i(v) = T(A_{i-1}, v) \cup D(\text{ind}=i; v);$$

The family

$$\mathbb{T}(\mathbb{A}, v) = \{TA_i(v)\}_{0 \leq i \leq k+1};$$

will be called the *track* of \mathbb{A} , and the family

$$(-v)_{[b, \lambda]}^{\rightsquigarrow}(\mathbb{A}) = \{TA_{i+1}(v) \cap f^{-1}(\lambda)\}_{0 \leq i \leq k}.$$

will be called the $(-v)_{[b, \lambda]}^{\rightsquigarrow}$ -*image* of \mathbb{A} . If the values b, λ are clear from the context, we shall abbreviate $(-v)_{[b, \lambda]}^{\rightsquigarrow}(\mathbb{A})$ to $(-v)^{\rightsquigarrow}(\mathbb{A})$.

Lemma 2.5.9. (1) $\mathbb{T}(\mathbb{A}, v)$ and $(-v)_{[b, \lambda]}^{\rightsquigarrow}(\mathbb{A})$ are compact families.

(2) *If λ is a regular value of f , then $(-v)_{[b, \lambda]}^{\rightsquigarrow}(\mathbb{A})$ is an s -submanifold of $f^{-1}(\lambda)$.*

Let us proceed to tracks of ts -submanifolds.

Definition 2.5.10. Let $\mathbf{A} = \{A_s(\delta)\}_{\delta \in (0, \delta_0), 0 \leq s \leq k}$ be a ts -submanifold of V_1 with core \mathbb{A} . Assume that v is δ_1 -separated. For $0 < \delta < \min(\delta_0, \delta_1)$ and $0 \leq s \leq k+1$, we put

$$TA_s(\delta, v) = T(A_{s-1}(\delta), v) \cup B_\delta(\text{ind}=s; v)$$

(by definition, $A_{-1}(\delta) = \emptyset$).

It may happen that in order to turn the family $\{TA_s(\delta, v)\}$ into a gfn -system, it is necessary to reduce its initial interval of definition.

Definition 2.5.11. Let $I = (0, \mu)$, and let $Z = \{Z(\delta)\}_{\delta \in I}$ be a family of subsets of some space X . Let $0 < \nu < \mu$. The family $\{Z(\delta)\}_{\delta \in (0, \nu)}$ will be called a *restriction* of Z to the interval $(0, \nu)$.

It is proved in [15, Lemma 2.11 and Proposition 2.12] that there is $\epsilon \in (0, \min(\delta_0, \delta_1))$ such that the restriction of the family $TA_s(\delta, v)$ to the interval $(0, \epsilon)$ is a *gfn*-system with the core $\mathbf{T}(\mathbf{A}, v)$.

Definition 2.5.12. This *gfn*-system will be denoted by $\mathbf{T}(\mathbf{A}, v)$; we call it the *track* of \mathbf{A} .

Observe that there is no canonical choice of an interval of definition for this system.

2.6. Ranging pairs. In this subsection, W is a Riemannian cobordism, $f : W \rightarrow [a, b]$ a Morse function, v an f -gradient satisfying the almost transversality condition, $n = \dim W$.

Definition 2.6.1. Let $\mathcal{V}_0 = (\phi_0, u_0, \mathcal{U}_0)$, $\mathcal{V}_1 = (\phi_1, u_1, \mathcal{U}_1)$ be \mathcal{AM} -flows on V_0 and V_1 . We say that $(\mathcal{V}_0, \mathcal{V}_1)$ is a *ranging pair* for (f, v) if $\mathbb{D}_a(v) \nmid \mathbb{D}(-u_0)$, $\mathbb{D}_b(-v) \nmid \mathbb{D}(u_1)$, and there is a number $\delta > 0$ such that for any $0 \leq s \leq n-1$ the following conditions are fulfilled:

(RP1) δ is in intervals of definition of $\mathbf{T}(\mathbf{D}(u_1), v)$ and of $\mathbf{T}(\mathbf{D}(-u_0), -v)$;

(RP2) the gradients v , u_0 , and u_1 are δ -separated;

(RP3) $\overline{\mathbf{T}(\mathbf{D}(u_1), v)_{(\leq s+1)}(\delta)} \cap V_0 \subset \mathbf{D}(u_0)_{(\leq s)}(\delta)$;

(RP4) $\overline{\mathbf{T}(\mathbf{D}(-u_0), -v)_{(\leq s+1)}(\delta)} \cap V_1 \subset \mathbf{D}(-u_1)_{(\leq s)}(\delta)$.

We say that v satisfies condition (RP) if (f, v) has a ranging pair.

The next theorem ([15, Theorem 4.3]) is one of the basic results of [15]. We will not comment on the proof (it occupies §3 and first 2 pages of §4 of [15]), since the techniques of the proof will not be used in the present paper.

Theorem 2.6.2. *Let $\epsilon > 0$. Then there is an f -gradient w satisfying the almost transversality condition and a ranging pair $(\mathcal{V}_0, \mathcal{V}_1)$ for (f, w) such that $\|w - v\| \leq \epsilon$. Moreover given a neighborhood U of ∂W we can assume that $\text{supp}(w - v) \subset U \setminus \partial W$.*

Assume now that (f, v) has a ranging pair $(\mathcal{V}_0, \mathcal{V}_1)$. We will now show that this ranging pair generates a family of ranging systems for (f, v) . Moreover, for every $p, q \in S(f)$ with $\text{ind } p = \text{ind } q + 1$, in this family there is a ranging system adapted to p, q .

Let $\Lambda = \{\lambda_0, \dots, \lambda_k\}$ be a set of regular values of f such that

$$\lambda_i < \lambda_{i+1}, \lambda_0 = a, \lambda_k = b,$$

and in each $[\lambda_i, \lambda_{i+1}]$ there is only one critical value of f . We choose some $\delta' > \delta$ in the interval of definition of $\mathbf{T}(\mathbf{D}(u_1), v)$ and $\mathbf{T}(\mathbf{D}(-u_0), -v)$ such

that for every $0 \leq s \leq n - 1$ we have

$$\begin{aligned} \overline{\mathbf{T}(\mathbf{D}(u_1), v)_{(\leq s+1)}(\delta')} \cap V_0 &\subset \mathbf{D}(u_0)_{(\leq s)}(\delta), \\ \overline{\mathbf{T}(\mathbf{D}(-u_0), -v)_{(\leq s+1)}(\delta')} \cap V_1 &\subset \mathbf{D}(-u_1)_{(\leq s)}(\delta) \end{aligned}$$

(such a number δ' exists because δ belongs to an interval of definition of $\mathbf{T}(\mathbf{D}(u_1), v)$ and of $\mathbf{T}(\mathbf{D}(-u_0), -v)$). Also, we require that the gradients v , u_0 , u_1 be δ' -separated.

Now, for each integer s with $0 \leq s \leq n$ we define compact subsets $A_\lambda^{(s)}$ and $B_\lambda^{(s)}$ of $f^{-1}(\lambda)$. We set $A_\lambda^{(0)} = B_\lambda^{(0)} = \emptyset$. Let

$$\delta = \delta_k < \delta_{k-1} < \cdots < \delta_1 < \delta_0 = \delta'$$

be a sequence of real numbers, and let $1 \leq s \leq n$. We put

$$\begin{aligned} A_{\lambda_l}^{(s)} &= \overline{\mathbf{T}(\mathbf{D}(u_1), v)_{(\leq s)}(\delta_l)} \cap f^{-1}(\lambda_l) \quad \text{for } 0 < l \leq k, \\ A_a^{(s)} &= \overline{\mathbf{D}(u_0)_{(\leq s-1)}(\delta)} = D_\delta(\text{ind} \leq s-1; u_0); \\ B_{\lambda_l}^{(s)} &= \overline{\mathbf{T}(\mathbf{D}(-u_0), -v)_{(\leq s)}(\delta_{k-l})} \cap f^{-1}(\lambda_l) \quad \text{for } 0 \leq l < k, \\ B_b^{(s)} &= \overline{\mathbf{D}(-u_1)_{(\leq s-1)}(\delta)} = D_\delta(\text{ind} \leq s-1; -u_1). \end{aligned}$$

Clearly, $A_b^{(s)} = D_\delta(\text{ind} \leq s-1; u_1)$ and $B_a^{(s)} = D_\delta(\text{ind} \leq s-1; -u_0)$.

The following lemma is proved in [15, Lemma 4.14].

Lemma 2.6.3. (1) $A_\lambda^{(r)} \cap B_\lambda^{(s)} = \emptyset$ if $r + s \leq n$.

(2) If $0 \leq s \leq n - 1$, then $\{(A_\lambda^{(s)}, B_\lambda^{(n-s-1)})\}_{\lambda \in \Lambda}$ is a ranging system for (f, v) adapted to every pair p, q of critical points of f with $\text{ind } p = s + 1$, $\text{ind } q = s$.

(3) For every $0 \leq s \leq n - 1$ the pair

$$(V_1 \setminus B_b^{(n-s-1)}, A_b^{(s)})$$

is homotopy equivalent to a finite CW-pair that has cells of dimension s only.

In the case when W is cyclic we can strengthen the results above so to construct cyclic ranging systems for C^0 -generic gradients.

Definition 2.6.4. Let $f: W \rightarrow [a, b]$ be a Morse function on a cyclic cobordism W , and let v be an f -gradient satisfying the almost transversality condition. An \mathcal{M} -flow $\mathcal{V}_0 = (f, v, \mathcal{U})$ on V_0 is called a *ranging flow* for (f, v) if $(\mathcal{V}_0, \Phi(\mathcal{V}_0))$ is a ranging pair for (f, v) .

We say that v satisfies condition (RF) if there is a ranging flow for (f, v) .

The proof of the following theorem is similar to the proof of Theorem 2.6.2.

Theorem 2.6.5. *Let $f : W \rightarrow [a, b]$ be a Morse function on a cyclic cobordism W . Let $\epsilon > 0$. There is an f -gradient w satisfying the almost transversality condition and a ranging flow $(\mathcal{V}_0, \mathcal{V}_1)$ for (f, w) such that Moreover, given a neighborhood U of ∂W we can assume that*

$$\text{supp}(w - v) \subset U \setminus \partial W.$$

Definition 2.6.6. A ranging system $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ for (f, v) is said to be *cyclic* if $\Phi(A_a) = A_b$ and $\Phi(B_a) = B_b$. A cyclic ranging system is said to be of *finite type* if $H_*(V_0 \setminus B_a, A_a)$ is finitely generated.

Similarly to the above for every ranging flow and any pair p, q of critical points of adjacent indices there is a finite type cyclic ranging system adapted to p, q . We arrive therefore at the following result which together with theorem 2.3.4 proves the Main Theorem of [15].

Theorem 2.6.7. *Let $f : M \rightarrow S^1$ be a Morse function. Let also p, q be critical points of f such that $\text{ind } p = \text{ind } q + 1$. Then the set of transverse f -gradients admitting a cyclic ranging system adapted to p, q is C^0 -dense in the set of all transverse f -gradients.*

3. MORSE MAPS $M \rightarrow S^1$

3.1. Algebraic preliminaries. We will work here with the terminology of §1.3.1.

Definition 3.1.1. We say that an element $\varkappa \in \mathbb{Z}(\pi_1 M)_\xi^-$ is of type (\mathcal{L}) , if there are $r, q \in G$, a natural number m , an $m \times m$ -matrix $\mathcal{A} = (a_{ij})_{1 \leq i, j \leq m}$ where $a_{ij} \in \mathbb{Z}G_{(-1)}$ and vectors $(X_i)_{1 \leq i \leq m}, (Y_i)_{1 \leq i \leq m}, X_i, Y_i \in \mathbb{Z}H$, such that

$$\varkappa = r \left(\sum_{s \geq 0} \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}} Y_i a_{ij}^{(s)} X_j \right) q$$

where $a_{ij}^{(s)}$ are the entries of \mathcal{A}^s .

The next lemma follows from the definition of the homomorphism ℓ .

Lemma 3.1.2. *The elements of type (\mathcal{L}) are contained in $\text{Im } \ell$.*

Proposition 3.1.3. *The elements of type (\mathcal{L}) are of exponential growth.*

Proof. It suffices to prove that every matrix entry of the matrix series $u = \sum_{s \geq 0} A^s$, where $A = (a_{ij})$ and $a_{ij} \in \mathbb{Z}G_{(-1)}$, is of exponential growth. For an $(m \times m)$ -matrix $B = (b_{ij})$ we denote by $\|B\|$ the number $\max_{i,j} \|b_{ij}\|$.

It is easy to check that $\|BC\| \leq \|B\| \cdot \|C\| \cdot m$. Let A be $(m \times m)$ -matrix. Then $\|A^s\| \leq \|A\|^s \cdot m^{s-1} \leq \|A\|^s \cdot m^s$. Let $1 \leq i, j \leq m$. Write $u_{ij} = \sum n_g g$. Then for $k \geq 0$ we have $\sum_{\xi(g)=-k} |n_g| \leq (m \cdot \|A\|)^k$. If $m \|A\| \leq 1$ this gives

$$\sum_{\xi(g) \geq -k} |n_g| \leq k + 1 \leq e^k. \text{ If } m \|A\| > 1 \text{ we have}$$

$$\sum_{\xi(g) \geq -k} |n_g| \leq \frac{(m \|A\|)^k - 1}{(m \|A\|) - 1} < D(m \|A\|)^k.$$

Therefore, in any case there are $c, d > 0$ such that for $k < 0, k \in \mathbb{Z}$ we have

$$\sum_{\xi(g) \geq k} |n_g| \leq c \cdot d^{-k}.$$

For $k \geq 0$ it is true obviously and this implies that u is of exponential growth. \square

3.2. Statement of Theorem 3.2.1. Theorem A follows immediately from the next theorem.

Theorem 3.2.1. *In the set $\mathcal{G}t(f)$ there is a subset $\mathcal{G}t_1(f)$ with the following properties:*

- (1) $\mathcal{G}t_1(f)$ is open and dense in $\mathcal{G}t(f)$ with respect to C^0 topology.
- (2) If $v \in \mathcal{G}t_1(f)$ then for every $x, y \in S(f)$ with $\text{ind } x = \text{ind } y + 1$ the incidence coefficient $\tilde{n}(\tilde{x}, \tilde{y}; v)$ is of type (\mathcal{L}) .
- (3) Let $v \in \mathcal{G}t_1(f)$. Let U be a neighborhood of $S(f)$. Then for every $w \in \mathcal{G}t_1(f)$ such that $w = v$ in U and w is sufficiently close to v in C^0 topology we have: $\tilde{n}(\tilde{x}, \tilde{y}; v) = \tilde{n}(\tilde{x}, \tilde{y}; w)$ for every $x, y \in S(f)$ such that $\text{ind } x = \text{ind } y + 1$.

3.3. Generalities on intersection indices. Let M be a manifold without boundary, $\mathcal{Q} : \widehat{M} \rightarrow M$ be a regular covering (not necessarily connected) with structure group H . We say, that a submanifold N of M is *lifted-oriented* (resp. *lifted-cooriented*) if a lift $\hat{i} : N \hookrightarrow \widehat{M}$ of the inclusion map $i : N \hookrightarrow M$ is fixed, and N is oriented (resp. cooriented). We shall denote $\hat{i}(N)$ by \widehat{N} .

Let $X \subset M$ and let N be a lifted-oriented submanifold of M such that $N \setminus \text{Int} X$ is compact. Then $\widehat{N} \setminus \text{Int} \mathcal{Q}^{-1}(X)$ is compact, and the orientation

class

$$\mu_{N \setminus \text{Int} \mathcal{Q}^{-1}(X)} \in H_n(N, N \cap \text{Int} \mathcal{Q}^{-1}(X))$$

is defined, where $n = \dim N$ (see [7, Theorem A8]). The image of this class in $H_n(\widehat{M}, \mathcal{Q}^{-1}(X))$ will be denoted by $[\widehat{N}]_{M,X}$ (or simply by $[\widehat{N}]$ if this does not lead to confusion).

Let L be a compact lifted-cooriented submanifold without boundary of M . Then there is the coorientation class

$$]\widehat{L}[\in H^{m-l}(\widehat{M}, \widehat{M} \setminus \widehat{L}),$$

where $l = \dim L$, $m = \dim M$.

Assume that $X \cap L = \emptyset$, $N \pitchfork L$ and $n + l = m$. Then \widehat{N} is transversal to \widehat{L} , the set $\widehat{N} \cap \widehat{L}$ is finite, and the intersection index $\widehat{N} \sharp \widehat{L} \in \mathbb{Z}$ is defined. Denote by j the inclusion $(\widehat{M}, \mathcal{Q}^{-1}(X)) \hookrightarrow (\widehat{M}, \widehat{M} \setminus \widehat{L})$. The next lemma is standard.

Lemma 3.3.1. $\widehat{N} \sharp \widehat{L} = j^*(]\widehat{L}[)([\widehat{N}])$. □

3.4. Lifts of ranging systems. Let $f : W \rightarrow [a, b]$ be a Morse function on a compact Riemannian cobordism, $f^{-1}(b) = V_1$, $f^{-1}(a) = V_0$, v be an f -gradient. Let $\mathcal{Q} : \widehat{W} \rightarrow W$ be a regular covering with a structure group H . The lift of v to \widehat{W} will be denoted by \widehat{v} . If $x \in W$ and $\gamma(x, t; v)$ is defined on $[0, a]$, and $\widehat{x} \in \mathcal{Q}^{-1}(x)$, then the lift to \widehat{W} of $\gamma(x, \cdot; v)$, starting at \widehat{x} is the \widehat{v} -trajectory $\gamma(x, \cdot; \widehat{v})$. It is easy to define with the help of this lifting procedure a diffeomorphism

$$\widehat{v}^{\rightsquigarrow} : \mathcal{Q}^{-1}(V_1 \setminus K(-v)) \rightarrow \mathcal{Q}^{-1}(V_0 \setminus K(v)).$$

For $X \subset V_1$ we denote by abuse of notation $\widehat{v}^{\rightsquigarrow}(X \setminus K(-v))$ by $\widehat{v}^{\rightsquigarrow}(X)$.

Let N be a oriented-lifted submanifold of V_1 . Then it is easy to see that $\widehat{v}^{\rightsquigarrow}(N)$ is a oriented-lifted submanifold of V_0 .

Proposition 3.4.1. *Let $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ be a ranging system for (f, v) . There is a homomorphism*

$$\widehat{H}(v) : H_*(\mathcal{Q}^{-1}(V_1 \setminus B_b), \mathcal{Q}^{-1}(A_b)) \rightarrow H_*(\mathcal{Q}^{-1}(V_0 \setminus B_a), \mathcal{Q}^{-1}(A_a))$$

of right $\mathbb{Z}H$ -modules, such that:

(1) *If N is an oriented-lifted submanifold of $V_1 \setminus B_b$, such that $N \setminus \text{Int} A_b$ is compact, then*

$$\widehat{H}(v)([\widehat{N}]) = [(\widehat{v}^{\rightsquigarrow}(N))^\wedge].$$

(2) *There is an $\epsilon > 0$ such that for every f -gradient w with $\|w - v\| < \epsilon$ we have $\widehat{H}(v) = \widehat{H}(w)$.*

Proof. An easy induction argument shows that it is sufficient to prove the proposition in the case $\text{card}\Lambda = 1$. Split the set $S(F)$ of all critical points of f in a disjoint union of two subsets $S1(f)$ and $S2(f)$ in such a way that

- for every $p \in S1(f)$ the condition (RS2) i) of Definition 2.2.1 holds,
- for every $p \in S2(f)$ the condition (RS2) ii) of Definition 2.2.1 holds.

Pick Morse functions $\phi_1, \phi_2 : W \rightarrow [a, b]$, adjusted to (f, v) , such that there is a regular value μ_1 of ϕ_1 and a regular value μ_2 of ϕ_2 satisfying the following:

- for every $p \in S1(f)$ we have: $\phi_1(p) < \mu_1$ and $\phi_2(p) > \mu_2$.
- for every $p \in S2(f)$ we have: $\phi_1(p) > \mu_1$ and $\phi_2(p) < \mu_2$.

Let $\delta > 0$. Put

$$D1_\delta(v) = V_0 \cap \bigcup_{p \in S1(f)} D_\delta(p, v), \quad D1_\delta(-v) = V_1 \cap \bigcup_{p \in S1(f)} D_\delta(p, -v).$$

By a certain abuse of notation the intersection of $\cup_{p \in S1(f)} D(p, v)$ with V_0 will be denoted by $D1_0(v)$. Put

$$\Delta(\delta, -v) = D2_\delta(-v) \cup (-v) \overset{\rightsquigarrow}{\rightsquigarrow} (B_a), \quad \nabla(\delta, v) = D1_\delta(v) \cup \overset{\rightsquigarrow}{\rightsquigarrow} v (A_b).$$

The similar notations like

$$D2_\delta(-v), \quad \Delta(0, -v), \quad D2_0(v)$$

etc. are now clear without special definition. For $\delta > 0$ sufficiently small we have

$$\begin{aligned} \forall p \in S1(f) : D_\delta(p) \subset \phi_1^{-1}((a, \mu_1)) \quad \text{and} \quad D_\delta(p) \subset \phi_2^{-1}((\mu_2, b)); \\ \forall p \in S2(f) : D_\delta(p) \subset \phi_1^{-1}((\mu_2, b)) \quad \text{and} \quad D_\delta(p) \subset \phi_2^{-1}((a, \mu_2)). \end{aligned} \quad (\text{D1})$$

It is easy to prove that for $\delta > 0$ sufficiently small we have:

$$\nabla(\delta, v) \subset \text{Int}A_a, \quad \Delta(\delta, -v) \subset \text{Int}B_b, \quad \Delta(\delta, -v) \cap D1_\delta(-v) = \emptyset. \quad (\text{D2})$$

Fix some $\delta > 0$ satisfying (D1) and (D2).

Let $0 \leq \mu' < \mu \leq \delta$ and U be any subset of V_1 such that

$$\Delta(0, -v) \subset U \subset B_b \quad \text{and} \quad U \cap D1_\delta(-v) = \emptyset.$$

(for example $U = \Delta(\delta, -v)$ will do). Consider the following sequence of homomorphisms

$$\begin{aligned} H_*(\mathcal{Q}^{-1}(V_1 \setminus B_b), \mathcal{Q}^{-1}(A_b)) &\xrightarrow{\hat{I}_*} H_*(\mathcal{Q}^{-1}(V_1 \setminus U), \\ \mathcal{Q}^{-1}(A_b \cup D1_\mu(-v))) &\xrightarrow{\text{Exc}^{-1}} H_*(\mathcal{Q}^{-1}(V_1 \setminus (U \cup D1_{\mu'}(-v))), \\ \mathcal{Q}^{-1}((A_b \cup D1_\mu(-v)) \setminus D1_{\mu'}(-v)) &\xrightarrow{\hat{v}_*} H_*(\mathcal{Q}^{-1}(V_0 \setminus B_a), \mathcal{Q}^{-1}(A_a)). \end{aligned}$$

Here \widehat{I} is the corresponding inclusion. Note that the last arrow is well defined since

$$(-v)^{\rightsquigarrow}(B_a) \subset U \quad \text{and} \quad D_0(-v) \cap V_1 \subset D2_0(-v) \cup D1_{\mu'}(-v).$$

All the three arrows are homomorphisms of right $\mathbb{Z}H$ -modules. (This is obvious for the first two arrows, as for the last just observe that $\widehat{v}^{\rightsquigarrow}$ commutes with the right action of H .)

The composition $\widehat{v}_*^{\rightsquigarrow} \circ \text{Exc}^{-1} \circ \widehat{I}_*$ of this sequence will be denoted by $\widehat{H}(v; \mu', \mu; \mathcal{U})$. An argument similar to the one in the beginning of the page 999 of [15] shows that this homomorphism does not depend neither on the choices of U, μ', μ, δ , nor on the choice of presentation

$$S(f) = S1(f) \sqcup S2(f)$$

(if there is more then one such presentation). Therefore this homomorphism is determined by v , the ranging system $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ and the covering $\mathcal{Q} : \widehat{W} \rightarrow W$. We shall denote it by $\widehat{H}(v)$.

The proof of properties (1) and (2) of $\widehat{H}(v)$ is similar of the proof of properties (1) and (2) of [15, Proposition 4.10] and will be omitted. \square

3.5. Lifts of equivariant ranging systems and the proof of Theorem 3.2.1.

We return here to the terminology of §1.3.1. We begin by some algebraic preliminaries. Let M, N be right $\mathbb{Z}H$ -modules and $f : M \rightarrow N$ be a homomorphism of abelian groups. We say that f is θ -semilinear, if we have $f(xh) = f(x)\theta h\theta^{-1}$ for every $x \in M$. If M, N are free finitely generated $\mathbb{Z}H$ -modules with bases $(e_j), (d_i)$, one can associate to each θ -semilinear homomorphism $f : M \rightarrow N$ a matrix $M(f) = (m_{ij})$ by the following rule: $f(e_j) = \sum_i d_i m_{ij}$.

If M is a right $\mathbb{Z}H$ -module and $f : M \rightarrow \mathbb{Z}$ is a homomorphism of abelian groups, then we shall say that f is of finite type, if for every $m \in M$ the set of $h \in H$, such that $f(mh) \neq 0$ is finite. If $f : M \rightarrow \mathbb{Z}$ is a homomorphism of a finite type, then we define a homomorphism $\tilde{f} : M \rightarrow \mathbb{Z}H$ of $\mathbb{Z}H$ -modules by $\tilde{f}(m) = \sum_{h \in H} f(mh)h^{-1}$.

Returning to Morse maps, let us assume that $f : M \rightarrow S^1$ belongs to an indivisible cohomology class in $H^1(M, \mathbb{Z})$. Further, denote by $\pi : \overline{M} \rightarrow M$ the (unique) infinite cyclic covering, such that $f \circ \pi \sim 0$. The universal covering $p : \widetilde{M} \rightarrow M$ factors as $p = \pi \circ \mathcal{Q}$ where $\mathcal{Q} : \widetilde{M} \rightarrow \overline{M}$ is a covering with structure group $H = \text{Ker } \xi$. Let u be an f -gradient, its lift to \overline{M} will be denoted by the same letter u . Let $\{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma}$ be a t -equivariant ranging system for (F, u) .

For $\nu, \mu \in \Sigma$, $\nu < \mu$ denote by $\widehat{H}_{[\mu, \nu]}(u)$ the homomorphism

$$\widehat{H}(u|_{F^{-1}([\nu, \mu])}),$$

associated by virtue of Proposition 3.4.1 to the ranging system

$$\{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma, \nu \leq \sigma \leq \mu}$$

and the covering \mathcal{Q} (restricted to the cobordism $F^{-1}([\nu, \mu])$). Denote by $\widehat{H}_{[\mu, \mu]}(u)$ the identity homomorphism of

$$H_*(\mathcal{Q}^{-1}(F^{-1}(\mu) \setminus B_\mu), \mathcal{Q}^{-1}(A_\mu))$$

to itself.

It follows from the construction that for every $g \in G$ with $\xi(g) = k \in \mathbb{Z}$ we have

$$\widehat{H}_{[\mu+k, \nu+k]}(u) = R(g) \circ \widehat{H}_{[\mu, \nu]}(u) \circ R(g^{-1}).$$

We have also

$$\widehat{H}_{[\nu, \theta]}(u) \circ \widehat{H}_{[\mu, \nu]}(u) = \widehat{H}_{[\mu, \theta]}(u).$$

For $\nu \in \Sigma$ put

$$\widehat{h}_\nu(u) = R(\theta^{-1}) \circ \widehat{H}_{[\nu, \nu-1]}(u).$$

Then $\widehat{h}_\nu(u)$ is a θ -semilinear endomorphism of

$$H_*(\mathcal{Q}^{-1}(F^{-1}(\nu) \setminus B_\nu), \mathcal{Q}^{-1}(A_\nu)).$$

We have obviously $\widehat{H}_{[\nu, \nu-k]}(u) = R(\theta^k) \circ (\widehat{h}_\nu(u))^k$. The next lemma follows from Proposition 3.4.1.

Lemma 3.5.1. *Let $\mu, \nu \in \Sigma$, $\nu \leq \mu$, and $k \in \mathbb{N}$. Let also N be oriented-lifted submanifold of $F^{-1}(\mu) \setminus B_\mu$ such that $N \setminus \text{Int}A_\mu$ is compact, and L be a cooriented-lifted compact submanifold of $F^{-1}(\nu) \setminus A_\nu$. Assume that $\dim N + \dim L = \dim M - 1$. Then:*

- (1) $N'_k = u_{[\mu, \nu-k]}^{\rightsquigarrow}(N)$ is an oriented-lifted submanifold of $F^{-1}(\nu - k) \setminus B_{\nu-k}$ such that $N'_k \setminus \text{Int}A_{\nu-k}$ is compact. If $N'_k \pitchfork Lt^k$, then $N'_k \cap Lt^k$ is finite and

$$\widehat{N}'_k \# \widehat{L}\theta^k = \widehat{i}^*(\widehat{L})\left((\widehat{h}_\nu(u))^k([\widehat{N}'_0])\right),$$

where \widehat{i} is the inclusion map

$$(\mathcal{Q}^{-1}(F^{-1}(\nu) \setminus B_\nu), \mathcal{Q}^{-1}(A_\nu)) \hookrightarrow \mathcal{Q}^{-1}(F^{-1}(\nu), \mathcal{Q}^{-1}(F^{-1}(\nu) \setminus \widehat{L})).$$

- (2) For every f -gradient w , sufficiently close to u in C^0 -topology, the family $\{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma}$ is also a t -equivariant ranging system for (F, w) and $\widehat{h}_\nu(u) = \widehat{h}_\nu(w)$, and $\widehat{H}_{[\mu, \nu]}(u) = \widehat{H}_{[\mu, \nu]}(w)$. \square

3.6. Proof of Theorem 3.2.1. It is easy to see that it suffices to prove our theorem for the case of indivisible homotopy class $[f] \in H^1(M, \mathbb{Z})$ and we make this assumption up to the end of this subsection.

Fix first two points $x, y \in S(f)$, and $x = \text{ind } y + 1$. Recall that we have chosen a lift $\tilde{x} \in \widetilde{M}$ for every $x \in S(f)$. Denote $\mathcal{Q}(\tilde{x})$ by \bar{x} . We can assume that $F(\bar{y}) < F(\bar{x}) \leq F(\bar{y}) + 1$. Denote $\dim M$ by n ; denote $\text{ind } x$ by $l + 1$, then $\text{ind } y = l$. Choose some set Σ of regular values of F , satisfying (\mathcal{S}) of Definition 2.3.1.

Denote by θ the maximal element of Σ with $\theta < F(\bar{x})$ and by $N(v)$ the intersection $D(\bar{x}, v) \cap F^{-1}(\theta)$. Let also $N(v)$ be an oriented submanifold of $F^{-1}(\theta)$ diffeomorphic to S^l . Denote by η the minimal element of Σ , satisfying $\eta > F(\bar{y})$. Then $\eta \leq \theta < \eta + 1$. Let $L(-v)$ be the intersection $D(\bar{y}, -v) \cap F^{-1}(\eta)$. Then $L(-v)$ is a cooriented submanifold of $F^{-1}(\eta)$, diffeomorphic to S^{n-1-l} .

Denote by W the cobordism $F^{-1}([\eta, \eta + 1])$. Then $\bar{x} \in W^\circ$.

Denote $F^{-1}(\eta)$ by V_0 , $F^{-1}(\eta + 1)$ by V_1 , and $\Sigma \cap [\eta, \eta + 1]$ by Λ .

Let $\mathcal{G}t_1(f; x, y)$ be the subset of $\mathcal{G}t(f)$, consisting of all the f -gradients v , such that there is an equivariant ranging system $\{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma}$ for (F, v) satisfying

$$N(v) \cap B_\theta = \emptyset, \quad L(-v) \cap A_\eta = \emptyset, \quad (3.1)$$

$$\left\{ \begin{array}{l} (F^{-1}(\eta) \setminus B_\eta, A_\eta) \text{ has a homotopy type of a finite CW-pair} \\ \text{having only } l\text{-dimensional cells.} \end{array} \right. \quad (3.2)$$

Now we shall prove three properties of the set $\mathcal{G}t_1(f; x, y)$.

1) $\mathcal{G}t_1(f; x, y)$ is an open and dense subset of $\mathcal{G}t(f)$ with respect to C^0 topology.

This can be proved in the same way as the open-and-dense property of $\mathcal{G}t_0(f; x, y)$ in [15, Proposition 4.12 and Theorem 4.16].

2) If $v \in \mathcal{G}t_1(f; x, y)$ then $\tilde{n}(\tilde{x}, \tilde{y}; v)$ satisfies the condition (\mathcal{L}) .

The lifts $x \mapsto \tilde{x}$ and $y \mapsto \tilde{y}$ define a lift $\widehat{N}(v)$ of $N(v)$ to $\mathcal{Q}^{-1}(F^{-1}(\theta))$ and $\widehat{L}(-v)$ of $L(-v)$ to $\mathcal{Q}^{-1}(F^{-1}(\eta))$. The $\mathbb{Z}H$ -module

$$\mathcal{H} = H_l\left(\mathcal{Q}^{-1}(F^{-1}(\eta) \setminus B_\eta), \mathcal{Q}^{-1}(A_\eta)\right)$$

is free. Choose some basis e_1, \dots, e_m of this module. The homomorphism $\widehat{h}_\eta(v)$ of this module is θ -semilinear. Denote by $\mathcal{B} = (b_{ij})$ its matrix, and denote by \mathcal{A} the matrix $(b_{ij}\theta)$. Let $a_{ij}^{(s)}$ be the coefficients of \mathcal{A}^s . Consider the element $\xi = \widehat{H}_{[\theta, \eta]}(v)([\widehat{N}(v)])$ of \mathcal{H} ; let $\xi = \sum e_i X_i$ with $X_i \in \mathbb{Z}H$. Consider $\beta = \widehat{i}^*([\widehat{L}(-v)])$ as a homomorphism of \mathcal{H} to \mathbb{Z} . It is of finite

type. Denote $\tilde{\beta}(e_j)$ by Y_j . Then $Y_j \in \mathbb{Z}H$. We claim that

$$\tilde{n}(\tilde{x}, \tilde{y}; v) = \sum_{\substack{s \geq 0 \\ 1 \leq i \leq m, 1 \leq j \leq m}} Y_i \cdot a_{ij}^{(s)} \cdot X_j. \quad (3.3)$$

To prove it write

$$\tilde{n}(\tilde{x}, \tilde{y}; v) = \sum_{s \geq 0} \left(\sum_{h \in H} \nu(\tilde{x}, \tilde{y}h\theta^s) \cdot h\theta^s \right)$$

(here $\nu(\tilde{x}, \tilde{y}h\theta^s)$ stands for the algebraic number of $(-v)$ -trajectories, joining \tilde{x} with $\tilde{y}h\theta^s$; note that since $F(\tilde{x}) \leq F(\tilde{y}) + 1$, there are no $(-v)$ -trajectories joining \tilde{x} to $\tilde{y}g$ if $\xi(g) > 0$).

To make the following computation more easy to comprehend, we introduce the following terminology conventions (valid only here). The homomorphism $\hat{h}_\eta(v) : \mathcal{H} \rightarrow \mathcal{H}$ will be denoted by μ . We identify the cohomology classes in

$$H^*(\mathcal{Q}^{-1}(F^{-1}(\nu)), \mathcal{Q}^{-1}(F^{-1}(\nu) \setminus \hat{L}))$$

with their images in $H^*(\mathcal{Q}^{-1}(F^{-1}(\nu) \setminus B_\nu), \mathcal{Q}^{-1}(A_\nu))$ (thus suppressing \hat{i}^* in the notation). We have:

$$\tilde{n}(\tilde{x}, \tilde{y}; v) = \sum_{s \geq 0} \left(\sum_{h \in H} ([\hat{L} \cdot h[(\mu^s(\xi))]h]) \theta^s \right)$$

(by Lemma 3.5.1). The latter expression equals

$$\begin{aligned} \sum_{s \geq 0} \left(\sum_{h \in H} ([\hat{L}[(\mu^s(\xi) \cdot h^{-1})]h]) \theta^s \right) &= \sum_{s \geq 0} \left(\sum_{h \in H} \beta(\mu^s(\xi) \cdot h^{-1})h \right) \theta^s = \\ &= \sum_{s \geq 0} \tilde{\beta}(\mu^s(\xi)) \cdot \theta^s. \end{aligned}$$

To obtain from this expression the formula (3.3) we need only a lemma, allowing to calculate $\mu^s(\xi)$ in terms of the coordinates of ξ and the matrix of μ (the expression differs slightly from the standard linear-algebraic one since μ is θ -semilinear).

Lemma 3.6.1. *Let F be a free $\mathbb{Z}H$ -module with a basis e_1, \dots, e_m and $\mu : F \rightarrow F$ be a θ -semilinear homomorphism of F . Let m_{ij} be its matrix. Denote by M the $m \times m$ matrix $(m_{ij} \cdot \theta)$. Let also $\xi \in F$ and $\xi = \sum e_j \xi_j$. Then for every natural $s \geq 0$ we have*

$$\mu^s(\xi) = \sum_{i,j} e_i [M^s]_{ij} \xi_j \theta^{-s}$$

Proof. We will use induction in s . We have

$$\begin{aligned}
\mu^{s+1}(\xi) &= \mu(\mu^s(\xi)) = \mu\left(\sum_i e_i \left(\sum_j [M^s]_{ij} \xi_j \theta^{-s}\right)\right) = \\
&= \sum_i \mu(e_i) \cdot \left(\theta \sum_j [M^s]_{ij} \xi_j \theta^{-s-1}\right) = \\
&= \sum_i \left(\sum_k e_k m_{ki}\right) \cdot \left(\theta \sum_j [M^s]_{ij} \xi_j \theta^{-s-1}\right) = \\
&= \sum_k e_k \left(\sum_i (m_{ki} \theta) \cdot [M^s]_{ij}\right) \xi_j \theta^{-s-1}. \quad \square
\end{aligned}$$

Now substitute the expression for $\mu^s(\xi)$ into the above formula, and the proof of (3.3) is over.

3) Stability with respect to perturbations of the gradient.

Let $v \in \mathcal{G}t_1(f; x, y)$ and U be a neighborhood of $S(f)$. We are going to prove that there is $\epsilon > 0$ such that for every $w \in \mathcal{G}t_1(f; x, y)$ with $\|w - v\| < \epsilon$ and $w|_U = v|_U$ we have: $\tilde{n}(\tilde{x}, \tilde{y}; v) = \tilde{n}(\tilde{x}, \tilde{y}; w)$.

Let w be an f -gradient, sufficiently close to v . Then $\{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma}$ is still a t -equivariant ranging system for (F, w) , satisfying the conditions (3.1) and (3.2). It is not difficult to see that

$$[\widehat{N}(w)] = [\widehat{N}(v)], \quad [\widehat{L}(-w)] = [\widehat{L}(-v)].$$

Then Lemma 3.5.1 together with the formula (3.3) completes the proof.

Define now the set $\mathcal{G}t_1(f)$ to be the intersection of all the sets $\mathcal{G}t_1(f; x, y)$ where x, y range over the critical points of f such that $\text{ind } x = \text{ind } y + 1$, and Theorem 3.2.1 is proved. \square

Remark 3.6.2. There is an obvious analogue of Theorem 3.2.1 or any regular covering $\phi : \widehat{M} \rightarrow M$ such that $f \circ \phi$ is homotopic to zero.

4. MORSE FORMS WITHIN ARBITRARY COHOMOLOGY CLASSES

Let ω be a Morse form of irrationality degree q , v be an ω -gradient. Let $Q : N \rightarrow M$ be the minimal free abelian covering such that $Q^* \omega$ is an exact form, that is, $Q^* \omega = dF$ where $F : N \rightarrow \mathbb{R}$ is a Morse function. The structure group of Q is isomorphic to \mathbb{Z}^m . If $m = 1$, the level surfaces $F^{-1}(a)$ and the cobordisms $F^{-1}([a, b])$ are compact. If $m > 1$ these manifolds are non-compact, and this is the main reason why the results of [15] do not carry over directly to this case $m > 1$.

The way round this difficulty consists in approximating the form ω by a rational 1-form ω_1 , so that $\omega_1 = d\phi$ where ϕ is a circle-valued Morse

function. Then the gradient v is approximated (in C^0 -topology) by a vector field w which is a gradient for both ω and ϕ and such that its incidence coefficients are rational functions.

This program is realized in the present section.

4.1. Algebraic preliminaries. We will need some lemmas about the ring $\mathbb{Z}[\mathbb{Z}^m]$ and its completions and localizations.

Definition 4.1.1. Let $\eta : \mathbb{Z}^m \rightarrow \mathbb{R}$ be a non-zero homomorphism. We extend it to a linear map $\mathbb{R}^m \rightarrow \mathbb{R}$, which will be denoted by the same letter. We say that a set $Z \subset \mathbb{R}^m$ is an η -cone, if there is a compact convex nonempty set $K \subset \eta^{-1}(-1)$ such that

$$Z = \{\lambda z \mid \lambda \in \mathbb{R}, \lambda \geq 0, z \in K\}.$$

We say that Z is (ξ, η) -cone if Z is ξ -cone and η -cone. We say that a set $Z \subset \mathbb{R}^m$ is an *integral* η -cone if there are $e_1, \dots, e_k \in \mathbb{Z}^m$, such that

- (1) $\text{rk}(e_1, \dots, e_k) = m$,
- (2) $\eta(e_i) < 0$,
- (3) $Z = \{\lambda_1 e_1 + \dots + \lambda_k e_k \mid \lambda_i \geq 0\}$.

We shall also write $Z = Z\langle e_1, \dots, e_k \rangle$. We say, that Z is an *integral* (ξ, η) -cone if the vectors e_1, \dots, e_k above satisfy $\xi(e_i) < 0$, $\eta(e_i) < 0$ for all i . Note that an integral (ξ, η) -cone is a (ξ, η) -cone.

Lemma 4.1.2. *Let Z be an (η_1, η_2) -cone. Then there is an integral (η_1, η_2) -cone $Z_0 \supset Z$.*

Proof. We assume that η_1, η_2 are linearly independent; the other case is considered similarly. Denote $\eta_1^{-1}(-1)$ by H , and the set $H \cap \{\eta_2(x) < 0\}$ by H_0 . Then H_0 is an open halfspace of H , containing $Z \cap H$. Denote by \mathcal{L} the set $\{\lambda x \mid \lambda \geq 0, x \in \mathbb{Z}^m\}$, then \mathcal{L} is everywhere dense in H and in H_0 . It is not difficult to prove that there is a finite subset $\mathcal{L}_0 \subset \mathcal{L}$ such that $\mathcal{L}_0 \subset H_0$ and $\langle \mathcal{L}_0 \rangle \supset Z \cap H$. We can choose \mathcal{L}_0 so that $\text{rk} \mathcal{L}_0 = m$ and the lemma is proved. \square

Lemma 4.1.3. *Let $\xi : \mathbb{R}^m \rightarrow \mathbb{R}$ be a non-zero linear form and $\epsilon > 0$. Then there is a finite set I of linear forms $\theta_i : \mathbb{R}^m \rightarrow \mathbb{R}, i \in I$ with $\|\theta_i\| < \epsilon$ such that*

- (1) *The set $\Gamma = \{x \in \mathbb{R}^m \mid (\xi + \theta_i)(x) \leq 0\}$ is a ξ -cone.*
- (2) *There is an integral ξ -cone Γ_0 such that for every family $\{A_i\}_{i \in I}$ of real numbers there is $b \in \mathbb{Z}^m$ with the property:*

$$\{x \in \mathbb{R}^m \mid (\xi + \theta_i)(x) \leq A_i\} \subset \Gamma_0 + b. \quad (4.1)$$

Proof. (1) Pick any m linearly independent linear forms

$$\alpha_1, \dots, \alpha_m : \mathbb{R}^m \rightarrow \mathbb{R}$$

with $\|\alpha_i\| < \min(\epsilon, \|\xi\|/2)$. I claim that the finite family

$$\{\alpha_1, -\alpha_1, \dots, \alpha_m, -\alpha_m\}$$

of linear forms satisfy the conclusions of the Lemma. Note first that

$$Z = \xi^{-1}(-1) \cap \{x \mid \forall i : (\xi + \alpha_i)(x) \leq 0, (\xi - \alpha_i)(x) \leq 0\}$$

is non empty and compact. (Indeed, let x be a vector such that $|x| = 1$ and $\xi(x) = \|\xi\|$. Then $a = -\frac{x}{\xi(x)} \in Z$. Further, if Z is not bounded, then there is a sequence $x_n \in Z$ such that $|x_n| \rightarrow \infty$. Consider the sequence $x_n/|x_n|$. We can assume that it converges to some v with $\|v\| = 1$. Since $\xi(x_n) = -1$, we have $\xi(v) = 0$. Further, for every i we have

$$(\xi + \alpha_i)(v) \leq 0, \quad (\xi - \alpha_i)(v) \leq 0,$$

therefore $\alpha_i(v) = 0$ and $v = 0$.) Further, $\Gamma \subset \xi^{-1}((-\infty, 0])$ and the intersection $\Gamma \cap \xi^{-1}(0)$ consists of 0. Therefore $x \in \Gamma$ and $x \neq 0$ implies $x = \lambda y, y \in Z$.

(2) Choose a vector $x_0 \in \mathbb{Z}^m$ such that $\xi(x_0) < 0$ and for every $1 \leq i \leq m$ we have

$$(\xi + \alpha_i)(x_0) < 0, \quad (\xi - \alpha_i)(x_0) < 0$$

(such a vector exists; it suffices to note that $(\xi \pm \alpha_i)(a) = -1 \mp \frac{\alpha_i(x)}{\xi(x)} < 0$, and to approximate a by an element $y_0 \in \mathbf{Q}^m$).

Denote $|(\xi \pm \alpha_i)(x_0)|$ by $\beta_i^\pm > 0$. Now $(\xi \pm \alpha_i)(x) \leq A_i$ implies

$$(\xi \pm \alpha_i)(x + px_0) \leq A_i - p\beta_i^\pm.$$

Therefore if p is sufficiently big, we obtain (4.1) with $b = -px_0$, or, equivalently, $\{x \in \mathbb{R}^m \mid (\xi + \theta_i)(x) \leq A_i\} \subset \Gamma + b$. Then apply Lemma 4.1.2 to obtain an integral ξ -cone Γ_0 , such that $\Gamma \subset \Gamma_0$. \square

We will use some terminology and results from [11]. Let $\eta : \mathbb{Z}^m \rightarrow \mathbb{R}$ be a homomorphism and $Z = \mathbb{Z}\langle e_1, \dots, e_k \rangle$ be an integral ξ -cone. Denote $\mathbb{Z}^m \cap Z$ by $[Z]$, and the set $\{n_1e_1 + \dots + n_ke_k \mid n_i \in \mathbb{N}\}$ by $[\tilde{Z}]$. Note that $[Z]$ and $[\tilde{Z}]$ are submonoids of \mathbb{Z}^m , $[\tilde{Z}] \subset [Z]$, and $[Z]$ is finitely generated over $[\tilde{Z}]$. Consider the group rings $\mathbb{Z}[Z]$ and $\mathbb{Z}[\tilde{Z}]$ of the monoids Z and \tilde{Z} ; since $\mathbb{Z}[\tilde{Z}]$ is an image of the ring $\mathbb{Z}[t_1, \dots, t_k]$, it is noetherian. The ring $\mathbb{Z}[Z]$ is finitely generated as a left module over $\mathbb{Z}[\tilde{Z}]$, therefore it is also noetherian.

Consider the submonoid $\{x \in Z \mid x \neq 0\}$ of Z and denote its group ring by \mathfrak{m}_Z . It is an ideal of $\mathbb{Z}[Z]$. Denote by $\mathbb{Z}[Z]^\wedge$ the \mathfrak{m}_Z -completion of $\mathbb{Z}[Z]$, and by S_Z the multiplicative set $1 + \mathfrak{m}_Z \subset \mathbb{Z}[Z]$. Then

$$(S_Z^{-1}\mathbb{Z}[Z])^\wedge = (\mathbb{Z}[Z])^\wedge$$

(see [2, Ch. 3, §3, Prop. 12]). The ring $(\mathbb{Z}[Z])^\wedge$ is easily identified with the ring of all the elements $\lambda \in (\mathbb{Z}[Z])^\wedge$ such that $\text{supp } \lambda \subset Z$ (this latter ring is obviously a subring of $\mathbb{Z}[Z]_\eta^-$). The ring $(\mathbb{Z}[Z])^\wedge$ is faithfully flat over $S_Z^{-1}\mathbb{Z}[Z]$ (ibid. Prop. 9). Therefore, if $P, Q \in \mathbb{Z}[Z]$ and there is $x \in (\mathbb{Z}[Z])^\wedge$ such that $P = Qx$, then $x \in S_Z^{-1}\mathbb{Z}[Z]$ (ibid. Ch. 1, §3).

Notice that $\sigma_Z := \{t^I \mid I \in Z\}$ is a multiplicative subset of $\mathbb{Z}[Z]$. It is easy to see that $\sigma_Z^{-1}\mathbb{Z}[Z] = \mathbb{Z}[\mathbb{Z}^m]$ and

$$\sigma_Z^{-1}(\mathbb{Z}[Z]^\wedge) = \{S \in (\mathbb{Z}[Z])^\wedge \mid \exists x \in \mathbb{Z}^m : \text{supp } S \subset Z + x\}.$$

The faithful flatness property cited above implies immediately that if $P, Q \in \mathbb{Z}[\mathbb{Z}^m]$ and there is $x \in \sigma_Z^{-1}(\mathbb{Z}[Z]^\wedge)$ such that $P = Qx$, then

$$x \in \sigma_Z^{-1}S_Z^{-1}\mathbb{Z}[Z].$$

Let η be a linear form. Put

$$\mathfrak{s}_\eta = \{1 + Q \mid \text{supp } Q \subset \eta^{-1}((-\infty, 0))\},$$

then \mathfrak{s}_η is a multiplicative subset of $\mathbb{Z}[\mathbb{Z}^m]$.

Lemma 4.1.4. *Let $\alpha : \mathbb{Z}^m \rightarrow \mathbb{Z}$ be an indivisible homomorphism. Denote $\text{Ker } \eta$ by H . Let A be $(k \times k)$ -matrix, such that $a_{ij} \in \mathbb{Z}[\mathbb{Z}^m]_{(-1)}$. Let $\xi = (\xi_1, \dots, \xi_k), \eta = (\eta_1, \dots, \eta_k)$ be vectors in $(\mathbb{Z}H)^k$. Denote by $a_{ij}^{(s)}$ the (ij) -coefficient of the matrix A^s . Then*

$$(1 - \det A) \sum_{s \geq 0} \left(\sum_{i,j} \xi_i a_{ij}^{(s)} \eta_j \right) \in \mathbb{Z}[\mathbb{Z}^m].$$

Proof. It suffices to prove that every coefficient of the $(k \times k)$ -matrix

$$(1 - \det A) \left(\sum_{s \geq 0} A^s \right)$$

belongs to $\mathbb{Z}[\mathbb{Z}^m]$. Consider the matrix $1 - A$. It is invertible in the ring $S_\alpha^{-1}\mathbb{Z}[\mathbb{Z}^m]$ and the Cramer rules imply $(1 - \det A)(1 - A)^{-1} \in \text{Mat}(\mathbb{Z}[\mathbb{Z}^m])$. On the other hand $(1 - A)^{-1} = \sum_{s \geq 0} A^s$. \square

4.2. Preliminaries on Morse forms and their gradients. In this subsection we will work with the terminology from 2.1. Let ω be a Morse form on a closed connected manifold M . Let $p : \widetilde{M} \rightarrow M$ be the universal covering and let $\widetilde{F} : \widetilde{M} \rightarrow \mathbb{R}$ be a Morse function such that $d\widetilde{F} = p^*\omega$. Note that $\widetilde{F}(xg) = \widetilde{F}(x) + \{\omega\}(g)$, where $g \in \pi_1 M$. Choose a Riemannian metric on M . Then \widetilde{M} obtains a $\pi_1 M$ -invariant Riemannian metric. If v is an ω -gradient we denote by the same letter v its lifts to \widetilde{M} and $\overline{\widetilde{M}}$, since there is no possibility of confusion. Assume that for every $x \in S(f)$ a lift \tilde{x} of x to \widetilde{M} is fixed. The length of a curve γ will be denoted by $l(\gamma)$. The following simple and useful lemma is known since the early 80s. I learned it from J.-Cl. Sikorav.

Lemma 4.2.1. *Let v be an ω -gradient. Then there are constants $A, B > 0$ such that for every $g \in \pi_1 M$ and every $(-v)$ -trajectory γ , joining \tilde{x} and $\tilde{y} \cdot g$ we have:*

$$l(\gamma) \leq A - B\{\omega\}(g).$$

Proof. Choose any $\delta > 0$ less than the injectivity radius of M and less than $\min_{p,q \in S(\omega)} \rho(p, q)$. Then any non-contractible piecewise smooth loop in M is longer than δ . Therefore for any $x \in \widetilde{M}, 1 \neq g \in \pi_1 M$, any piecewise smooth path in \widetilde{M} , joining a point of $D(x, \delta/3)$ with a point of $D(xg, \delta/3)$ has the length $\geq \delta/3$. Also for $x, y \in \widetilde{M}$ with $p(x), p(y) \in S(\omega)$ any piecewise smooth path joining joining a point of $D(x, \delta/3)$ with a point of $D(y, \delta/3)$ has the length $\geq \frac{\delta}{3}$.

Let⁴

$$\{\Phi_p : U_p \rightarrow B^n(0, r)\}_{p \in S(\omega)}$$

be an ω -chart-system such that $\mathcal{G}(\overline{U}_p, \Phi_p) \leq C$ for every $p \in S(\omega)$ and that $rC < \frac{\delta}{12}$. This condition implies in particular that $U_p \subset D(p, \frac{\delta}{12})$. Choose some lifts \tilde{U}_p of neighborhoods U_p , extending $x \mapsto \tilde{x}$. Let $D > 0$ be less than $\min_{x \notin \cup U_p} \omega(v(x))$. Denote $\|v\|$ by E .

Now let γ be a v -trajectory, joining \tilde{x} with $\tilde{y}g$, and let

$$A_0 = \tilde{x}, A_1, \dots, A_n, \tilde{y} = A_{n+1}$$

be the points in $p^{-1}(S(\omega))$ such that γ intersects \tilde{U}_{A_i} . Then

$$(n + 1)\delta/3 \leq l(\gamma).$$

⁴ Recall from [15, p. 975] that for a chart $\Phi : U \rightarrow V \subset \mathbb{R}^n$ of M and $x \in U$ we denote by $\mathcal{G}(x, \Phi)$ the number $\sup_{x \in M, h \in T_x M, h \neq 0} (\max(|h|_\rho / |\Phi_* h|_e, |\Phi_* h|_e / |h|_\rho))$, where ρ stands for the metric on M and e for the euclidean metric in \mathbb{R}^n .

The length of the part of γ inside of $\bigcup_{i=0}^{n+1} \tilde{U}_{A_i}$ is not more than $2rC(n+1)$.

Denote by t_i and τ_i the moments when γ enters and leaves \tilde{U}_{A_i} . We have

$$\int_{\tau_i}^{t_{i+1}} \omega(\dot{\gamma}) dt = - \int_{\tau_i}^{t_{i+1}} \omega(v) dt = \tilde{F}(\gamma(t_{i+1})) - \tilde{F}(\gamma(\tau_i)).$$

Therefore the total time which γ can spend outside $\bigcup_p \tilde{U}_{A_i}$ is not more than $|\tilde{F}(\tilde{y}g) - \tilde{F}(\tilde{x})|/D$, and the length of the corresponding part of the curve is at most $\frac{E}{D} |\tilde{F}(\tilde{y}g) - \tilde{F}(\tilde{x})|$. Since γ joins \tilde{x} with $\tilde{y}g$, the last expression is at most

$$\frac{E}{D} (\tilde{F}(\tilde{x}) - \tilde{F}(\tilde{y}) - \{\omega\}(g)).$$

Therefore

$$\begin{aligned} l(\gamma) &\leq 2rC(n+1) + \frac{E}{D} (\tilde{F}(\tilde{x}) - \tilde{F}(\tilde{y})) - \frac{E}{D} \{\omega\}(g) \leq \\ &\leq \frac{6rC}{\delta} l(\gamma) + \frac{A}{2} - \frac{B}{2} \{\omega\}(g), \end{aligned}$$

where A is chosen so that $A/2 \geq |\tilde{F}(\tilde{x}) - \tilde{F}(\tilde{y})|$ for every $x, y \in S(f)$ and $B = 2E/D$. The inequality $l(\gamma) \leq A - B\{\omega\}(g)$ follows. \square

Let ω be a Morse form, and $\{\Phi_p : U_p \rightarrow B^n(0, r_p)\}_{p \in S(\omega)}$ be an ω -chart-system. Choose a basis a_1, \dots, a_m in $H_1(M, \mathbb{Z})/\text{Tors}$.

Lemma 4.2.2. *There exist closed 1-forms $\lambda_1, \dots, \lambda_m$ on M , such that*

$$\langle [\lambda_i], a_j \rangle = \delta_{ij}$$

and $\text{supp } \lambda_i \cap \bar{U}_p = \emptyset$ for every i and every $p \in S(\omega)$.

Proof. Let θ be any closed 1-form. Let

$$\{\tilde{\Phi}_p : U'_p \rightarrow B^n(0, r'_p)\}_{p \in S(\omega)}$$

be some standard extension of

$$\{\Phi_p : U_p \rightarrow B^n(0, r_p)\}_{p \in S(\omega)}$$

and let ϕ_p be a C^∞ function which equals to 1 in a neighborhood of \bar{U}_p and $\text{supp } \phi_p \subset U'_p$. Let F_p be a function on U'_p , such that $dF_p = \theta$. Consider the form $\theta' = \theta - \sum d(\phi_p F_p)$ where the summation ranges over all zeros of ω . Then $[\theta'] = [\theta]$ and $\theta(x) = 0$ in every \bar{U}_p . \square

Choose and fix closed 1-forms $\lambda_1, \dots, \lambda_m$ on M satisfying the conclusions of the previous lemma. For $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m) \in \mathbb{R}^m$ denote by $\boldsymbol{\mu} \cdot \boldsymbol{\lambda}$ the 1-form $\sum_{i=1}^m \mu_i \lambda_i$, and by $\omega_{\boldsymbol{\mu}}$ the 1-form $\omega + \boldsymbol{\mu} \cdot \boldsymbol{\lambda}$. For $\epsilon > 0$ set

$$\Omega_\epsilon = \{\omega_{\boldsymbol{\mu}} \mid |\boldsymbol{\mu}| = \max_i |\mu_i| \leq \epsilon\}.$$

Definition 4.2.3. We shall say that ω_ϵ is a *Morse family*, if for every $\bar{\epsilon}$ with $|\boldsymbol{\mu}| \leq \epsilon$ the form $\omega_{\boldsymbol{\mu}}$ is a Morse form and $S(\omega_{\boldsymbol{\mu}}) = S(\omega)$. Let ω_ϵ be a Morse family, and v be a vector field. We say, that v is an ω_ϵ -*gradient*, if v is an ω_ϵ -gradient for each $\omega_{\boldsymbol{\mu}} \in \omega_\epsilon$.

- Lemma 4.2.4.** 1) *There is $\epsilon > 0$, such that ω_ϵ is a Morse family.*
 2) *Let v be an ω -gradient. Then there is $\epsilon > 0$ such that v is an ω_ϵ -gradient.*
 3) *Let v be an ω_ϵ -gradient. Then there is $\delta > 0$ such that every ω -gradient u with $\|u - v\| < \delta$ is an ω_ϵ -gradient.*

Proof. 1) Put

$$\lambda = \sup_{\substack{1 \leq i \leq m \\ x \in M}} \|\lambda_i(x)\|, \quad \eta = \min_{x \in M \setminus \cup_p U_p} \|\omega(x)\|.$$

Then $\epsilon = \frac{\eta}{2m\lambda}$ will do.

2) Put

$$\bar{\lambda} = \sup_{\substack{1 \leq i \leq m \\ x \in M}} \|\lambda_i(v)(x)\|, \quad \bar{\eta} = \min_{x \in M \setminus \cup_p U_p} \omega(v)(x).$$

3) Denote by Q the compact set $(M \setminus \cup_p U_p) \times [-\epsilon, \epsilon]^m$ and by $F : Q \rightarrow \mathbb{R}$ the map $F : (x, \mu_1, \dots, \mu_m) \mapsto \omega_{\boldsymbol{\mu}}(v)(x)$. Since $\text{Im } F \subset (0, \infty)$, there is $\beta > 0$ such that $\text{Im } F \subset [\beta, \infty)$. Let u be any ω -gradient such that

$$\|u - v\| \cdot (m\lambda\epsilon + \|\omega\|) \leq \frac{\beta}{2}.$$

We claim that u is an ω_ϵ -gradient. Indeed, note first that $\omega_{\boldsymbol{\mu}}(u)(x) > 0$ for any $x \in (\cup_p U_p \setminus S(\omega))$ and any $|\boldsymbol{\mu}| \leq \epsilon$. Further, if $x \in (M \setminus \cup_p U_p)$, we have

$$\begin{aligned} |\omega_{\boldsymbol{\mu}}(u)(x) - \omega_{\boldsymbol{\mu}}(v)(x)| &= |(\omega + \sum_i \mu_i \lambda_i)(u - v)(x)| \leq \\ &\leq \|u - v\| \cdot (\|\omega\| + \epsilon\lambda m) \leq \frac{\beta}{2}, \end{aligned}$$

therefore $\omega_{\boldsymbol{\mu}}(u)(x) > 0$. Finally, u has a standard form with respect to some ω -chart-system. A suitable restriction of this system will be an $\omega_{\boldsymbol{\mu}}$ -chart-system for any $\boldsymbol{\mu}$ with $|\boldsymbol{\mu}| < \epsilon$. \square

Now we can define the incidence coefficients with respect to the universal cover. The preceding lemma implies that v is an $\bar{\omega}$ -gradient for some 1-form $\bar{\omega}$ which cohomology class is rational. Therefore (see [12]) for every $g \in \pi_1(M)$ there is at most finite set of $(-v)$ -trajectories joining \tilde{x} with $\tilde{y}g$ if $\text{ind } x = \text{ind } y + 1$. Choose orientations of descending discs. For each such trajectory we denote by $\epsilon(\gamma)$ the sign of intersection of $D(\tilde{x}, v)$ with $D(\tilde{y}g, -v)$ along γ .

Definition 4.2.5. The element $\sum_{\gamma} \epsilon(\gamma)$ is denoted by $\nu(\tilde{x}, \tilde{y}g)$ and we set⁵

$$\tilde{n}(\tilde{x}, \tilde{y}; v) = \sum_{g \in G} \nu(\tilde{x}, \tilde{y}g)g \in (\mathbb{Z}[\pi_1 M])^{\wedge \wedge}.$$

Lemma 4.2.6. $\tilde{n}(\tilde{x}, \tilde{y}; v) \in (\mathbb{Z}[\pi_1 M])_{\{\omega\}}^-$.

Proof. Recall that Novikov ring $\mathbb{Z}([\pi_1 M])_{\{\omega\}}^-$ consists of all $\lambda \in (\mathbb{Z}[\pi_1 M])^{\wedge \wedge}$ such that for every $c \in \mathbb{R}$ we have: $\text{supp } \lambda \cap \{\omega\}^{-1}([c, \infty))$ is finite. Then our lemma follows from Lemma 4.2.1. \square

Remark 4.2.7. Note that the analogs of Lemmas 4.2.1 and 4.2.6 are obviously true for any regular covering $p' : \widehat{M} \rightarrow M$ such that $(p')^*[\omega] = 0$.

4.3. Incidence coefficients with respect to a free abelian cover.

Next we pass to free abelian covers. Assume here the terminology of Subsection 1.3.2 of Introduction. The universal covering factors then as $p = \mathcal{P} \circ \mathcal{Q}$, where $\mathcal{Q} : \widetilde{M} \rightarrow \overline{\overline{M}}$. The epimorphism $\pi_1(M) \rightarrow H_1(M, \mathbb{Z})/\text{Tors}$ will be denoted by Q . The de Rham cohomology class $[\omega]$ of ω defines a homomorphism $H^1(M, \mathbb{Z})/\text{Tors} \rightarrow \mathbb{R}$, which will be denoted by the same letter $[\omega]$. Note that $\{\omega\} = [\omega] \circ Q$. Since $\mathcal{P}^*(H^1(M, \mathbb{R})) = 0$, there is a Morse function $\overline{\overline{F}} : \overline{\overline{M}} \rightarrow \mathbb{R}$, such that $d\overline{\overline{F}} = \mathcal{P}^*\omega$, and we shall assume that $\tilde{F} = \overline{\overline{F}} \circ Q$.

We have chosen a Riemannian metric on M , whence the manifold $\overline{\overline{M}}$ obtains a Riemannian metric, which is \mathbb{Z}^m -invariant. We have also chosen a basis (a_1, \dots, a_m) in $H_1(M, \mathbb{Z})/\text{Tors}$. Therefore this group is identified with \mathbb{Z}^m , and the vector space $H^1(M, \mathbb{R})$ with the dual space of linear forms $\mathbb{R}^m \rightarrow \mathbb{R}$. Choose the L_1 -norm in \mathbb{R}^m . Then the dual space obtains the *sup*-norm. (That is $\|\sum_i \alpha_i a_i\| = \sum_i |\alpha_i|$ and $\|\sum_i \beta_i a_i^*\| = \max_j |\beta_j|$, where $\{a_i^*\}$ is the base dual to $\{a_i\}$.)

⁵ Recall that $(\mathbb{Z}[\pi_1 M])^{\wedge \wedge}$ is the abelian group of all the formal linear combinations (infinite in general) of the elements of G with integer coefficients.

Corollary 4.3.1. *Let v be an ω -gradient. There is $\epsilon > 0$, such that every linear form $\eta : \mathbb{R}^m \rightarrow \mathbb{R}$ with $\|[\omega] - \eta\| \leq \epsilon$ is a cohomology class of a Morse form $\omega(\eta)$ such that v is an ω -gradient.*

Proof. Every linear form $\eta : \mathbb{R}^m \rightarrow \mathbb{R}$ with $\|[\omega] - \eta\| \leq \epsilon$ can be written as $\eta = [\omega] + \sum \eta_i a_i^*$ where $\eta_i \in \mathbb{R}$, $|\eta_i| \leq \epsilon$. Now let $\epsilon > 0$ be so small that ω_ϵ is a Morse family and v is an ω_ϵ -gradient and set $\omega(\eta) = \omega + \sum \eta_i \lambda_i$. \square

Definition 4.3.2. For two critical points $x, y \in S(\omega)$ and an ω -gradient v we set

$$I(x, y; v) = \{g \in \pi_1(M) \mid \text{there is a } (-v)\text{-trajectory } \gamma \text{ joining } \tilde{x} \text{ to } \tilde{y} \cdot g\}.$$

If the set of $(-v)$ -trajectories joining \tilde{x} to $\tilde{y} \cdot g$ is finite, we denote by $N(\tilde{x}, \tilde{y}, g; v)$ its cardinality. (We identify here two trajectories which differ by a parameter change.)

Remark 4.3.3. If v is a good ω -gradient and elements $x, y \in S(\omega)$ satisfy $\text{ind } x = \text{ind } y + 1$, then $\text{supp}(\tilde{n}(\tilde{x}, \tilde{y}; v)) \subset I(x, y; v)$ and for every $g \in \pi_1 M$ the set $N(\tilde{x}, \tilde{y}, g; v)$ is finite.

Lemma 4.3.4. *There is an integral $[\omega]$ -cone Γ and a vector $a \in \mathbb{Z}^m$ such that*

$$Q(I(x, y; v)) \subset \Gamma + a.$$

Proof. Note that if v is a \varkappa -gradient for some Morse form \varkappa , then

$$Q(I(x, y; v)) \subset [\varkappa^{-1}]((-\infty, A])$$

for some A . By 4.3.1 there is $\epsilon > 0$, such that every linear form $\eta : \mathbb{R}^m \rightarrow \mathbb{R}$ with $\|[\omega] - \eta\| \leq \epsilon$ is the cohomology class of a Morse form $\omega(\eta)$ such that v is an $\omega(\eta)$ -gradient. Choose then the linear forms η_i so as to satisfy Lemma 4.1.3 and obtain the conclusion of the Lemma. \square

Lemma 4.3.5. *Let γ, γ' be two Morse forms with $[\gamma] \neq 0, [\gamma'] \neq 0$. Assume that a vector field v satisfying the transversality assumption is γ -gradient and γ' -gradient. Let $x, y \in S(\gamma), \text{ind } x = \text{ind } y + 1$. Then*

- 1) *If $[\gamma] = \alpha[\gamma']$ with $\alpha < 0$, then $I(x, y; v)$ is finite.*
- 2) *If there is no $\alpha < 0$ with $[\gamma] = \alpha[\gamma']$, then there is an integral $([\gamma], [\gamma'])$ -cone Δ and $b \in \mathbb{Z}^m$ such that $Q(I(x, y; v)) \subset \Delta + b$.*

Proof. 1) Obvious.

2) If $[\gamma]$ and $[\gamma']$ are linearly dependent, then $[\gamma] = \alpha[\gamma']$ with $\alpha > 0$ and our lemma follows from Lemma 4.3.4. Therefore we can assume that $[\gamma]$ and $[\gamma']$ are linearly independent, which imply that there is $h \in \mathbb{Z}^m$

with $[\gamma](h), [\gamma'](h) < 0$. From Lemma 4.3.4 we know that there are integral $[\gamma]$ -cone Γ_1 and $a_1 \in \mathbb{Z}^m$ such that $Q(I(x, y; v)) \subset \Gamma_1 + a_1$.

Also there are integral $[\gamma']$ -cone Γ_2 and $a_2 \in \mathbb{Z}^m$ such that $Q(I(x, y; v)) \subset \Gamma_2 + a_2$. Adding to the generators of Γ_1 and Γ_2 some integral vectors we can assume that $h \in \text{Int}\Gamma_1, h \in \text{Int}\Gamma_2$. Then there exists $N \in \mathbb{N}$ such that $\Gamma_1 + a_1 \subset \Gamma_1 - Nh$ and $\Gamma_2 - Nh \supset \Gamma_2 + a_2$. Thus

$$Q(I(x, y; v)) \subset (\Gamma_1 - Nh) \cap (\Gamma_2 - Nh) = \Gamma_1 \cap \Gamma_2 - Nh.$$

The set $\Gamma_1 \cap \Gamma_2$ is a $([\gamma], [\gamma'])$ -cone, and by (3.1) there is an integral $([\gamma], [\gamma'])$ -cone Δ such that $\Gamma_1 \cap \Gamma_2 \subset \Delta$ therefore $Q(I(x, y; v)) \subset \Delta - Nh$. \square

By Remark 4.2.7 the incidence coefficient $\bar{n}(\bar{x}, \bar{y}; v) \in (\mathbb{Z}[\mathbb{Z}^m])_{[\omega]}^-$ is defined. We shall assume that the lifts \bar{x} of points $x \in S(\omega)$ are chosen so that $Q(\tilde{x}) = \bar{x}$. Note that obviously $\text{supp}(\bar{n}(\bar{x}, \bar{y}; v)) \subset Q(\text{supp} \tilde{n}(\tilde{x}, \tilde{y}; v))$.

4.4. Proof of Theorem B. For a Morse form ξ such that $[\xi] \in H^1(M, \mathbf{Q})$, $[\xi] \neq 0$ we denote by ξ_0 the (unique) Morse form, such that $[\xi_0]$ is an indivisible class in $H^1(M, \mathbb{Z})$ and that $\xi_0 = \mu\xi$ with $\mu > 0$. The map $M \rightarrow S^1$, corresponding to ξ_0 , will be denoted by $f_0\langle\xi\rangle$.

Definition 4.4.1. Define now $\mathcal{G}t_1(\omega)$ as the set of all ω -gradients $v \in \mathcal{G}t(\omega)$, satisfying the following property:

(C) There is $\epsilon : 0 < \epsilon \leq \|\omega\|/2$ such that v is an ω_ϵ -gradient and there is a Morse form $\xi \in \omega_\epsilon$ with $[\xi] \in H^1(M, \mathbf{Q})$ such that $v \in \mathcal{G}t_1(f_0\langle\xi\rangle)$.

We shall now prove the properties of $\mathcal{G}t_1(\omega)$ required in the statement.

1) $\mathcal{G}t_1(\omega)$ is C^0 -open in $\mathcal{G}t(\omega)$. Indeed, if v satisfies (C) then every ω -gradient u , sufficiently close to v , is also an ω_ϵ -gradient (due to Lemma 4.2.4) and $u \in \mathcal{G}t_1(f_0\langle\xi\rangle)$ since $\mathcal{G}t_1(f_0\langle\xi\rangle)$ is C^0 -open in $\mathcal{G}t(f_0\langle\xi\rangle)$ (by Theorem 3.2.1).

2) $\mathcal{G}t_1(\omega)$ is C^0 -dense in $\mathcal{G}t(\omega)$. Indeed, if $v \in \mathcal{G}t(\omega)$, then there exists an $\epsilon > 0$ such that v is an ω_ϵ -gradient. Choose any form $\omega' \in \omega_\epsilon$ with $[\omega'] \in H^1(M, \mathbf{Q})$. Then by Theorem 3.2.1 arbitrarily close to v one can find an ω' -gradient $u \in \mathcal{G}t_1(f_0\langle\omega'\rangle)$. By Lemma 4.2.4, u can be chosen so as to be an ω_ϵ -gradient.

3) If $v \in \mathcal{G}t_1(\omega)$, then $\bar{n}(\bar{x}, \bar{y}; v) \in \mathfrak{s}_{[\omega]}^{-1}\mathbb{Z}[\mathbb{Z}^m]$. Indeed, v is an ω -gradient and a ξ -gradient for some $\xi \in \omega_\epsilon$ with $[\xi] \in H^1(M, \mathbf{Q})$. Note that if $[\xi]$ and $[\omega]$ are linearly dependent, then $[\xi] = \alpha[\omega]$ with $\alpha > 0$. Indeed, $[\xi] = \alpha[\omega]$ with $\alpha < 0$ would imply

$$(1 - \alpha)[\omega] + \sum \epsilon_i a_i^* = 0,$$

which contradicts $\epsilon \leq \|\omega\|/2$. Therefore Lemma 4.3.5 implies that there is an integral $([\xi], [\omega])$ -cone Δ , such that

$$\text{supp } \bar{n}(\bar{x}, \bar{y}; v) \subset \Delta + b$$

for some $b \in \mathbb{Z}^m$. Consider $\bar{n}(\bar{x}, \bar{y}; v)$ as an element of $(\mathbb{Z}[\mathbb{Z}^m])_{[\xi]}^-$. Then Lemma 4.1.4 together with Theorem 3.2.1 imply that it belongs to the localization $\mathfrak{s}_{[\xi]}^{-1}\mathbb{Z}[\mathbb{Z}^m]$, therefore there are $P, Q \in \mathbb{Z}[\mathbb{Z}^m]$ such that

$$P = Q \cdot \bar{n}(\bar{x}, \bar{y}; v).$$

Since $\bar{n}(\bar{x}, \bar{y}; v) \in \sigma_{\Delta}^{-1}\mathbb{Z}[\Delta]^{\wedge}$ the faithful flatness property imply

$$\bar{n}(\bar{x}, \bar{y}; v) \in \sigma_{\Delta}^{-1}S_{\Delta}^{-1}\mathbb{Z}[\Delta] \subset \mathfrak{s}_{[\omega]}^{-1}\mathbb{Z}[\mathbb{Z}^m].$$

5. AN EXAMPLE

In this section we shall construct a Morse map $M \rightarrow S^1$ on a closed 3-manifold M , having two critical points: x of index 2 and y of index 1, such that

$$n(\bar{x}, \bar{y}; v) = \sum_{k \geq 0} n_k t^k$$

with $n_k \sim \alpha \cdot \beta^k$ with $\alpha < 0, \beta > 0$. For any f -gradient w sufficiently close to v in C^0 topology, we shall have: $n_k(\bar{x}, \bar{y}; w) = n_k(\bar{x}, \bar{y}; v)$.

We start with a torus T^2 , remove from it two open discs and obtain a surface S with two components of boundary. Choose and fix a parallel β and a meridian α of this twice punctured torus. The copies of this surface (resp. the copies of α, β etc.) will be denoted by the same letter S (resp. α, β , etc.) adding indices in order to distinguish between them. The corresponding discs will be denoted by D_1, D_2 . We glue a copy of S , denoted by $S(1, 1)$ to a copy of S , denoted by $S(1, 2)$ and close the boundary by two discs $D_1(1), D_1(2)$. The resulting surface is called N . We glue three copies of S successively, close the boundary and obtain a closed surface

$$L = D_1(\frac{1}{2}) \cup S(\frac{1}{2}, 1) \cup S(\frac{1}{2}, 0) \cup S(\frac{1}{2}, 2) \cup D_2(\frac{1}{2}).$$

The surface $K = D_1(0) \cup S(0, 1) \cup S(0, 2) \cup D_2(0)$ is constructed similarly to N . The surface L is depicted on Figure 5.1 below.

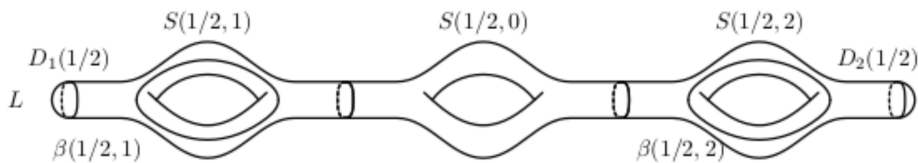


FIGURE 5.1.

Let W_1 be the cobordism between N and L , corresponding to the surgery on the circle $\beta(\frac{1}{2}, 2)$. Introduce on W_1 the corresponding Morse function $F_1 : W_1 \rightarrow [\frac{1}{2}, 1]$ with one critical point x of index 2, and such that

$$F_1^{-1}(1) = N, \quad F_1^{-1}(\frac{1}{2}) = L.$$

We can find an F_1 -gradient v_1 such that

- $(-v_1)_{[\frac{1}{2}, 1]}^{\rightsquigarrow}$ restricted to

$$D_1(\frac{1}{2}) \cup S(\frac{1}{2}, 1) \cup S(\frac{1}{2}, 0)$$

is a diffeomorphism of this surface onto

$$D_1(1) \cup S(1, 1) \cup S(1, 2)$$

which identifies $S(\frac{1}{2}, 0)$ with $S(1, 2)$ and $S(\frac{1}{2}, 1)$ with $S(1, 1)$, slightly diminished from the left, so that the image of $D_1(\frac{1}{2})$ contains $D_1(1)$ in its interior, and therefore

$$(v_1)_{[\frac{1}{2}, 1]}^{\rightsquigarrow}(D_1(1)) \subset \text{Int}D_1(\frac{1}{2});$$

- $(-v_1)_{[\frac{1}{2}, 1]}^{\rightsquigarrow}(S(\frac{1}{2}, 2) \cup D_2(\frac{1}{2})) = D_2(1)$,
- $(-v_1)_{[\frac{1}{2}, 1]}^{\rightsquigarrow}(D_2(\frac{1}{2})) \subset \text{Int}D_2(1)$,
- and for some $\delta > 0$ the $(-v_1)_{[\frac{1}{2}, 1]}^{\rightsquigarrow}$ -image of a δ -tubular neighborhood of $\beta(\frac{1}{2}, 2)$ is in $\text{Int}D_2(1)$ (this image equals to $D_\delta(-v_1) \cap F_1^{-1}(1)$).

Let W_0 be the cobordism between K and L , corresponding to the surgery on the circle $\beta(\frac{1}{2}, 1)$. Introduce on W_0 the corresponding Morse function $F_0 : W_0 \rightarrow [0, \frac{1}{2}]$ with one critical point y of index 1, and such that

$$F_0^{-1}(\frac{1}{2}) = L, \quad F_0^{-1}(0) = K.$$

We can find an F_0 -gradient v_0 such that

- $(v_0)_{[\frac{1}{2}, 0]}^{\rightsquigarrow}$ restricted to

$$S(\frac{1}{2}, 0) \cup S(\frac{1}{2}, 2) \cup D_2(\frac{1}{2})$$

is a diffeomorphism of this surface onto

$$D_1(0) \cup S(0, 1) \cup S(0, 2)$$

which identifies $S(\frac{1}{2}, 0)$ with $S(0, 1)$ and $S(\frac{1}{2}, 2)$ with $S(0, 2)$ slightly diminished from the right so that the image of $D_2(\frac{1}{2})$ contains $D_2(0)$ in its interior, and therefore

$$(-v_0)_{[\frac{1}{2}, 0]}^{\rightsquigarrow}(D_2(0)) \subset \text{Int}D_2(\frac{1}{2});$$

- $(v_0)_{[\frac{1}{2},0]}^{\rightsquigarrow}(S(\frac{1}{2}, 1) \cup D_1(\frac{1}{2})) = D_1(0)$;
- $(v_0)_{[\frac{1}{2},0]}^{\rightsquigarrow}(D_1(\frac{1}{2})) \subset \text{Int}D_1(0)$;
- and for some $\delta > 0$ the $(v_0)_{[\frac{1}{2},0]}^{\rightsquigarrow}$ -image of a δ -tubular neighborhood of $\beta(\frac{1}{2}, 1)$ (this image equals to $D_\delta(v_0) \cap F_0^{-1}(0)$) is in $\text{Int}D_1(0)$.

Glue together W_0 and W_1 along L , denote the resulting cobordism by W . Then we have $\partial W = K \cup N$. Glue the functions F_1 and F_0 to a Morse function $F : W \rightarrow [0, 1]$ and the vector fields v_1, v_0 to an F -gradient v . We shall now define a ranging system for (F, v) . Consider the set $\Lambda = \{0, \frac{1}{2}, 1\}$ of regular values, and set

$$\begin{aligned} A_1 &= D_1(1), & A_{\frac{1}{2}} &= D_1(\frac{1}{2}), & A_0 &= D_1(0), \\ B_1 &= D_2(1), & B_{\frac{1}{2}} &= D_2(\frac{1}{2}), & B_0 &= D_2(0). \end{aligned}$$

The fact that $\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ is a ranging system for (F, v) follows immediately from the properties of v_0, v_1 cited above.

It is not difficult to compute the homomorphism

$$H(v) : H_1(N \setminus D_2(1), D_1(1)) \rightarrow H_1(K \setminus D_2(0), D_1(0)).$$

Namely,

$$[\alpha(1, 1)] \mapsto 0, \quad [\beta(1, 1)] \mapsto 0, \quad [\alpha(1, 2)] \mapsto [\alpha(0, 1)], \quad [\beta(1, 2)] \mapsto [\beta(0, 1)].$$

It is also obvious that the homology class of $D(x, v) \cap K$ equals to $[\beta(0, 2)]$ and the homology class of $D(y, -v) \cap N$ equals to $[\beta(1, 1)]$.

Consider the embedded curves

$$\alpha(0, 1), \quad \beta(0, 1), \quad \alpha(0, 2), \quad \beta(0, 2)$$

in K . Their homology classes form a symplectic basis in K . Consider the isomorphism of $H_1(K, \mathbb{Z}) \approx \mathbb{Z}^4$ given in this basis by the matrix

$$\begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 3 & 0 & -2 \end{pmatrix}$$

It preserves the intersection form, therefore it can be realized by a diffeomorphism $\Phi : K \rightarrow K$ and it is easy to see that we can assume that $\Phi(x) = x$ for $x \in D_1(0) \cup D_2(0)$. Denote by Ψ the composition of Φ with the subsequent identification of K with N .

Denote by M the 3-dimensional manifold obtained by gluing of K to N by means of Ψ , and let $f : M \rightarrow S^1$ be the Morse function obtained from F .

The corresponding cyclic covering \overline{M} is a union of countably many copies of W , denoted by $W[i], i \in \mathbb{Z}$, glued together by the diffeomorphisms

$$\Psi : K[i] \rightarrow N[i - 1]$$

of the components of boundaries.

The identification $W[k] \rightarrow W$ will be denoted by J_k . For $X \subset W$ we denote $J_k^{-1}(X)$ by $X[k]$. Define a lift $\bar{F} : \overline{M} \rightarrow \mathbb{R}$ of f by setting

$$\bar{F}|_{W[k]} = F \circ J_k + k.$$

Set

- $\Sigma = \{n/2 \mid n \in \mathbb{Z}\}$;
- $A_n = D_1(0)[n], B_n = D_2(0)[n]$ for $n \in \mathbb{Z}$;
- $A_n = D_1(\frac{1}{2})[k], B_n = D_2(\frac{1}{2})[k]$ for $n = k + \frac{1}{2}, k \in \mathbb{Z}$.

It is obvious that $\{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma}$ is a t -equivariant ranging system for (\bar{F}, v) . Put $\bar{x} = J_0^{-1}(x)$ and $\bar{y} = J_0^{-1}(y)$. Then we have:

$$n_0(\bar{x}, \bar{y}; v) = 0, \quad n_1(\bar{x}, \bar{y}; v) = \Psi_*([\beta(0, 2)]) \# [\beta(1, 1)] = 0.$$

Denote by D the matrix of $\Psi_* \circ H(v)$. Then

$$D = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 3 \end{pmatrix}$$

Then for $k \geq 1$ we have

$$n_{k+1}(\bar{x}, \bar{y}; v) = (\Psi_* \circ H(v))^k (\Psi_*([\beta(0, 2)]) \# [\beta(1, 1)]).$$

To abbreviate the notation we denote $[\alpha(1, 1)]$ by a_1 , $[\alpha(1, 2)]$ by a_2 , $[\beta(1, 1)]$ by b_1 , $[\beta(1, 2)]$ by b_2 . Then $n_{k+1}(\bar{x}, \bar{y}; v)$ is the coefficient of a_1 in

$$D^k(b_1 - 2b_2) = (-2)D^k(b_2).$$

To find $D^k(b_2)$ assume by induction that

$$D^k(b_2) = \alpha_k a_1 + \gamma_k a_2 + \beta_k b_2.$$

Then

$$D^{k+1}(b_2) = -\gamma_k b_2 + \beta_l(2a_1 + 3\beta_2 + a_2) = 2\beta_k a_1 + (3\beta_k - \gamma_k)b_2 + \beta_k a_2.$$

Therefore the vectors (β_k, γ_k) satisfy

$$\begin{pmatrix} \beta_{k+1} \\ \gamma_{k+1} \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_k \\ \gamma_k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \beta_0 \\ \gamma_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The coefficient $\alpha_{k+1} = 2\beta_k$. Denote

$$\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$$

by C . Then the explicit computation shows

$$C^n = \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & \lambda_2^n - \lambda_1^n \\ \lambda_1^n - \lambda_2^n & \lambda_2^{n-1} - \lambda_1^{n-1} \end{pmatrix},$$

where $\lambda_1 = \frac{3+\sqrt{5}}{2}$, $\lambda_2 = \frac{3-\sqrt{5}}{2}$. Therefore

$$n_{k+1}(\bar{x}, \bar{y}; v) = -\frac{4}{\sqrt{5}} \left(\left(\frac{3+\sqrt{5}}{2} \right)^k - \left(\frac{3-\sqrt{5}}{2} \right)^k \right).$$

5.1. Incidence coefficients with respect to the universal cover: the exponential estimate.

Lemma 5.1.1. *Let $\xi, \eta : \mathbb{R}^m \rightarrow \mathbb{R}$ be non-zero linear forms, and let Γ be a (ξ, η) -integral cone. Then there is $A > 0$ such that for every $b \in \mathbb{R}^m$ there is $B \in \mathbb{R}$ such that for every $c \in \mathbb{R}$ we have:*

$$(\Gamma + b) \cap \xi^{-1}([c, \infty)) \subset (\Gamma + b) \cap \eta^{-1}([Ac + B, \infty)). \tag{5.1}$$

Proof. Abbreviate $\xi^{-1}([c, \infty))$ by $\{\xi \geq c\}$. It is sufficient to consider the case $b = 0$, that is, to prove that there is $A \geq 0$ such that

$$\Gamma \cap \{\xi \geq c\} \subset \Gamma \cap \{\eta \geq Ac\} \tag{5.2}$$

(the property (5.1) with arbitrary b follows then from $B = \eta(b) - A\xi(b)$).

To prove (5.2) let e_i be the generators of Γ , and choose $A > 0$ such that

$$\eta(e_i) \geq A\xi(e_i)$$

for all i . Then $x \in \Gamma \cap \{\xi \geq c\}$ means that $x = \sum \lambda_i e_i$, where $\lambda_i \geq 0$ and $\sum \lambda_i \xi(e_i) \geq c$. This implies

$$\eta(x) = \sum \lambda_i \eta(e_i) \geq \sum \lambda_i A\xi(e_i) \geq A\xi(x). \quad \square$$

Definition 5.1.2. Let $\lambda = \sum_g n_g g \in \mathbb{Z}[\pi_1 M]_{\xi}^-$ and $c \in \mathbb{R}$. Put

$$\lambda[c] = \sum_{\xi(g) \geq c} n_g \cdot g \in \mathbb{Z}[\pi_1 M], \quad N_c(\lambda) = \sum_{\xi(g) \geq c} |n_g|.$$

5.2. Proof of Theorem C. By the definition of $\mathcal{G}t_1(\omega)$ there exists an

$$\epsilon \in \left(0, \frac{1}{2} \|\omega\|\right)$$

and a Morse form $\xi \in \omega_\epsilon$, such that

$$[\xi] \in H^1(M, \mathbf{Q}), \quad v \in \mathcal{G}t_1(f_0\langle \xi \rangle).$$

Therefore there is an integral $([\xi], [\omega])$ -cone Δ and $b \in \mathbb{Z}^m$, such that

$$Q(\text{supp } n(\tilde{x}, \tilde{y}; v)) \subset \Delta + b.$$

Lemma 5.1.1 implies then that

$$\begin{aligned} \text{supp } \tilde{n}(\tilde{x}, \tilde{y}; v) \cap (\{\omega\})^{-1}([c, \infty)) &\subset \\ &\subset \text{supp } \tilde{n}(\tilde{x}, \tilde{y}; v) \cap (\{\xi\})^{-1}([Ac + B, \infty)), \end{aligned}$$

and since v is a ξ -gradient, our estimate follows from Theorem 3.2.1 and Proposition 3.1.3. \square

5.3. On the Novikov complex for a Morse form. In this paper we do not use the notion of the Novikov complex, and work only with the incidence coefficients. The latter were introduced however in [9] as the matrix entries of the boundary operators in the Novikov complex. In this subsection we use the results of Subsection 4.2 and 5.2 to give a simple proof of the fact that the boundary operators above indeed turn the graded module into a chain complex (that is, $\partial^2 = 0$). We reduce the proof to the corresponding statement about rational Morse forms, proved in [12].

We assume here the terminology of Subsection 1.2 of the Introduction. Moreover, if Δ is an integral $[\omega]$ -cone, we denote by λ_Δ the subset of $\mathbb{Z}[\pi_1 M]_{\{\omega\}}^-$, defined by

$$\lambda_\Delta = \{\lambda \mid \text{there exists } b \in \mathbb{Z}^m \text{ such that } Q(\text{supp } \lambda) \subset \Delta + b\}.$$

It is not difficult to see that λ_Δ is a subring of $\mathbb{Z}[\pi_1 M]_{\{\omega\}}^-$.

Now let v be an ω -gradient satisfying transversality assumption and $\epsilon > 0$ so small that ω_ϵ is a Morse family and v is an ω_ϵ -gradient. Let $\xi \in \omega_\epsilon$ be a Morse form such that $[\xi] \in H^1(M, \mathbf{Q})$. Then it follows from Lemma 4.3.5 that there is an integral $([\xi], [\omega])$ -cone Δ , such that for every $x, y \in S(\omega)$ with $\text{ind } x = \text{ind } y + 1$ we have

$$\tilde{n}(\tilde{x}, \tilde{y}; v) \subset \lambda_\Delta.$$

Therefore the homomorphism

$$\partial : C_*(\omega, v) \rightarrow C_{*-1}(\omega, v)$$

is defined actually over the ring λ_Δ and to verify that $\partial^2 = 0$ it is sufficient to verify it over the ring $\mathbb{Z}[\pi_1 M]_{\{\xi\}}^-$, and this is done in [12].

6. EXPONENTIAL ESTIMATES OF THE ABSOLUTE NUMBER OF
TRAJECTORIES

6.1. Bunches of ranging systems. Let $f : W \rightarrow [a, b]$ be a Morse function on a cobordism W such that

$$f^{-1}(b) = V_1, \quad f^{-1}(a) = V_0, \quad \dim W = n.$$

Let v be an f -gradient, $\Lambda = \{\lambda_0, \dots, \lambda_k\}$ be a set of regular values of f , such that

$$\lambda_0 = a < \lambda_1 < \dots < \lambda_{k-1} < \lambda_k = b$$

and between any two adjacent values λ_i and λ_{i+1} there is exactly one critical value of f .

Definition 6.1.1. Assume that for each integer $s : 0 \leq s \leq n$ and every $\lambda \in \Lambda$ there are given compacts $A_\lambda^{(s)}, B_\lambda^{(s)}$ in $f^{-1}(\lambda)$. We shall say that

$$\{(A_\lambda^{(s)}, B_\lambda^{(s)})\}_{\lambda \in \Lambda, 0 \leq s \leq n}$$

is a *bunch of ranging systems for (f, v)* (abbreviation: BRS for (f, v)) if the following conditions hold:

- (1) $A_\lambda^{(0)} = B_\lambda^{(0)} = \emptyset$.
- (2) $r \leq s \Rightarrow (A_\lambda^{(r)} \subset A_\lambda^{(s)}, B_\lambda^{(r)} \subset B_\lambda^{(s)})$.
- (3) $A_\lambda^{(r)} \cap B_\lambda^{(s)} = \emptyset$ for $r + s \leq n$,
- (4) If $\lambda, \mu \in \Lambda$ are adjacent, then for every r, s :

$$v_{[\mu, \lambda]} \rightsquigarrow (A_\mu^{(r)}) \subset \text{Int} A_\lambda^{(r)} \quad \text{and} \quad (-v)_{[\lambda, \mu]} \rightsquigarrow (B_\lambda^{(s)}) \subset \text{Int} B_\mu^{(s)}.$$

- (5) Let $\lambda, \mu \in \Lambda$ be adjacent. Then for every $p \in S(f) \cap f^{-1}([\lambda, \mu])$ we have:

- i) $D(p, v) \cap f^{-1}(\lambda) \subset \text{Int} A_\lambda^{(\text{ind } p)}$,
- ii) $D(p, -v) \cap f^{-1}(\mu) \subset \text{Int} B_\mu^{(n - \text{ind } p)}$.

Let $\{(A_\lambda^{(s)}, B_\lambda^{(s)})\}_{\lambda \in \Lambda, 0 \leq s \leq n}$ be a BRS for (f, v) . The following properties are either obvious or proved similarly to the corresponding properties of ranging systems (see Subsection 2.2 and [15, §4]).

- i) For every $s : 0 \leq s \leq n$ the family $\{(A_\lambda^{(s)}, B_\lambda^{(n-s-1)})\}_{\lambda \in \Lambda}$ is a ranging system for (f, v) .
- ii) v is an almost good f -gradient.
- iii) There is $\epsilon > 0$ such that for every f -gradient w with $\|w - v\| < \epsilon$ the system $\{(A_\lambda^{(s)}, B_\lambda^{(s)})\}_{\lambda \in \Lambda, 0 \leq s \leq n}$ is a BRS for (f, w) .

iv) For every $\delta > 0$ sufficiently small the following strengthening of (5) holds:

$$\begin{cases} D_\delta(p, v) \cap f^{-1}(\lambda) \subset \text{Int}A_\lambda^{(\text{ind } p)}, \\ D_\delta(p, -v) \cap f^{-1}(\mu) \subset \text{Int}B_\mu^{(n - \text{ind } p)}. \end{cases} \quad (6.1)$$

For $p \in S(f)$ we shall denote by λ_p and μ_p the adjacent elements of Λ such that $\lambda_p < f(p) < \mu_p$. Choose $\rho > 0$ so small that for every $p \in S(f)$ we have:

$$\lambda_p < \lambda_p + \rho < f(p) - \rho < f(p) + \rho < \mu_p - \rho.$$

Let $\delta > 0$ satisfy (6.1) above and assume further that

$$\begin{cases} (1) \text{ If } p, q \in S(f) \text{ and } f(p) = f(q) \text{ then:} \\ \quad D_\delta(p, v) \cap D_\delta(q, v) \cap f^{-1}([\lambda_p, \mu_p]) = \emptyset, \\ \quad D_\delta(p, -v) \cap D_\delta(q, -v) \cap f^{-1}([\lambda_p, \mu_p]) = \emptyset. \\ (2) \text{ For every } p \in S(f) \text{ we have} \\ \quad D_\delta(p) \subset f^{-1}([f(p) - \rho, f(p) + \rho]). \end{cases} \quad (6.2)$$

Let $\epsilon > 0$ satisfy iii) above. The manifold

$$S_p = D(p, -v) \cap f^{-1}(\mu_p)$$

is an embedded sphere of dimension $n - \text{ind } p - 1$. The normal bundle to S_p in $f^{-1}(\mu_p)$ is trivial. Chose an embedding

$$\psi : S_p \times B^{\text{ind } p}(0, \theta) \rightarrow f^{-1}(\mu_p)$$

(where $\theta > 0$) which is a diffeomorphism onto its image, and such that $\psi|_{(S_p \times \{0\})} = \text{id}$.

For $\varkappa \in (0; \theta]$ the set $\psi(S_p \times B^{\text{ind } p}(0, \varkappa))$ will be denoted by $\text{Tub}(S_p, \varkappa)$ and for $\eta \in B^{\text{ind } p}(0, \theta)$ the embedded sphere $\psi(S_p \times \{\eta\})$ by $S_p(\eta)$. Assume that θ is so small that

$$\overline{\text{Im } \psi} \subset B_\delta(p, -v).$$

Choose $\tau > 0$ so small that for every $z \in f^{-1}(\mu_p)$ the v -trajectory $\gamma(z, \cdot; v)$ is defined on the interval $[-\tau, 0]$ and its image is in $f^{-1}((\mu_p - \rho, \mu_p])$. Define a map

$$\Psi : [-\tau, 0] \times S_p \times B^{\text{ind } p}(0, \theta) \rightarrow W$$

by

$$\Psi(t, s, h) = \gamma(\psi(s, h), t; v).$$

Then Ψ is a diffeomorphism onto its image, and the vector field $\Psi_*^{-1}(v)$ equals $(1, 0, 0)$.

Let $\xi \in B^{\text{ind } p}(0, \frac{\theta}{2})$ and let $H_t \langle \xi \rangle$, $t \in [-\tau, 0]$, be an isotopy of $B^{\text{ind } p}(0, \theta)$ having the following properties:

$$(1) \quad \frac{d}{dt} H_t \langle \xi \rangle(x) = 0 \text{ for } t \in [-\tau, -\frac{2}{3}\tau] \cup [-\frac{1}{3}\tau, 0].$$

(2) $H_t\langle\xi\rangle(x) = x$ for $x \in B^{\text{ind}p}(0, \theta) \setminus B^{\text{ind}p}(0, \frac{\theta}{2})$, and for $t \in [-\tau, -\frac{2}{3}\tau]$.

(3) $H_0\langle\xi\rangle(0) = \xi$.

Then $\frac{d}{dt}H_t\langle\xi\rangle$ is a time-dependent vector field on $B^{\text{ind}p}(0, \theta)$. Define a vector field $w\langle\xi\rangle$ on $[-\tau, 0] \times S_p \times B^{\text{ind}p}(0, \theta)$ by

$$w\langle\xi\rangle(t, x, y) = (1, 0, \frac{d}{dt}H_t\langle\xi\rangle).$$

Remark 6.1.2. Observe that for every $x \in S_p \times B^{\text{ind}p}(0, \frac{\theta}{2})$ we have

$$\gamma(x, -\tau; w\langle\xi\rangle) \in \{-\tau\} \times S_p \times B^{\text{ind}p}(0, \frac{\theta}{2}). \tag{6.3}$$

Choose $\alpha \in (0, \frac{\theta}{2})$ so small that for each $\xi \in B^{\text{ind}p}(0, \alpha)$ there is an isotopy $H_t\langle\xi\rangle$ such that

$$\|v - \Psi_*(w\langle\xi\rangle)\| < \epsilon. \tag{6.4}$$

We perform this construction for every $p \in S(f)$, and obtain a diffeomorphism Ψ_p , a vector ξ_p , and a vector field $w_p\langle\xi_p\rangle$. We assume that $\theta_p, \alpha_p, \tau_p$ are chosen to be independent of p , and we denote them by θ, α, τ omitting p in the notation. The set $\{\xi_p\}_{p \in S(f)}$ will be denoted by $\vec{\xi}$. Define a new vector field $u = v\langle\vec{\xi}\rangle$ setting

$$u(x) = \begin{cases} v(x) & \text{for } x \in W \setminus \bigcup_p \text{Im } \Psi_p, \\ (\Psi_p)_*(w_p\langle\xi_p\rangle)(x) & \text{for } x \in \text{Im } \Psi_p. \end{cases}$$

It follows from the construction that for every $\vec{\xi}$ the vector field $v\langle\vec{\xi}\rangle$ is an f -gradient. Note also that for $p \in S(f)$ we have

$$D(p, -v\langle\vec{\xi}\rangle) \cap f^{-1}(\mu_p) = S_p(\xi_p).$$

The condition (6.4) implies that $\{(A_\lambda^{(s)}, B_\lambda^{(s)})\}_{\lambda \in \Lambda, 0 \leq s \leq n}$ is a BRS for $(f, v\langle\vec{\xi}\rangle)$. Therefore $v\langle\vec{\xi}\rangle$ is an almost good f -gradient. It will be called *regular perturbation of v corresponding to $\vec{\xi}$* .

Proposition 6.1.3. 1) *Let $p, q \in S(f)$ with $\text{ind } p = \text{ind } q + 1$. The set of $(-v\langle\vec{\xi}\rangle)$ -trajectories⁶ joining p with q is in a bijective correspondence with the set*

$$(D(p, v) \cap f^{-1}(\mu_q)) \cap S_q(\xi_q).$$

2) *The vector field $v\langle\vec{\xi}\rangle$ is a good f -gradient if for every $p, q \in S(f)$ with $\text{ind } p = \text{ind } q + 1$ the submanifold $D(p, v) \cap f^{-1}(\mu_q)$ of $f^{-1}(\mu_q)$ is transversal to $S_q(\xi_q)$.*

⁶ We identify here two trajectories which differ by a parameter change.

Proof. 1) Let γ be a $(-v\langle\vec{\xi}\rangle)$ -trajectory joining p with q and $\gamma(t_0) \in \text{inf}^{-1}(\mu_q)$, where $t_0 \in \mathbb{R}$. I claim that for $t < t_0$ we have

$$\gamma(t) \notin \text{supp}(v - v\langle\vec{\xi}\rangle).$$

Indeed, if the opposite is true, let $t_1 < t_0$ be the first moment when γ intersect $\text{supp}(v - v\langle\vec{\xi}\rangle)$. Then there is $s \in S(f)$ such that $\gamma(t_1) \in \text{Tub}(S_s, \frac{\theta}{2})$. Note that

$$\text{Tub}(S_s, \frac{\theta}{2}) \subset D_\delta(s, -v) \cap f^{-1}(\mu_s) \subset \text{Int}B_{\mu_s}^{(n-\text{ind } s)},$$

therefore $\gamma(t_1)$ belongs to the intersection of $A_{\mu_s}^{(\text{ind } p)}$ with $B_{\mu_s}^{(n-\text{ind } s)}$ and $\text{ind } p \geq \text{ind } s + 1$.

Further, $f(s) > f(q)$ and $\lambda_s \geq \mu_q$. Denote by t_2 the moment when γ intersects $f^{-1}(\lambda_s)$. Then $t_0 \geq t_2 > t_1 + \tau$, and it is easy to see that $\gamma|_{[t_1 + \tau, t_2]}$ does not intersect $\text{supp}(v - v\langle\vec{\xi}\rangle)$. The property (6.3) implies that

$$\gamma(t_1 + \tau) \in D_\delta(s, -v),$$

and, therefore,

$$\gamma(t_2) \in D_\delta(s, v) \cap f^{-1}(\lambda_s) \subset \text{Int}A_{\lambda_s}^{(\text{ind } s)},$$

which implies $\gamma(t_0) \in \text{Int}A_{\mu_q}^{(\text{ind } s)}$. Hence

$$A_{\mu_q}^{(\text{ind } s)} \cap D_\delta(q, -v) \neq \emptyset$$

and thus

$$A_{\mu_q}^{(\text{ind } s)} \cap B_{\mu_q}^{(n-\text{ind } q)} \neq \emptyset,$$

which implies

$$\text{ind } s > \text{ind } q \quad \text{and} \quad \text{ind } p \geq \text{ind } s + 1 > \text{ind } q + 1.$$

This gives a contradiction.

Therefore $\gamma|_{(-\infty, t_0]}$ is a $(-v)$ -trajectory. In the same way one can show that every $(-v)$ -trajectory joining p with a point of $D_\delta(q, -v) \cap f^{-1}(\mu_q)$ does not intersect $\text{supp}(v - v\langle\vec{\xi}\rangle)$. That proves the property 1).

2) Note that $v\langle\vec{\xi}\rangle$ is almost good. Therefore, $v\langle\vec{\xi}\rangle$ is good if and only if

$$\left(\text{ind } p = \text{ind } q + 1\right) \Rightarrow \left(D(p, v\langle\vec{\xi}\rangle) \pitchfork D(q, -v\langle\vec{\xi}\rangle)\right).$$

Let $p, q \in S(f)$, $\text{ind } p = \text{ind } q + 1$. It suffices to prove that the intersection $D(p, v\langle\vec{\xi}\rangle) \cap f^{-1}(\mu_q)$ is transversal to $S_q(\xi_q)$. Let x be a point in the intersection of these manifolds. In the part 1) we have proved that there is a $(-v)$ -trajectory, joining p with x and not intersecting $\text{supp}(v - v\langle\vec{\xi}\rangle)$. Then

a small neighborhood of this trajectory does not intersect $\text{supp}(v - v\langle \bar{\xi} \rangle)$ and the transversality sought follows from

$$(D(p, v) \cap f^{-1}(\mu_q)) \pitchfork S_q(\xi_q). \quad \square$$

6.2. Volume estimates. We will use here the terminology of the previous subsection. Let

$$\{(A_\lambda^{(s)}, B_\lambda^{(s)})\}_{\lambda \in \Lambda, 0 \leq s \leq n}$$

be a BRS for (f, v) . Assume that $\delta > 0$ satisfies (6.2) from Subsection 6.1. Fix an integer $s : 0 \leq s \leq n$. Let $\lambda < \mu$ be adjacent elements of Λ . Denote by S' the subset of all critical points of f of index $\leq s$ and with critical values in (λ, μ) . The set

$$\mathcal{D}_\mu^{(s)} = f^{-1}(\mu) \setminus \left(\text{Int}A_\mu^{(s)} \cup \text{Int}B_\mu^{(n-s-1)} \cup \left(\bigcup_{p \in S'} B_\delta(p, -v) \cap f^{-1}(\mu) \right) \right)$$

is a compact subset of the domain of definition of $v_{[\mu, \lambda]}^{\rightsquigarrow}$. Denote by $N_\mu^{(s)}$ the norm of the derivative of $v_{[\mu, \lambda]}^{\rightsquigarrow}$ restricted to $\mathcal{D}_\mu^{(s)}$:

$$N_\mu^{(s)} = \sup_{x \in \mathcal{D}_\mu^{(s)}} \left\| (v_{[\mu, \lambda]}^{\rightsquigarrow})'(x) \right\|$$

and by A the maximum of $(N_\mu^{(s)})^k$ over all $\mu \in \Lambda, 0 \leq s \leq n$, and $0 \leq k \leq n$. For $p \in S(f)$ denote by $B_{(p)}$ the norm of the derivative of

$$\psi_p^{-1} \Big|_{\psi_p(S_p \times D^{\text{ind } p}(0, \frac{\theta}{2}))}$$

(where the manifold $S_p \times B^{\text{ind } p}(0, \theta)$ is endowed with the product Riemannian metric). Denote by B the maximum of $B_{(p)}$ over all $p \in S(f)$.

Lemma 6.2.1. *Let $\lambda, \mu \in \Lambda, \lambda < \mu$, and s be an integer with $0 \leq s \leq n$. Let also N be a submanifold of $f^{-1}(\mu) \setminus B_\mu^{(n-s-1)}$, such that $N \setminus A_\mu^{(s)}$ is compact. Denote by k the number of elements of Λ belonging to $[\lambda, \mu]$. Then $N' = v_{[\mu, \lambda]}^{\rightsquigarrow}(N)$ is a submanifold of $f^{-1}(\lambda) \setminus B_\lambda^{(n-s-1)}$, such that $N' \setminus A_\lambda^{(s)}$ is compact, and*

- 1) $\text{vol}(N' \setminus A_\lambda^{(s)}) \leq A^{k-1} \cdot \text{vol}(N \setminus A_\mu^{(s)})$;
- 2) if $p \in S(f)$ with $\text{ind } p = s$ and $\mu_p = \lambda$, then

$$\text{vol}(\psi_p^{-1}(N' \cap \text{Tub}(S_p, \frac{\theta}{2}))) \leq BA^{k-1} \text{vol}(N \setminus A_\mu^{(s)}).$$

Proof. For the proof of 1) note that it suffices to consider the case $k = 2$. For this case we have obviously $N' \setminus A_\lambda^{(s)} \subset \rightsquigarrow v(N \cap \mathcal{D}_\mu^{(s)})$. Statement 2) follows from 1) since $\text{Tub}(S_p, \frac{\theta}{2}) \subset B_\delta(p, -v)$ does not intersect $A_\lambda^{(s)}$. \square

6.3. Bunches of equivariant ranging systems. We will use here the terminology of §1.3.1 of Introduction. Assume that the homotopy class of f is indivisible, so that $F(xt) = F(x) - 1$.

Definition 6.3.1. Let Σ be the set of regular values of F such that

- ($\Sigma 1$) $\sigma \in \Sigma \Rightarrow \sigma + n \in \Sigma$ for every $n \in \mathbb{Z}$.
- ($\Sigma 2$) If $\sigma_1, \sigma_2 \in \Sigma$ are adjacent, there is exactly one critical value of F in (σ_1, σ_2) .
- ($\Sigma 3$) For every $A, B \in \mathbb{R}$ the set $\Sigma \cap [A, B]$ is finite.

Assume that for every integer $s : 0 \leq s \leq n$ and every $\sigma \in \Sigma$ there are given compacts $A_\sigma^{(s)}, B_\sigma^{(s)}$ in $F^{-1}(\sigma)$. We shall say that

$$\{(A_\sigma^{(s)}, B_\sigma^{(s)})\}_{\sigma \in \Sigma, 0 \leq s \leq n}$$

is a *bunch of t -equivariant ranging systems* (abbreviation: BERS for (F, v)) if:

- (1) For every $\mu, \nu \in \Sigma, \mu < \nu$ the system $\{(A_\sigma^{(s)}, B_\sigma^{(s)})\}_{\sigma \in \Sigma, \mu \leq \sigma \leq \nu}$ is a BRS for $(F|_{F^{-1}([\mu, \nu])}, v)$;
- (2) $A_{\sigma-n}^{(s)} = A_\sigma^{(s)} \cdot t^n$ and $B_{\sigma-n}^{(s)} = B_\sigma^{(s)} \cdot t^n$ for every $n \in \mathbb{Z}$.

Up to the end of this subsection we assume that (F, v) has a BERS $\{(A_\sigma^{(s)}, B_\sigma^{(s)})\}_{\sigma \in \Sigma, 0 \leq s \leq n}$.

Choose any $\sigma \in \Sigma$ and denote by W the cobordism $F^{-1}([\sigma, \sigma + 1])$ and by Λ the set $\Sigma \cap [\sigma, \sigma + 1]$. We apply the constructions of Subsections 6.1 and 6.2 to W and then extend the results to the whole of \overline{M} in the t -invariant way, thus obtaining the sets $\text{Tub}(S_q, \varkappa) \subset F^{-1}(\mu_q)$ for every $q \in S(F)$. Let $p, q \in S(F)$. Assume that $F(p) > F(q)$ and $\text{ind } p = \text{ind } q + 1$. Denote by $N_{p,q}$ the submanifold of dimension $\text{ind } q$ of $S_q \times B^{\text{ind } q}(0, \frac{\theta}{2})$, defined by

$$N_{p,q} = \psi_q^{-1} \left((D(p, v) \cap F^{-1}(\mu_q)) \cap (\text{Tub}(S_q, \frac{\theta}{2})) \right).$$

The next lemma follows from Lemma 6.2.1.

Lemma 6.3.2. *There are constants $C, D > 0$ such that for every $p, q \in S(F)$ with $\text{ind } p = \text{ind } q + 1$ and $F(p) > F(q)$ we have:*

$$\text{vol}(N_{p,q}) \leq C \cdot D^{[F(p)-F(q)]}.$$

The next lemma is a direct consequence of Sard Theorem.

Lemma 6.3.3. *Let $q \in S(F)$. Then there is a residual subset*

$$Q \subset B^{\text{ind } q}(0, \alpha)$$

such that for every $\xi \in Q$ and every $p \in S(F)$ with $\text{ind } p = \text{ind } q + 1$ we have: $D(p, v) \pitchfork S_q(\xi)$. \square

The next proposition follows from (5.2) by arguments due to V. I. Arnold (see [1, p. 81]).

Proposition 6.3.4. *Let $q \in S(F)$. Then there is a subset $Q \subset B^{\text{ind } q}(0, \alpha)$ of full measure such that for every $\xi \in Q$ we have:*

- *For every $p \in S(F)$ with $\text{ind } p = \text{ind } q + 1$ there are constants $K, R > 0$, such that for every integer $l \geq 0$ we have:*

$$\sharp(N_{pt^{-l}, q} \cap (S_q \times \{\xi\})) \leq K \cdot R^l.$$

Corollary 6.3.5. *Let $\nu > 0$. Then there exists a good f -gradient u with $\|u - v\| < \nu$ and constants $K, L > 0$ such that for every $p, q \in S(f)$ with $\text{ind } p = \text{ind } q + 1$ we have $N_l(p, q; \nu) \leq L \cdot K^l$.*

Proof. The construction of regular perturbation, described in §6.1, applied to $v|_W$, gives for every $\vec{\xi} = \{\xi_p\}_{p \in S(F) \cap W}$, $\xi_p \in B^{\text{ind } p}(0, \alpha)$, an $F|_W$ -gradient $v\langle \vec{\xi} \rangle$. Since $\text{supp}(v - v\langle \vec{\xi} \rangle)$ does not intersect ∂W , we can extend $v\langle \vec{\xi} \rangle$ t -invariantly to \bar{M} and obtain a t -invariant F -gradient, which will be denoted by the same symbol $v\langle \vec{\xi} \rangle$. Note that since

$$\{(A_\sigma^{(s)}, B_\sigma^{(s)})\}_{\sigma \in \Sigma, 0 \leq s \leq n}$$

is a BERS for $(F, v\langle \vec{\xi} \rangle)$, the vector field $v\langle \vec{\xi} \rangle$ is an almost good F -gradient.

If all $\xi_q \in B^{\text{ind } q}(0, \alpha)$ are sufficiently small, we will have $\|u - v\langle \vec{\xi} \rangle\| < \nu$. Lemma 6.3.3 and Proposition 6.3.4 imply that we can choose the components ξ_q of $\vec{\xi}$ in such a way that

- (1) For every $p \in S(F)$, $\text{ind } p = \text{ind } q + 1$ we have:

$$D(p, v) \cap F^{-1}(\mu_q) \pitchfork S_q(\xi_q).$$

- (2) There are $K, R > 0$ such that for every $p \in S(F)$ with $\text{ind } p = \text{ind } q + 1$ we have:

$$\sharp(D(pt^{-l}, v) \cap S_q(\xi_q)) \leq K \cdot R^l.$$

Apply Proposition 6.1.3 and deduce that $u = v\langle \vec{\xi} \rangle$ satisfy the conclusions of the Corollary. \square

6.4. Proof of Theorem D. We can assume that $S(f) \neq \emptyset$ and that the homotopy class of f is indivisible. In view of Corollary 6.3.5 it is sufficient to prove that the set of f -gradients, having a BERS, is C^0 -dense in $\mathcal{G}(f)$. To prove that, let v be any f -gradient and $\epsilon > 0$. Choose any regular value λ of F and denote by W the cobordism $F^{-1}([\lambda, \lambda + 1])$. Choose any set of regular values of F , satisfying $(\Sigma 1)$ - $(\Sigma 3)$ of Definition 6.3.1. Theorem 2.6.5 implies

that there is an almost good $F|_W$ -gradient u and a ranging pair $(\mathcal{V}_0, \mathcal{V}_1)$ for $(F|_W, u)$ such that $\|v - u\| \leq \epsilon$ and $v = u$ near ∂W . We can also assume that $\mathcal{V}_0 t = \mathcal{V}_1$. Then the procedure, described in [15, Construction 4.13], page 1000 defines a bunch of ranging systems for $(F|_W, u)$. Extend u to a t -invariant F -gradient. The BRS constructed is easily extended to a BERS for (F, u) . \square

We mention here an obvious corollary of Theorem D which will be of use in the proof of Theorem E. For a good f -gradient $v, c \in \mathbb{R}$ and $x, y \in S(f)$ with $\text{ind } x = \text{ind } y + 1$ we denote by $N_{\geq c}(x, y; v)$ the sum $\sum_{-k \geq c} N_k(x, y; v)$.

Corollary 6.4.1. *Let $v \in \mathcal{G}_0(f)$. Then there are constants $R, Q > 0$ such that for every critical points $x, y \in S(f)$ with $\text{ind } x = \text{ind } y + 1$ and every $c \in \mathbb{R}$ we have: $N_{\geq c}(x, y; v) \leq R \cdot Q^{-c}$.* \square

6.5. Proof of the Theorem E. By Lemma 4.2.4 find $\epsilon > 0$ and $\delta > 0$ such that ω_ϵ is a Morse family, v is an ω_ϵ -gradient and every ω -gradient u with $\|u - v\| < \delta$ is an Ω_ϵ -gradient. Choose some $\xi \in \Omega_\epsilon$ with $[\xi] \in H^1(M, \mathbf{Q})$ and choose (by Theorem D) a good ξ -gradient u , satisfying the condition (2) of Theorem D and such that $\|u - v\| < \delta$. Then u is an ω_ϵ -gradient. By Lemma 4.3.5 for every $x, y \in S(\omega)$ there is an integral $([\xi], [\omega])$ -cone Δ and $b \in \mathbb{Z}^m$ such that

$$Q(I(\tilde{x}, \tilde{y}; u)) \subset \Delta + b.$$

Lemma 5.1.1 implies that there are A, B such that for every $\lambda \in \mathbb{R}$ we have

$$I(\tilde{x}, \tilde{y}; u) \cap \{\omega\}^{-1}([\lambda, \infty)) \subset I(\tilde{x}, \tilde{y}; u) \cap \{\xi\}^{-1}([A\lambda + B, \infty)),$$

and this together with Corollary 6.4.1 implies the conclusion. \square

7. ACKNOWLEDGMENTS

The author thanks the anonymous referee for many helpful remarks, that have lead to a substantial improvement of the manuscript.

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Received: January 15, 2020, accepted: October 23, 2020.

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