

# (In)homogeneous invariant compact convex sets of probability measures

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**Abstract.** It is proved that for any iterated function system of contractions on a complete metric space there exists an invariant compact convex sets of probability measures of compact support on this space. A similar result is proved for the inhomogeneous compact convex sets of probability measures of compact support.

**Анотація.** Математичні підвалини теорії фракталів запропонував Дж. Гатчінсон у 80-х роках минулого століття. Зокрема, він означив поняття атрактора (або інваріантного об'єкта) для ітерованої системи стискуючих відображень (скорочено IFS) на повному метричному просторі і довів існування таких атракторів у гіперпросторі (просторові непорожніх компактних підмножин) та просторі ймовірнісних мір з компактними носіями на повному метричному просторі. Доведення Гатчінсона використовують принцип стискуючих відображень і, зокрема, потребують відповідної метризації простору ймовірнісних мір.

Незабаром аналогічні результати було отримано і для неоднорідних атракторів (тобто атракторів з приєднаними ущільнюючими множинами), які є природними узагальненнями інваріантних множин та інваріантних мір.

У цій статті ми запроваджуємо поняття інваріантного об'єкта для IFS у просторі компактних опуклих множин ймовірнісних мір з компактними носіями у повному метричному просторі. Такі компактні опуклі множини мір мають численні застосування у теорії очікуваної корисності. На відміну від гіперпростору компактних опуклих підмножин повного метричного простору, інваріантні об'єкти в якому виглядають регулярними, атрактори IFS у просторі компактних опуклих множин мір зберігають іррегулярність, притаманну фрактальним множинам.

Одним з основних результатів є теорема існування та єдиності інваріантної компактної опуклої множини ймовірнісних мір з компактними

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носіями у повному метричному просторі. Окрім традиційного підходу до доведення такого типу результатів, що використовує принцип стискуючих відображень, ми пропонуємо також і функціональний підхід, який не опирається на метризацію, а натомість використовує функціональне зображення компактних опуклих підмножин у просторах ймовірнісних мір.

Аналогічні результати отримано і для випадку неоднорідних інваріантних опуклих множин ймовірнісних мір.

## 1. INTRODUCTION

Hutchinson [7] proved the existence of invariant sets and invariant probability measures for the iterated function systems in the complete metric spaces. The structure of these two proofs is similar and it exploits, in particular, the functoriality of the constructions involved (i.e., the hyperspaces and spaces of probability measures) as well as existence of special metrizations. This led to several generalizations of the existence results, in particular, to the cases of inclusion hyperspaces (i.e., two-valued measures) [11] and idempotent measures on ultrametric spaces [9].

Another approach is applied in [10] and it is proved therein that there exists an invariant idempotent measure (see [18] for topological aspects of the theory of idempotent measures) for an iterated function system on a complete metric space.

Recently, a related notion of inhomogeneous invariant set and measure was introduced in [15]. The properties of these sets and measures were studied in various publications (see, e.g., [5, 1, 13]).

The compact convex sets of probability measures are used in the maxmin expected utility (MEU) theory [6].

## 2. PRELIMINARIES

**2.1. Hyperspaces.** Let  $\exp X$  denote the set of all nonempty compact subsets of a Tychonov space  $X$ . A base of the Vietoris topology on  $\exp X$  consists of the sets of the form

$$\langle U_1, \dots, U_n \rangle = \{A \in \exp X \mid A \subset \bigcup_{i=1}^n U_i, A \cap U_i \neq \emptyset \text{ for all } i\},$$

where  $n \in \mathbb{N}$  and  $U_1, \dots, U_n$  are open sets in  $X$ . The obtained space is called the hyperspace of  $X$ .

Actually,  $\exp$  is a functor in the category of Tychonov spaces and continuous maps. Given a map  $f: X \rightarrow Y$ , the map  $\exp f: \exp X \rightarrow \exp Y$  acts as follows:  $\exp f(A) = f(A)$ ,  $A \in \exp X$ .

If  $(X, d)$  is a metric space, then the Vietoris topology on  $\exp X$  is induced by the Hausdorff metric  $d_H$ ,

$$d_H(A, B) = \inf\{r > 0 \mid A \subset O_r(B), B \subset O_r(A)\},$$

where  $O_r(C)$  stands for the open  $r$ -neighborhood of a subset  $C$ .

By  $u_X: \exp \exp X = \exp^2 X \rightarrow \exp X$  we denote the union map. This map is known to be well defined and, in the case of metric space, nonexpanding.

**2.2. Kantorovich metric.** By  $P(X)$  we denote the space of probability measures on a compact Hausdorff space  $X$ . We regard the set of probability measures on  $X$  also as a set of normed linear functionals on the Banach space  $C(X)$  of continuous real-valued functions on  $X$ . Given  $\mu \in P(X)$ , we let  $\mu(\varphi) = \int_X \varphi d\mu$ ,  $\varphi \in C(X)$ .

The set  $P(X)$  is endowed with the weak\* topology. A base of this topology is comprised by the sets of the form

$$O\langle \mu_0; \varphi_1, \dots, \varphi_n; \varepsilon \rangle = \{\mu \in P(X) \mid |\mu(\varphi_i) - \mu_0(\varphi_i)| < \varepsilon, i = 1, \dots, n\},$$

where  $\mu_0 \in P(X)$ ,  $\varphi_1, \dots, \varphi_n \in C(X)$ ,  $\varepsilon > 0$ .

Let  $(X, d)$  be a compact metric space. By  $1\text{-LIP}(X)$  we denote the set of all nonexpanding functions on  $X$ , i.e. functions  $\varphi: X \rightarrow \mathbb{R}$  satisfying

$$|\varphi(x) - \varphi(y)| \leq d(x, y)$$

for all  $x, y \in X$ . The Kantorovich metric  $d_K$  on the space of probability measures  $P(X)$  is defined as follows:

$$d_K(\mu, \nu) = \sup\{|\mu(\varphi) - \nu(\varphi)| \mid \varphi \in 1\text{-LIP}(X)\}.$$

Every continuous map  $f: X \rightarrow Y$  between compact spaces induces the map

$$P(f): P(X) \rightarrow P(Y)$$

defined by  $P(f)(\mu)(A) = \mu(f^{-1}(A))$  for any  $\mu \in P(X)$  and any measurable subset  $A \subset Y$ . In terms of functionals,  $P(f)(\mu)(\varphi) = \mu(\varphi \circ f)$  for all  $\mu \in P(X)$  and  $\varphi \in C(Y)$ .

Actually,  $P$  is a functor in the category **Comp** of compact Hausdorff spaces.

There is a procedure of extensions of functors from the category **Comp** to the category of Tychonov spaces [4]. In the case of the functor  $P$ , this procedure gives the space of probability measures of compact support. Recall that the *support* of  $\mu \in P(X)$  is the minimal closed set  $A \subset X$  such that  $\mu(X \setminus A) = 0$ . Alternatively, the support of  $\mu$  is the minimal closed set  $A \subset X$  with the property that, for all  $\varphi, \psi \in C(X)$ ,  $\varphi|_A = \psi|_A$  implies  $\mu(\varphi) = \mu(\psi)$ .

**2.3. Convex sets of probability measures.** Let  $X$  be a compact Hausdorff space. Denote by  $\text{ccP}(X)$  the hyperspace of closed convex subsets of the space  $P(X)$ . Given a continuous map  $f: X \rightarrow Y$  between compact spaces, we define the map  $\text{ccP}(f) : \text{ccP}(X) \rightarrow \text{ccP}(Y)$  as follows:

$$\text{ccP}(f)(A) = \{P(f)(\mu) \mid \mu \in A\}, \quad A \in \text{ccP}(X).$$

It is known that  $\text{ccP}$  is a functor on the category **Comp** (see, e.g. [16]). Given  $A \in \text{ccP}(X)$ , we say that the set  $\bigcup\{\text{supp}(\mu) \mid \mu \in A\}$  is the *support of  $A$*  (denoted  $\text{supp}(A)$ ). (Hereafter, for any set  $Y$  in a topological space, we denote by  $\bar{Y}$  its closure). Again, applying construction from [4] we extend the functor  $\text{ccP}$  onto the category of Tychonov spaces. We preserve the notation  $\text{ccP}$  for this extension.

For any metrizable space  $X$ , the space  $\text{ccP}(X)$  is exactly the hyperspace of closed convex subsets  $A$  of  $P(X)$  such that  $\text{supp}(A)$  is compact.

Now, assume that  $X$  is compact and define a map

$$\theta_X : \text{ccP}^2(X) \rightarrow \text{ccP}(X),$$

as follows, see [12]. First, for any compact convex subset  $K$  of a locally convex space, denote by  $b_K : P(K) \rightarrow K$  the barycenter map. Since  $P(X)$  is a subset of the dual space  $C(X)'$  endowed with the weak\* topology, the hyperspace  $\text{ccP}(X)$  can be regarded as a compact convex subset of a locally convex space [14] and therefore one can consider the barycenter map

$$b_{\text{ccP}(X)} : P(\text{ccP}(X)) \rightarrow \text{ccP}(X).$$

Finally, define  $\theta_X$  by the formula

$$\theta_X(\mathfrak{A}) = \bigcup_{M \in \mathfrak{A}} b_{\text{ccP}(X)}(M), \quad \mathfrak{A} \in \text{ccP}^2(X).$$

Note that the continuity of  $\theta_X$  is a consequence of the continuity of the barycenter map [3, Chapt. III, §3, Corollary of Proposition 9] and the union map [17, Proposition 5.2].

In the case when  $\mathfrak{B}$  is a compact convex subset of the convex hull of a set  $\{M_1, \dots, M_n\}$ , where  $M_1, \dots, M_n \in P(\text{ccP}(X))$ , we have

$$\theta_X(\mathfrak{B}) = \left\{ \sum_{i=1}^n \alpha_i b_{\text{ccP}(X)}(M_i) \mid \alpha_1, \dots, \alpha_n \geq 0, \sum_{i=1}^n \alpha_i = 1, \sum_{i=1}^n \alpha_i M_i \in \mathfrak{B} \right\}.$$

Now, let  $(X, d)$  be a metric space. We endow  $\text{ccP}(X)$  with the Hausdorff metric induced by the Kantorovich metric on  $P(X)$ . By [8, Proposition 3.2], the map  $\theta_X : \text{ccP}^2(X) \rightarrow \text{ccP}(X)$  is nonexpanding.

Let  $c > 0$ . A map  $f : X \rightarrow Y$  from a metric space  $(X, d)$  to a metric space  $(Y, \varrho)$  is called *c-Lipschitz* if  $\varrho(f(x), f(y)) \leq c \cdot d(x, y)$  for all  $x, y \in X$ . As mentioned above, the 1-Lipschitz maps are also called *nonexpanding*.

**Proposition 2.4.** *Let  $f: X \rightarrow Y$  be a  $c$ -Lipschitz map. Then  $\text{ccP}(f)$  is also a  $c$ -Lipschitz map.*

**Proof.** The proof is a consequence of known results on the estimations of constants for the maps of hyperspaces [7, 2.4 (i)] and of spaces of probability measures [7, Theorem 4.4 (1)(i)].  $\square$

### 3. RESULTS

Let  $(X, d)$  be a complete metric space and  $\{f_1, f_2, \dots, f_n\}$  be a finite family of contractions on  $X$  (that is, an iterated function system, IFS). Let us consider the discrete topology on the set  $\{1, 2, \dots, n\}$ . Then the space  $P(\{1, 2, \dots, n\})$  can be regarded as the standard  $(n-1)$ -dimensional simplex  $\Delta^{n-1}$  in  $\mathbb{R}^n$ ,

$$\Delta^{n-1} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum_{i=1}^n x_i = 1 \right\}$$

by identifying  $\sum_{i=1}^n \alpha_i \delta_i \in P(\{1, 2, \dots, n\})$  with  $(\alpha_1, \dots, \alpha_n) \in \Delta^{n-1}$ .

For  $B \in \text{ccP}(\{1, 2, \dots, n\})$  define the map  $\Phi_B: \text{ccP}(X) \rightarrow \text{ccP}(X)$  as follows. Let  $A \in \text{ccP}(X)$  and  $g_A: \{1, 2, \dots, n\} \rightarrow \text{ccP}(X)$  be the map sending  $i$  to  $\text{ccP}(f_i)(A)$ . Then we set

$$\Phi_B(A) = \theta_X(\text{ccP}(g_A)(B)).$$

We say that  $A \in \text{ccP}(X)$  is an *invariant set of probability measures* for  $\{f_1, f_2, \dots, f_n\}$  and  $B$  whenever  $A = \Phi_B(A)$ .

**Theorem 3.1.** *For any IFS  $\{f_1, f_2, \dots, f_n\}$  and  $B \in \text{ccP}(\{1, 2, \dots, n\})$  there exists a unique invariant closed convex set of probability measures.*

**Proof.** We first consider the case of compact space  $X$ . Note that the map  $\Phi_B$  is a contraction. This follows from the fact that the functor  $\text{ccP}$  preserves  $c$ -maps and the map  $\theta_X$  is nonexpanding. By the Banach Contraction Principle, there exists a unique  $A \in \text{ccP}(X)$  such that  $A = \Phi_B(A)$ .

In the case of noncompact space  $X$ , consider the map  $\Psi: \exp X \rightarrow \exp X$  defined as follows:  $\Psi(D) = \bigcup_{i=1}^n f_i(D)$ . It follows from [7, 3.1 (3)(viii)] that

the set  $\overline{\bigcup_{i=1}^{\infty} \Psi^i(D)}$  is compact for any  $D \in \exp X$ .

Now, consider an arbitrary  $C \in \text{ccP}(X)$  and let  $K = \text{supp}(C)$ . Then the set  $Y = \overline{\bigcup_{i=1}^{\infty} \Psi^i(K)}$  is compact. Note that  $f_i(Y) \subset Y$ ,  $i = 1, \dots, n$ . Since

$$C \in \text{ccP}(Y) \subset \text{ccP}(X),$$

the above arguments show that there exists an invariant closed convex set of probability measures  $A_0 \in \text{ccP}(Y) \subset \text{ccP}(X)$ .  $\square$

Suppose that we are given an IFS  $\{f_1, f_2, \dots, f_n\}$  on  $X$ ,  $B$  is an element of  $\text{ccP}(\{0, 1, \dots, n\})$ , and  $C \in \text{ccP}(X)$ . For any  $A \in \text{ccP}(X)$  let

$$g'_{A,C}: \{0, 1, 2, \dots, n\} \rightarrow \text{ccP}(X)$$

be defined by the formulas:

$$g'_{A,C}(0) = C, \quad g'_{A,C}(i) = \text{ccP}(f_i)(A), \quad (i = 1, \dots, n).$$

Define  $\Phi'_{B,C}: \text{ccP}(X) \rightarrow \text{ccP}(X)$  by

$$\Phi'_{B,C}(A) = \theta_X(\text{ccP}(g'_{A,C})(B)).$$

Then the set  $A$  satisfying  $A = \Phi'(A)$  is called an *inhomogeneous invariant convex set of probability measures*.

**Theorem 3.2.** *For any IFS  $\{f_1, f_2, \dots, f_n\}$ ,  $B \in \text{ccP}(\{0, 1, \dots, n\})$  and  $C \in \text{ccP}(X)$  there exists a unique inhomogeneous invariant convex set of probability measures.*

**Proof.** Similarly to the proof of the previous theorem, in the compact case we apply the Banach Contraction Principle to the map  $\Phi'$ . The non-compact case can be reduced to the compact one similarly as in the proof of Theorem 3.1.  $\square$

**Proposition 3.3.** *If the set  $B \in \text{ccP}(\{1, 2, \dots, n\})$  from the definition of invariant convex set of probability measures is a singleton, then the obtained invariant convex set of probability measures is a singleton as well.*

**Proof.** Let  $B = \{\mu\} \in \text{ccP}(\{1, 2, \dots, n\})$ , for some  $\mu \in P(\{1, 2, \dots, n\})$ , where  $\mu = \sum_{i=1}^n \alpha_i \delta_i$ . We start with  $A_0 = \{\nu_0\} \in \text{ccP}(X)$ . Then clearly

$$A_1 = \Phi(A_0) = \left\{ \sum_{i=1}^n \alpha_i P(f_i)(\nu_0) \right\} = \{\nu_1\}$$

and this easily implies that the invariant set of probability measures  $A_\infty$  in this case is  $\{\nu_\infty\}$ , where  $\nu_\infty$  is the invariant measure in the sense of [7] corresponding to the IFS  $\{f_1, \dots, f_n\}$  and  $\mu = \sum_{i=1}^n \alpha_i \delta_i$ .  $\square$

A similar statement can be formulated and proved in the inhomogeneous case. Therefore our considerations are in some sense extensions of known results from [7] and [15] on probability measures.

## 4. FUNCTIONAL APPROACH

Let  $X$  be a compact Hausdorff space. Every  $A \in \text{ccP}(X)$  determines a functional  $F_A: C(X) \rightarrow \mathbb{R}$  defined as follows:

$$F_A(\varphi) = \sup_{\mu \in A} \mu(\varphi), \quad \varphi \in C(X).$$

**Proposition 4.1.** *If  $A, B \in \text{ccP}(X)$  and  $A \neq B$ , then  $F_A \neq F_B$ .*

**Proof.** Denote by

$$\iota: P(X) \rightarrow \prod_{\varphi \in C(X)} \mathbb{R}_\varphi$$

the canonical embedding  $\iota(\mu) = (\mu(\varphi))_{\varphi \in C(X)}$ , where  $\mathbb{R}_\varphi$  is a copy of  $\mathbb{R}$  for every  $\varphi \in C(X)$ .

Without loss of generality one may assume that there exists  $\mu \in A \setminus B$ . Since  $B$  is compact, there are  $\varphi_1, \dots, \varphi_k \in C(X)$ , for some  $k \in \mathbb{N}$ , such that  $p(\mu) \notin p(B)$ , where  $p: \prod_{\varphi \in C(X)} \mathbb{R}_\varphi \rightarrow \prod_{i=1}^k \mathbb{R}_{\varphi_i}$  is the canonical projection.

Since  $p(B)$  is compact and convex, it follows from the hyperplane separation theorem that there exists a linear functional  $l: \prod_{i=1}^k \mathbb{R}_{\varphi_i} \rightarrow \mathbb{R}$  such that  $\sup_{\nu \in B} l(p(\nu)) < l(p(\mu))$ .

Then there exists  $(l_1, \dots, l_k) \in \mathbb{R}^k$  such that

$$l(x_1, \dots, x_k) = \sum_{i=1}^k l_i x_i, \quad (x_1, \dots, x_k) \in \mathbb{R}^k.$$

Now, let  $\psi = \sum_{i=1}^k l_i \varphi_i$ . Then for each  $\nu \in P(X)$

$$\nu(\psi) = \nu \left( \sum_{i=1}^k l_i \varphi_i \right) = \sum_{i=1}^k l_i \nu(\varphi_i) = l(p(\iota(\nu))).$$

Therefore,  $\mu(\psi) > \sup_{\nu \in B} \nu(\psi) = F_B(\psi)$ . Hence  $F_A(\psi) \geq \mu(\psi) > F_B(\psi)$ , i.e.  $F_A \neq F_B$ .  $\square$

Let  $\tau^*$  be the weak\* topology on the set  $\mathcal{F} = \{F_A \mid A \in \text{ccP}(X)\}$ , i.e., the topology induced from the product topology on  $\mathbb{R}^{C(X)}$ . A base of this topology is comprised by the sets of the form

$$\begin{aligned} O' \langle F_{A_0}; \varphi_1, \dots, \varphi_n; \varepsilon \rangle = \\ = \{F_A \mid A \in \text{ccP}(X), |F_A(\varphi_i) - F_{A_0}(\varphi_i)| < \varepsilon, i = \overline{1, n}\}, \end{aligned}$$

where  $A_0 \in \text{ccP}(X)$ ,  $\varphi_1, \dots, \varphi_n \in C(X)$ ,  $\varepsilon > 0$ .

**Proposition 4.2.** *The map  $A \mapsto F_A: \text{ccP}(X) \rightarrow \mathcal{F}$  is continuous.*

**Proof.** Note that the sets  $O'\langle F_{A_0}; \varphi_0; \varepsilon \rangle$ , where  $A_0 \in \text{ccP}(X)$ ,  $\varphi_0 \in C(X)$ , and  $\varepsilon > 0$ , comprise a subbase for the topology  $\tau^*$ .

Given such a set  $O'\langle F_{A_0}; \varphi_0; \varepsilon \rangle$  and  $\mu \in A_0$ , consider the set  $O\langle \mu; \varphi_0; \varepsilon \rangle$ . Then the family  $\{O\langle \mu; \varphi_0; \varepsilon \rangle \mid \mu \in A_0\}$  is an open cover of  $A_0$ . Let

$$\{O\langle \mu_i; \varphi_0; \varepsilon \rangle \mid i = 1, \dots, n\}$$

be a finite subcover of this cover. Then  $\langle O\langle \mu_1; \varphi_0; \varepsilon \rangle, \dots, O\langle \mu_n; \varphi_0; \varepsilon \rangle \rangle$  is a neighborhood of  $A_0$  in  $\text{ccP}(X)$ .

We are going to show that for each  $A \in \langle O\langle \mu_1; \varphi_0; \varepsilon \rangle, \dots, O\langle \mu_n; \varphi_0; \varepsilon \rangle \rangle$  the functional  $F_A \in O'\langle F_{A_0}; \varphi_0; \varepsilon \rangle$ . This will finish the proof.

If  $\max\{\mu(\varphi_0) \mid \mu \in A\} = \mu_0(\varphi_0)$  for some  $\mu_0 \in A$ , then there exists  $i \in \{1, \dots, n\}$  such that  $\mu_0 \in O\langle \mu_i; \varphi_0; \varepsilon \rangle$ . Then

$$F_A(\varphi_0) = \mu_0(\varphi_0) < \mu_i(\varphi_0) + \varepsilon \leq F_{A_0}(\varphi_0) + \varepsilon.$$

One can similarly prove that  $F_{A_0}(\varphi_0) \leq F_A(\varphi_0) + \varepsilon$ . □

**Corollary 4.3.** *The map  $A \mapsto F_A: \text{ccP}(X) \rightarrow \mathcal{F}$  is a homeomorphism.*

**Proof.** Due to compactness of  $X$ , the space  $\text{ccP}(X)$  is compact, and the assertion follows from the hausdorffness of  $\mathcal{F}$  and Proposition 4.1. □

Now the mentioned functional representation  $A \mapsto F_A$  of compact convex sets of probability measures allows us to obtain a purely functional proof of the main results of this paper in the spirit of [10, Theorem 1].

## 5. REMARKS

In the case when  $X = \mathbb{R}^n$  and the maps  $f_1, \dots, f_n$  are similarities, one can find many pictures of invariant and inhomogeneous sets in the literature.

The invariant probability measures can be visualized in a gray scale by using the random iteration algorithm (see [2, Chapt. IX] for details). An open problem is that of visualization of invariant convex sets of probability measures.

## REFERENCES

- [1] Simon Baker, Jonathan M. Fraser, András Máthé. Inhomogeneous self-similar sets with overlaps. *Ergodic Theory Dynam. Systems*, 39(1):1–18, 2019, doi: 10.1017/etds.2017.13.
- [2] Michael F. Barnsley. *Fractals everywhere*. Academic Press Professional, Boston, MA, second edition, 1993. Revised with the assistance of and with a foreword by Hawley Rising, III.

- [3] Nicolas Bourbaki. *Integration. I. Chapters 1–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2004. Translated from the 1959, 1965 and 1967 French originals by Sterling K. Berberian.
- [4] A. Ch. Chigogidze. Extension of normal functors. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, (6):23–26, 110, 1984.
- [5] Jonathan M. Fraser. Inhomogeneous self-similar sets and box dimensions. *Studia Math.*, 213(2):133–156, 2012, doi: 10.4064/sm213-2-2.
- [6] Itzhak Gilboa, David Schmeidler. Maxmin expected utility with nonunique prior. *J. Math. Econom.*, 18(2):141–153, 1989, doi: 10.1016/0304-4068(89)90018-9.
- [7] John E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981, doi: 10.1512/iumj.1981.30.30055.
- [8] V. Levyts'ka, M. Zarichnyi. Spaces of nonexpanding maps: categorical properties. *Mat. Stud.*, 16(1):3–12, 2001.
- [9] N. Mazurenko, M. Zarichnyi. Idempotent ultrametric fractals. *Visnyk of the Lviv Univ. Series Mech. Math.*, 79:111–118, 2014.
- [10] N. Mazurenko, M. Zarichnyi. Invariant idempotent measures. *Carpathian Math. Publ.*, 10(1):172–178, 2018, doi: 10.15330/cmp.10.1.172-178.
- [11] N. I. Melnychuk. On invariant inclusion hyperspaces for iterated function systems. *Mat. Stud.*, 17(2):211–214, 2002.
- [12] O. R. Nykyforchyn. *Probability measures, measurable maps, and convexity: categorical properties*. PhD thesis, Lviv University, 1996. (in Ukrainian).
- [13] L. Olsen, N. Snigireva. Multifractal spectra of in-homogenous self-similar measures. *Indiana Univ. Math. J.*, 57(4):1789–1843, 2008, doi: 10.1512/iumj.2008.57.3622.
- [14] Hans Rådström. An embedding theorem for spaces of convex sets. *Proc. Amer. Math. Soc.*, 3:165–169, 1952, doi: 10.2307/2032477.
- [15] N. Snigireva. *Inhomogeneous self-similar sets and measures*. PhD thesis, University of St Andrews, 2008.
- [16] A. Teleiko, M. Zarichnyi. *Categorical topology of compact Hausdorff spaces*, volume 5 of *Mathematical Studies Monograph Series*. VNTL Publishers, L'viv, 1999.
- [17] Keith R. Wicks. *Fractals and hyperspaces*, volume 1492 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1991, doi: 10.1007/BFb0089156.
- [18] M. Zarichnyi. Spaces and mappings of idempotent measures. *Izvestiya: Math.*, 74(3):481–499, 2010, doi: 10.4213/im2785.

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