Commutative morphic rings of stable range 2

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Abstract It is know that a left quasi-morphic ring R is a ring of stable range 1 if and only if $\dim R = 0$. In this paper it is shown that a commutative morphic ring R is a ring of stable range 2 if and only if $\dim R = 1$.

Keywords Stable range 1, stable range 2.

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Introduction

In his pioneering paper [3] Kaplansky has raised the question: If $aR = bR$ in a ring R, are a and b necessarily right associates? He remarked that for a commutative ring the property holds for the principal ideal rings and artinian rings.

Developing these ideas Canfell [2] to introduce the concept of the set of principal ideals $\{a_iR | i = 1, 2, \ldots, n\}$ is uniquely generated if whenever $a_iR = b_iR$ there exist elements $u_i \in R$ such that $a_i = b_iu_i$, $i = 1, 2, \ldots, n$, and $u_1R + u_2R + \ldots + u_nR = R$. The dimension of a commutative ring R, denoted by $\dim R$, is the least integer n such that every set of $n+1$ principal ideals is uniquely generated. In [3] Canfell obtained characterizations of the n-dimensional F-spaces in terms of the rings of continuous real-valued and complex-valued functions defined on the space. By extending the notion of uniqueness of generators of principals he gave an algebraic characterization of the concept "n-dimensional".

In [1] Bass introduced the notion of stable range for a ring. A ring R with stable range one is both left and right uniquely generated [6]. The converse does not hold true in general. Then $\mathbb{Z}$ is uniquely generated. Indeed, the ring $\mathbb{Z}$ is uniquely generated, but $\mathbb{Z}$ does not have stable range one. The stable range of $\mathbb{Z}$ is equal 2 [6].

In the case of a left quasi morphic ring we have that property a left uniquely generated is equivalent to the property of stable range one [5].

In this article we will show that in the case of a commutative morphic ring we see that the property $\dim R = 1$ is equivalent to the property of the stable range 2.

1 Results

All our rings are commutative with identity. A ring R is said to have stable range 2 if for any $a, b, c \in R$ such that $aR+bR+cR = R$ there exist $x, y \in R$ such that $(a + cx)R + (b + cy)R = R$ [6] (in the notion $\mathrm{st.r}(R) = 2$). We say that a ring R is an almost Baer ring if for each $x \in R$ there exists an element $y \in R$ such that $\mathrm{Ann}(xR) = yR$ where $\mathrm{Ann}(xR) = \{z|zxr = 0, r \in R\}$. By a Bezout ring we mean a ring in which all finitely generated ideals are principal. Two rectangular matrices A and B are equivalent if there exist invertible matrices P and Q of adequate size such that $B = PAQ$. A ring R is Hermite if every rectangular matrix A over R is equivalent to an upper or a lover triangular matrix [3]. Any Hermite ring is a Bezout ring [2]. In [6] we have the following theorem.

Theorem 1 *A commutative Bezout ring R is an Hermite ring if and only if* $\mathrm{st.r}(R) = 2$.

A commutative ring R is called a morphic ring if for any $a \in R$ there is an isomorphism $R/Ra \cong \mathrm{Ann}(a)$ as of R modules.

Theorem 2 *Let R be a commutative Bezout ring and* $\dim R = 1$. *Then* $\mathrm{st.r}(R) = 2$.

Proof Let $a, b \in R$. Since R is a commutative Bezout ring, $aR + bR = dR$, for some element $d \in R$. There exist $a_0, b_0 \in R$ and $u, v \in R$ such that $a = da_0$, $b = db_0$ and $d = au + bv = a_0ud + b_0vd$. Put $q = 1 - a_0u - b_0v$.

Then $dq = 0$ and for any elements $t_1, t_2 \in R$ we $(a_0+t_1q)d = a$, $(b_0+t_2q)d = b$. We will choose the t_i, $i = 1, 2$. So that the elements $a_0+t_1q = a_1$ and $b_0+t_2q = b_2$ generate R.

Then $a_1x + b_1y = 1$ for some elements $x, y \in R$ and $a = a_1d$, $b = b_1d$. By [3] R is an Hermite ring and by Theorem 1 we have $\mathrm{st.r}(R) = 2$. The theorem is proved.

Theorem 3 *Let R be a commutative almost Baer Bezout ring of stable range 2. Then* $\dim(R) = 1$.

Proof Let $a_1R = b_1R$ and $a_2R = b_2R$. Then $a_1 = x_1b_1$, $a_2 = x_2b_2$ and $b_1 = y_1a_1$, $b_2 = y_2a_2$ for some $x_1, x_2, y_1, y_2 \in R$. Then $b_1(1 - x_1y_1) = 0$, $b_2(1 - x_2y_2) = 0$ and $1 - x_1y_1 \in \mathrm{Ann}(b_1R)$, $1 - x_2y_2 \in \mathrm{Ann}(b_2R)$. Let $\mathrm{Ann}(b_1R) = \alpha_1R$ and $\mathrm{Ann}(b_2R) = \alpha_2R$ for some $\alpha_1, \alpha_2 \in R$.

Since $1 - x_1y_1 \in \alpha_1R$ and $1 - x_2y_2 \in \alpha_2R$, we have $x_1R + \alpha_1R = R$ and $x_2R+\alpha_2R = R$. Obviously, $x_1R+x_2R+\alpha_1\alpha_2R = R$. Since $\mathrm{st.r}(R) = 2$, we have $(x_1 + \alpha_1\alpha_2s)R + (x_2 + \alpha_1\alpha_2t)R = R$ for some $st \in R$. Since $(x_1 + \alpha_1\alpha_2t)b_1 = x_1b_1+\alpha_2t\alpha_1b = x_1b_1 = a_1$, $(x_2+\alpha_1\alpha_2s)b_2 = x_2b_2+\alpha_1s\alpha_2b = x_2b_2 = a_2$. Denote $x_1 + \alpha_1\alpha_2t = u_1$, $x_2 + \alpha_1\alpha_2s = u_2$.

We proved $u_1b_1 = a_1$, $u_2b_2 = a_2$ and $u_1R + u_2R = R$, that is $\dim(R) = 1$. The theorem is proved.

A commutative morphic ring is an obvious example of an almost Baer Bezout ring [4].

As a consequence of Theorem 2 and Theorem 3 we obtain the following result.

Theorem 4 *A commutative morphic ring R is a ring of stable range 2 if and only if* $\dim(R) = 1$.

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