

# Asymptotic properties of the (convex) hyperspaces

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**Abstract** It is known that the hyperspaces of compact sets and compact convex set of the Euclidean space  $\mathbb{R}^n$ ,  $n \geq 2$ , both are homeomorphic to the pictured Hilbert cube. The main result of this note states that these hyperspaces are not coarsely equivalent.

**Keywords** Hyperspace, convex set, coarse equivalence

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## 1 Introduction

Asymptotic topology (coarse geometry) deals with the large scale properties of metric spaces and more general objects (coarse spaces; see [2, 4]). The results of asymptotic topology have their applications in geometric group theory, differential and algebraic topology as well as in another parts of mathematics. One of the main concepts in the asymptotic topology is that of coarse equivalence (see the definition below).

For a metric space  $(X, \rho)$  the hyperspace  $\exp(X)$  (the set of all nonempty compact subsets in  $X$ ) is endowed with the Hausdorff metric  $\rho_H$ :

$$\rho_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B), B \subset O_\varepsilon(A)\}.$$

By  $cc(\mathbb{R}^n)$  we denote the subset of  $\exp \mathbb{R}^n$  consisting of all convex subsets.

It is known that the hyperspaces  $\exp \mathbb{R}^n$  and  $cc(\mathbb{R}^n)$ ,  $n \geq 2$ , are both homeomorphic to  $Q \setminus \{*\}$ , where  $Q$  is the Hilbert cube (see respectively [1] and [3]).

In this note we show that, for every natural  $n$ , the hyperspaces  $\exp \mathbb{R}^n$  and  $\text{cc}(\mathbb{R}^n)$  are not coarsely equivalent.

## 2 Preliminaries

We first recall the necessary definitions; see, e.g., [2] for details. A metric space is called proper if every closed ball in it is compact. A map of metric spaces is called metrically proper if the preimage of every bounded set is bounded.

A map  $f: X \rightarrow Y$  between metric spaces  $(X, d)$  and  $(Y, \varrho)$  is called asymptotically Lipschitz if there exist  $\lambda, s > 0$  such that

$$\varrho(f(x), f(y)) \leq \lambda d(x, y) + s, \quad x, y \in X.$$

A map  $f: X \rightarrow Y$  between metric spaces  $(X, d)$  and  $(Y, \varrho)$  is an asymptotically bi-Lipschitz embedding if there exist  $C > 0, D \geq 0$  such that

$$\frac{1}{C}d(x, y) - D \leq \varrho(f(x), f(y)) \leq Cd(x, y) + D$$

for every  $x, y \in X$ .

A map  $f: X \rightarrow Y$  between metric spaces  $(X, d)$  and  $(Y, \varrho)$  is called a coarse embedding if there exist nondecreasing functions  $\varphi_1, \varphi_2: [0, \infty) \rightarrow [0, \infty)$  such that

$$\varphi_1(d(x, y)) \leq \varrho(f(x), f(y)) \leq \varphi_2(d(x, y)), \quad x, y \in X.$$

A map  $f: X \rightarrow Y$  is a coarse equivalence if  $f$  is a coarse embedding and there exists  $r > 0$  such that the  $r$ -neighborhood of  $f(X)$  equals  $Y$ . Equivalently, one can describe coarse equivalences in categorical terms. J. Roe [4] introduced the category whose objects are proper metric spaces and whose morphisms are coarsely uniform, metrically proper maps. Two spaces  $X, Y$  are coarsely equivalent if there are morphisms  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  such that the compositions  $gf$  and  $fg$  are of finite distance to the identity maps  $1_X$  and  $1_Y$  respectively.

A metric space  $(X, d)$  is called geodesic if, for every  $x, y \in X$ , there exists an isometric embedding  $\alpha: [0, d(x, y)] \rightarrow X$  such that  $\alpha(0) = x, \alpha(d(x, y)) = y$ . It is known (see [2]) that any coarsely uniform map  $f: X \rightarrow Y$  of metric spaces is asymptotically Lipschitz provided  $X$  is a geodesic space.

**Lemma 1** *The space  $\text{cc}(\mathbb{R}^n)$  is a geodesic space.*

*Proof* Let  $A, B \in \text{cc}(\mathbb{R}^n)$  and  $d_H(A, B) = c$ . It is easy to see that the map  $\alpha: [0, c] \rightarrow \text{cc}(\mathbb{R}^n)$ ,

$$t \mapsto \frac{c-t}{c}A + \frac{t}{c}B, \quad t \in [0, c],$$

is an isometric embedding.

**Lemma 2** *The space  $\exp \mathbb{R}^n$  is a geodesic space.*

*Proof* Let  $A, B \in \exp \mathbb{R}^n$  and  $d_H(A, B) = c$ . Given  $t \in [0, c]$ , define

$$C_t = \left\{ \frac{c-t}{c}a + \frac{t}{c}b \mid a \in A, b \in B, d(a, b) \leq c \right\}.$$

Then the map  $t \mapsto C_t$  is an isometric embedding of the segment  $[0, c]$  into  $\exp \mathbb{R}^n$  such that  $0 \mapsto A$  and  $c \mapsto B$ .

If  $x_0$  is a given base point of a metric space  $(X, d)$ , we say that  $\|x\| = d(x, x_0)$  is the norm of  $x \in X$ . We choose  $\{0\}$  as the base point in both  $\text{cc}(\mathbb{R}^n)$  and  $\exp \mathbb{R}^n$ .

Let  $C \geq 1, r, R > 0$ . Given a metric space  $X$ , define

$$\Phi_{C,r}(R) = \Phi_{C,r}^X(R) = \max\{|A| \mid A \subset \bar{B}_{Cr}(x) \text{ is } r\text{-discrete,} \\ \|x\| \leq R\}.$$

Let  $D > 0$ . A subset  $A$  of a metric space  $X$  is said to be  $D$ -discrete if  $d(x, y) \geq D$ , for every  $x, y \in A, x \neq y$ . It is a folklore (and easy to see) that any asymptotically Lipschitz map of a  $D$ -discrete space (for some  $D > 0$ ) is Lipschitz.

**Proposition 1** *Suppose that there exists an asymptotically Lipschitz embedding of a metric space  $X$  into a metric space  $Y$ . Then there exist  $K > 0$  and  $r_0 > 0$  such that, for every  $C \geq 1, r \geq r_0$ , there exists  $R_0 > 0$  such that*

$$\Phi_{C,r}^X \leq \Phi_{KC, \frac{r}{K}}^Y(KR)$$

for every  $R > R_0$ .

*Proof* Given an asymptotically Lipschitz embedding  $f: X \rightarrow Y$ , find  $K > 0$  and  $r_0 > 0$  such that, for any  $K$ -discrete subspace  $X'$  of  $X$  the following holds:

$$\frac{1}{K}d(x, y) \leq \varrho(f(x), f(y)) \leq Kd(x, y)$$

for every  $x, y \in X'$ .

The conclusion of the proposition follows from the fact that the image of any  $r$ -discrete set where  $r \geq r_0$ , in  $X$ , is an  $(r/K)$ -discrete subset of  $Y$ .

### 3 Main result

**Lemma 1** *Let  $r > 0$ . The cardinality of any family of  $r$ -disjoint convex sets in the  $C_1r$ -neighborhood of any  $W \in \text{cc}(\mathbb{R}^n)$  with  $\|W\| \leq R$  (in the space  $\text{cc}(\mathbb{R}^n)$ ) is  $\leq C_2 2^{R^{n-1}}$ , where  $C_2 = C_2(r, C_1)$ .*

*In other words,  $\Phi_{C,r}^{\text{cc}(\mathbb{R}^n)}(R) \leq C_2 2^{R^{n-1}}$ .*

*Proof* Let  $A$  be a maximal (with respect to inclusion)  $(r/3)$ -discrete subset of  $U$ , where by  $U$  we denote the  $2C_1r$ -neighborhood of the set  $\partial W$ .

Suppose that  $A_1, A_2 \in \text{cc}(\mathbb{R}^n)$ ,  $A_1 \neq A_2$ ,  $d_H(A_i, \bar{B}_R(0)) < C_1r$ ,  $i = 1, 2$ . Then also  $d_H(\partial A_i, S_R^{n-1}) < C_1r$ . Since  $A_1 \neq A_2$ ,  $d_H(A_1, A_2) \geq r$  and therefore without loss of generality we may assume that there exists  $x \in U$  such that  $x \in A_1$  and  $B_r(x) \cap A_2 = \emptyset$ . Then there exists  $a \in A$  such that  $a \in B_{r/2}(x) \subset O_{r/2}(A_1)$ . Clearly,  $a \notin O_{r/2}(A_2)$ .

Denote by  $\mathcal{A}$  a family of  $r$ -disjoint convex sets in a  $C_1r$ -neighborhood of the set  $\partial W$  (in the space  $\text{cc}(\mathbb{R}^n)$ ). We conclude from what is shown above that there exists an embedding  $K \mapsto A \cap O_{r/2}(K)$  of  $\mathcal{A}$  into the set  $2^A$ .

It easily follows from the properties of the volume of the outer convex bodies that the maximum of the volume of  $U$  is attained when  $W = \bar{B}_R(0)$ . Since in this case the volume of  $U$  is  $C_3R^{n-1}$ , the cardinality of  $A$  is  $C_4R^{n-1}$  (here,  $C_i = C_i(C_1, r)$ ,  $i = 3, 4$ ) and we obtain that the cardinality of the family  $\mathcal{A}$  is  $\leq 2^{C_4R^{n-1}} = C_2 2^{R^{n-1}}$ .

**Lemma 2** *Let  $C \geq 1$ . Then  $\Phi_{C,r}^{\text{exp } \mathbb{R}^n}(R) \geq C_3 2^{R^n}$ ,  $R \geq R_0$ , for some  $C_3 = C_3(C, r)$  and some  $R_0 \geq 0$ .*

*Proof* Suppose that  $A(R) = 3r\mathbb{Z}^n \cap \bar{B}_R(0)$ . Then  $A$  is a  $3r$ -discrete set. For any  $A \subset A(R)$ , let  $A_A(R) = \bar{B}_R(0) \setminus O_r(A)$ . The map  $A \mapsto A_A(R)$  is injective and its image is an  $r$ -discrete subset of the  $r$ -neighborhood of  $\{0\}$  in  $\text{exp } \mathbb{R}^n$ .

Since the number of points of  $A(R)$  is at least  $C'R^n$ , for some  $C' > 0$  and  $R \geq R_0$ , for some  $R_0$ , we are done.

**Theorem 1** *The spaces  $\text{exp } \mathbb{R}^n$  and  $\text{cc}(\mathbb{R}^n)$  are not coarsely equivalent.*

*Proof* Assume the contrary, then there exist coarsely uniform maps  $f: \text{exp } \mathbb{R}^n \rightarrow \text{cc}(\mathbb{R}^n)$  and  $g: \text{cc}(\mathbb{R}^n) \rightarrow \text{exp } \mathbb{R}^n$  such that  $gf$  and  $fg$  are of finite distance to  $1_{\text{exp } \mathbb{R}^n}$  and  $1_{\text{cc}(\mathbb{R}^n)}$  respectively. Since the spaces  $\text{exp } \mathbb{R}^n$  and  $\text{cc}(\mathbb{R}^n)$  are proven to be geodesic, the maps  $f$  and  $g$  are asymptotically Lipschitz. Moreover, the map  $f$  is an asymptotically Lipschitz embedding.

Now we use Lemmas 1, 2, and Proposition 1 in order to conclude that there exist constants  $C_2, C_3 > 0$  and  $R_0 \geq 0$  such that  $C_2 2^{R^n} \leq C_3 2^{R^{n-1}}$ , whenever  $R \geq R_0$ . This gives us a contradiction.

#### 4 Remarks

For  $n \geq 1$  the result follows from the dimensional arguments: the space  $\exp \mathbb{R}$  is infinite-dimensional (in the sense of asymptotic dimension; see [2]) while the asymptotic dimension of  $\text{cc}(\mathbb{R})$  is 2.

By  $\exp^c X$  we denote the hyperspace of continua of a space  $X$ , this is a subspace in  $\exp X$  whose elements are compact connected sunsets in  $X$ . It is not known whether the spaces  $\text{cc}(\mathbb{R}^n)$  and  $\exp^c \mathbb{R}^n$  are coarsely equivalent.

The following question concerns the hyperbolic space  $\mathbb{H}^n$ : are the spaces  $\text{cc}(\mathbb{H}^n)$  and  $\exp \mathbb{H}^n$  coarsely equivalent?

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