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Moyal and Rankin-Cohen deformations of algebras

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Abstract. It is proven that Rankin-Cohen brackets form an associative deformation of the algebra of polynomials whose coefficients are holomorphic functions on the upper half-plane.

Анотація. Будується вкладення добутку верхньої напівплощини на комплексну площину з виколотим нулем в комплексний простір $\mathbb{C}^2$ з координатами (q, p). Уздовж нього переноситься бівекторне поле

$$M = \partial_p \otimes \partial_q - \partial_q \otimes \partial_p,$$

двоїсте до стандартної симпліціальної форми на $\mathbb{C}^2$. Подібним чином з деформації Муаяля алгебри поліномів від двох змінних отримано асоціативну деформацію алгебри поліномів, коефіцієнти яких є голоморфними функціями на верхній напівплощині. Мета і основний результат: доведено, що n-й член цієї деформації збігається з n-ю дужкою Ранкіна-Коена. Це пов'язує деформацію з модулярними формами. В доведенні використовується результат Ель Градекі про єдиність з точністю до числового множника $SL(2, \mathbb{R})$-еквіваріантного бідиференціального оператора на просторах голоморфних функцій на верхній напівплощині з певною дією групи $SL(2, \mathbb{R})$. Як члени згаданої деформації, так і дужки Ранкіна-Коена задовольняють ці умови, отже, є пропорційними. Перевіряється, що коефіцієнт пропорційності дорівнює 1. Наведене доведення основного твердження коротше за існуючі: П. Коен, Маніна і Загіра; Конна і Московічі; В. Овсієнка.

Аннотация. Доказано, что скобки Ранкина-Коэна образуют ассоциативную деформацию алгебры многочленов, коэффициенты которых являются голоморфными функциями на верхней полуплоскости.

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1. Rankin-Cohen brackets

Denote $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$, $\mathbb{C}^\times = \mathbb{C} - \{0\}$. Then the left action

$$SL(2,\mathbb{R}) \times (\mathbb{H} \times \mathbb{C}^\times) \to \mathbb{H} \times \mathbb{C}^\times, \qquad \gamma.(z,X) = \Big(\frac{az+b}{cz+d}, \frac{X}{cz+d}\Big),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, induces the right action of $SL(2,\mathbb{R})$ on the algebra of holomorphic functions $\mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times)$. The restriction of this action to the subspace $X^k \mathcal{H}ol(\mathbb{H})$, $k \in \mathbb{Z}$, equips $\mathcal{H}ol(\mathbb{H})$ with the right action $|_k$ of $SL(2,\mathbb{R})$. Recall that a modular (automorphic) form of weight k [10, Definition 2.1] is, in particular, a fixed point of this action. Rankin described in [8] $SL(2,\mathbb{R})$-invariant polydifferential operators on $\mathbb{H}$. Earlier Gordan proposed bidifferential $SL(2,\mathbb{R})$-invariant operators and called them transvectants [6]. In modular form theory these operators are known as Rankin-Cohen brackets. They were defined by H. Cohen as follows in [1]. Let $k, l \in \mathbb{Z}$, $n \in \mathbb{N} = \mathbb{Z}_{\geqslant 0}$, $f, g \in \mathcal{H}ol(\mathbb{H})$. Then

$$[X^k f, X^l g]^{RC}_n = X^{k+l+2n} \sum_{r+s=n}^{r,s\geqslant 0} (-1)^s \binom{k+n-1}{s} \binom{l+n-1}{r} f^{(r)} g^{(s)},$$

where

$$f^{(r)} = \partial_z^r f, \qquad \partial_z f = (\partial_x f - i\partial_y f)/2.$$

H. Cohen proved in [1, Theorem 7.1] that the operation $[\_,\_]^{RC}_n$ on the algebra $\mathcal{H}ol(\mathbb{H})[X^{-1}, X]$ is $SL(2,\mathbb{R})$-equivariant.

2. Deformations

A *deformation* of an associative $\mathbb{C}$-algebra $(A, m : A \otimes A \to A)$ is a $\mathbb{C}$-bilinear map

$$\mu : A \times A \to A[[\hbar]], \qquad \mu(a,b) = \sum_{n=0}^{\infty} \hbar^n \mu_n(a,b)$$

such that

$$\mu_0(a,b) = m(a \otimes b) = ab, \qquad \mu(1,a) = a = \mu(a,1),$$

and the $\mathbb{C}[[\hbar]]$-bilinear map $\tilde{\mu} : A[[\hbar]] \times A[[\hbar]] \to A[[\hbar]]$, which extends μ by $\mathbb{C}[\hbar]$-bilinearity and $\hbar$-adic continuity, is associative.

Let $\mathfrak{a} \subset \mathrm{Der} A$ be an abelian $\mathbb{C}$-subalgebra of the Lie $\mathbb{C}$-algebra of derivations of A. View an element

$$P = \sum_{i=1}^{l} \xi_i \otimes \eta_i \in \mathfrak{a} \otimes_{\mathbb{C}} \mathfrak{a}$$

as a $\mathbb{C}$-linear operator $A^{\otimes 2} \to A^{\otimes 2}$. It is well-known that

$$a \star_P b \equiv \mu_P(a,b) = m \exp(\hbar P).(a \otimes b) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} m P^n.(a \otimes b)$$

is a deformation of A. An example is provided by the Moyal deformation of $A = \mathcal{H}ol(\mathbb{C}^2)$, $\mathbb{C}^2 = \{(q,p)\}$, $P = M = \partial_p \otimes \partial_q - \partial_q \otimes \partial_p$. The point of this note is to show that the Rankin-Cohen brackets give a deformation of the algebra $\mathcal{H}ol(\mathbb{H})[X]$, related to the Moyal deformation.

Following a remark of Eholzer, Zagier noticed in [12] that many identities involving operations $[,]_n^{RC}$ follow from the statement that these form a deformation of the algebra of modular forms. This was further elaborated by P. Cohen, Manin, Zagier [2], where they show that the deformed algebra of modular forms is isomorphic to certain algebra of invariant pseudodifferential operators.

Connes and Moscovici discovered certain Hopf algebra $\mathcal{H}_1$ during their work on foliations [3]. They described also its action [4] on the algebra of modular forms from which a deformation of the latter is obtained. Moreover the Rankin-Cohen deformation generalizes to a universal deformation formula of $\mathcal{H}_1$ as shown by Tang and Yao [11]. These results are reviewed in [9].

Our proof of the main result is concise. It is most close to the proof by V. Ovsienko [7] with the difference that he looks for a symplectomorphism whereas we find an open embedding of $\mathbb{H} \times \mathbb{C}^\times$ into $\mathbb{C}^2$, compatible with the Poisson structure.

3. Rankin-Cohen brackets as deformation

Consider the open embedding

$$\Psi = \big(\mathbb{H} \times \mathbb{C}^\times \hookrightarrow \mathbb{C} \times \mathbb{C}^\times \xrightarrow[\cong]{\psi} \mathbb{C} \times \mathbb{C}^\times \hookrightarrow \mathbb{C}^2\big),$$

where $\psi(z,X) = (zX^{-1}, X^{-1}) = (q,p)$. Vector field ∂_p (resp. ∂_q) on $\mathbb{C}^2$ is lifted along Ψ to vector field $\xi = -X^2\partial_X - zX\partial_z$ (resp. $\eta = X\partial_z$) on $\mathbb{H} \times \mathbb{C}^\times$. Thus,

$$M = \partial_p \otimes \partial_q - \partial_q \otimes \partial_p$$

is lifted to

$$P = RC = \xi \otimes \eta - \eta \otimes \xi$$

and ξ and η commute. Hence,

$$F \star_{RC} G = m \exp(\hbar RC).(F \otimes G) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} m (RC)^n.(F \otimes G)$$

is a deformation of $\mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times)$.

Theorem 3.1. *Let $k, l \in \mathbb{Z}$, $n \in \mathbb{N}$. Assume that one of the following conditions holds:*

- $k > 0$;
- $l > 0$;
- $k, l \leqslant 0$ *and* $n < \max\{1-k, 1-l\}$;
- $k, l \leqslant 0$, $n > 1-k-l$.

Then for all $f, g \in \mathcal{H}ol(\mathbb{H})$ and $F = X^k f$, $G = X^l g$ we have

$$(n!)^{-1} m(RC)^n.(F \otimes G) = [F, G]_n^{RC}.$$

Proof. The embedding $\Psi : \mathbb{H} \times \mathbb{C}^\times \hookrightarrow \mathbb{C}^2$ is $SL(2,\mathbb{R})$-equivariant, where the left action on $\mathbb{C}^2 = \{(q,p)\}$ is the standard action of matrices on vectors. The mapping

$$M = \partial_p \otimes \partial_q - \partial_q \otimes \partial_p \ \in \ \mathrm{End}_{\mathbb{C}}(\mathcal{H}ol(\mathbb{C}^2) \otimes \mathcal{H}ol(\mathbb{C}^2))$$

commutes with the action of $SL(2,\mathbb{R})$, or, equivalently, of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Hence, the Moyal deformation is $SL(2,\mathbb{R})$-equivariant. This implies and can be checked directly that

$$RC = \xi \otimes \eta - \eta \otimes \xi \ \in \ \mathrm{End}_{\mathbb{C}}(\mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times) \otimes \mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times))$$

commutes with the action of $\mathfrak{sl}(2,\mathbb{R})$ and $SL(2,\mathbb{R})$. Thus,

$$\star_{RC} : \mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times) \otimes \mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times) \to \mathcal{H}ol(\mathbb{H} \times \mathbb{C}^\times)[[\hbar]]$$

is a homomorphism of $SL(2,\mathbb{R})$-modules. Accordingly to El Gradechi [5, Proposition 3.11] if the hypotheses on (k,l,n) hold, then there is only 1-dimensional vector space of $SL(2,\mathbb{R})$-equivariant bidifferential operators

$$(\mathcal{H}ol(\mathbb{H}), |_k) \otimes (\mathcal{H}ol(\mathbb{H}), |_l) \to (\mathcal{H}ol(\mathbb{H}), |_{k+l+2n}).$$

Both maps

$$f \otimes g \mapsto (n!)^{-1} m(RC)^n.(X^k f \otimes X^l g), \qquad f \otimes g \mapsto [X^k f, X^l g]_n^{RC}$$

belong to this space, therefore, they are proportional. One checks that the proportionality constant is 1. In fact, evaluate both operators on the following $f \otimes g$.

If $k = l = n = 0$, take $f = g = 1$.

If $k > 0$ or $k < 0$, $k \leqslant l \leqslant 0$, $n < 1-k$, take $f = 1$, $g = z^n$.

If $l > 0$ or $l < 0$, $l \leqslant k \leqslant 0$, $n < 1-l$, take $f = z^n$, $g = 1$.

If $k, l \leqslant 0$ and $n > 1-k-l$ choose $r, s \in \mathbb{N}$, such that $r+s = n$, $r+k > 0$ and $s+l > 0$, and take $f(z) = z^r$ and $g(z) = z^s$. (Hint: use coordinates (q,p) to make the computations easier). $\square$

Corollary 3.2. *The Rankin-Cohen deformation $\star_{RC}$ restricted to $A^{\otimes 2}$, $A = \mathcal{H}ol(\mathbb{H})[X]$ coincides with the map*

$$A \otimes A \to A[[\hbar]], \qquad X^k f \otimes X^l g \mapsto \sum_{n=0}^{\infty} \hbar^n [X^k f, X^l g]_n^{RC}.$$

Proof. For all $(k,l,n) \in \mathbb{N}^3$ the statement follows from Theorem 3.1 except for $(k,l,n) = (0,0,1)$. In the latter case $[f,g]_1 = 0 = m(RC)(f \otimes g)$ for all $f, g \in \mathcal{H}ol(\mathbb{H})$. □

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