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METHOD FOR ANALYSING NON-ERGODIC MARKOV SYSTEMS

МЕТОД АНАЛІЗУ НЕЕРГОДИЧНИХ СИСТЕМИ МАРКОВА

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Abstract. *The object of study is a non-ergodic Markov system functioning under conditions when there is a subset of absorbing states among the set of possible states. The relevance of the problem is determined by the impossibility of analysing such systems by traditional methods. A high practical need for solving these problems arises to ensure the required efficiency of complex systems. Accordingly, the aim of the research is to develop a simple, convenient and efficient method for analysing non-ergodic Markov systems. The method of solving the problem proposed in this paper consists in transforming the original non-ergodic system into an ergodic one. The essence of the method consists in adding to the set of possible transitions of the system fictitious transitions from absorbing states to return states. In this case, the original non-ergodic system is transformed to an ergodic system available for analysis in the standard way. Thus, the proposed method solves the problem. The advantage of the method is simplicity of implementation and practical independence of the computational procedure from the dimensionality of the problem.*

Анотація. *Об'єкт дослідження - неергодична марковська система, що функціонує в умовах, коли серед множини можливих станів є підмножина поглинаючих. Актуальність проблеми визначається неможливістю аналізу таких систем традиційними методами. Висока практична потреба розв'язання цих задач виникає для забезпечення необхідної ефективності складних систем. Відповідно до цього мета дослідження - розроблення простого, зручного й ефективного методу аналізу неергодичних марковських систем. Запропонований у роботі метод розв'язання поставленої задачі полягає в перетворенні вихідної неергодичної системи на ергодичну. Сутність методу полягає в додаванні до множини можливих переходів системи фіктивних переходів зі станів, що поглинають, у зворотні. При цьому вихідна неергодична система трансформується до ергодичної, доступної для аналізу стандартним шляхом. Таким чином, запропонований метод вирішує поставлене завдання. Перевага методу - простота реалізації та практична незалежність обчислювальної процедури від розмірності задачі.*

Key words: analysis of non-ergodic Markov systems, transformation of a non-ergodic Markov chain into an ergodic one.
Ключові слова: аналіз неергодичних марковських систем, перетворення неергодичного марковського ланцюга в ергодичний.

Introduction. The most important element of the procedure for analysing Markov systems is the calculation of the probability distribution of states at any moment of time [1]. At the same time, the final distribution of these probabilities is of great practical interest. If each state of the system is reachable from any other state of this system, then the system is ergodic and to obtain the final probability distribution of states, the system of linear algebraic equations Kolmogorov [2] is solved, which is formed as follows. The system in the process of functioning can be in one of its states.Possibilities of transition of the system to other states are determined by the matrix of transition probabilities $P=(p_{ij})$,where p_{ij} probability of transition of the system from the state $i, i=1,2,..,n,$ in the state of $j, j=1,2,..,n,$ in one step. Let's introduce the set $\mathrm{p}=(\mathrm{p}_1,\mathrm{p}_2,...,\mathrm{p}_n)$ of the final state probabilities. Then the system of algebraic equations with respect to unknown final probabilities p_{i_1} $i=1,2,..,n$ is in the form of:

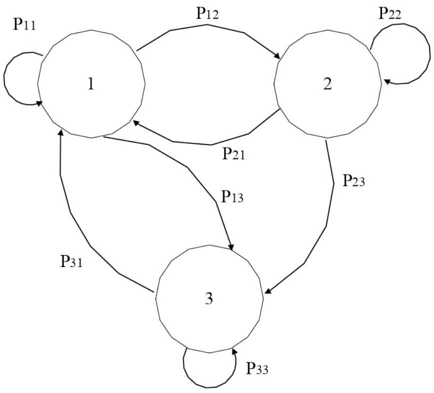
$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} + \ldots + \pi_n p_{n1},$

$\pi_2 = \pi_1 p_{12} + \pi_2 p_{22} + \ldots + \pi_n p_{n2},$

....

$\pi_n = \pi_1 p_{1n} + \pi_2 p_{2n} + \ldots + \pi_n p_{nn}.$

This system of equations is solved by known methods, for example, by Kramer's formulas [3]. Let us consider an example. Let the system can be in one of three states. The graph of states and the transition probability matrix are shown in Fig. 1.



$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & 0 & P_{33} \end{pmatrix}$$

Fig.1. State graph and transition probability matrix
Рис.1. Граф станів та матриця ймовірностей переходу

Let us introduce a vector of final probabilities of the system states $\pi = (\pi_1, \pi_2, \pi_3)$. The system of equations for finding the vector of final probabilities is as follows:

$\pi = (\pi_1 \pi_2 \pi_3) = (\pi_1 \pi_2 \pi_3) P.$

Let us write this system of equations in scalar form:

$\pi_1 = \pi_1 P_{11} + \pi_2 P_{21} + \pi_3 P_{31},$

$\pi_2 = \pi_1 P_{12} + \pi_2 P_{22},$

$\pi_3 = \pi_1 P_{13} + \pi_2 P_{23} + \pi_3 P_{33}.$

The resulting system of equations is linearly dependent [3]. Indeed, summing up the left and right parts of these equations, we obtain the identity:

$\pi_1 + \pi_2 + \pi_3 = \pi_1 (P_{11} + P_{12} + P_{13}) + \pi_2 (P_{21} + P_{22} + P_{23}) + \pi_3 (P_{31} + P_{32}).$

In this connection, let us replace one of these equations by the normalisation condition. In this case, we obtain

$$\pi_1 = \pi_1 P_{11} + \pi_2 P_{21} + \pi_3 P_{31},$$
$$\pi_2 = \pi_1 P_{12} + \pi_2 P_{22},$$
$$\pi_1 + \pi_2 + \pi_3 = 1.$$

Using the second equation of the system, express the variable π_1 through π_2.

We have

$\pi_2 (1 - P_{22}) = \pi_1 P_{12},$

from where

$\pi_1 = \dfrac{1 - P_{22}}{P_{11}} \pi_2.$

Let's substitute the obtained expressions into the first equation of the system, which we will bring to the form:

$$\pi_1 = (1 - P_{11}) = \pi_2 P_{21} + \pi_3 P_{31}.$$

Then

$\dfrac{1 - P_{22}}{1 - P_{11}} \pi_2 - \pi_2 P_{21} = \pi_3 P_{31}$

or

$\pi_2 (\dfrac{1 - P_{22}}{1 - P_{11}} - P_{21}) = \pi_3 P_{31}$

or

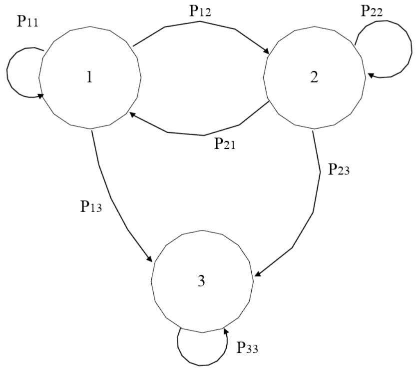
$$\pi_2 (\frac{(1 - P_{22}) - P_{21}(1 - P_{11})}{1 - P_{11}}) = \pi_3 P_{31}$$

or

$$\pi_2 = \pi_3 \frac{P_{31}(1-P_{11})}{(1-P_{22}) - P_{21}(1-P_{11})}. \quad (1)$$

Now, substituting the obtained relations for π_1 and π_2 in the normalisation condition, we find the value of π_3 through which we obtain relations for calculation of final probabilities of states π_1, π_2, π_3.

If among the possible states of the system there is at least one absorbing state, the system becomes non-ergodic, and its analysis by traditional methods is possible in the case when such a state is the only one. Let us consider an example. The graph of the system states and the transition probability matrix are shown in Fig. 2.



$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

Fig.2. State graph and transition probability matrix
Рис.2. Граф станів та матриця ймовірностей переходів

Let us introduce the vector of final probabilities of the system $\pi = (\pi_1 \pi_2 \pi_3)$;

To find the vector of final probabilities, let us form a system of Kolmogorov equations

$$\lambda = \pi P. \quad (2)$$

Let us reduce the matrix system of equations (2) to the scalar form

$$(\pi_1 \pi_2 \pi_3) = (\pi_1 \pi_2 \pi_3) \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

From here

$$\begin{aligned} \pi_1 &= \pi_1 P_{11} + \pi_2 P_{21}, \\ \pi_2 &= \pi_1 P_{12} + \pi_2 P_{22}, \\ \pi_3 &= \pi_1 P_{13} + \pi_2 P_{23} + \pi_3. \end{aligned} \quad (3)$$

The resulting system is linearly dependent. Indeed, let us sum up the first two equations of the system (3).

$$\pi_1 + \pi_2 = \pi_1 (P_{11} + P_{12}) + \pi_2 (P_{21} + P_{22}),$$

or

$$\pi_1 + \pi_2 = \pi_1 (1 - P_{13}) + \pi_2 (1 - P_{23}),$$

or

$$\pi_1 + \pi_2 = \pi_1 - \pi_1 P_{13} + \pi_2 - \pi_2 P_{23}.$$

From here

$$\pi_1 P_{13} + \pi_2 P_{23} = 0 .$$

The obtained equation coincides with the third equation of the system (3). Therefore, we remove the third equation from the system (3) and add the normalisation condition instead. Thus we obtain:

$$\begin{aligned} \pi_1 &= \pi_1 P_{11} + \pi_2 P_{21}, \\ \pi_2 &= \pi_1 P_{12} + \pi_2 P_{22}, \\ \pi_1 &+ \pi_2 + \pi_3 = 1. \end{aligned} \quad (4)$$

From here

$$\begin{aligned} &\pi_1 = (1 - P_{11}) - \pi_2 P_{21} = 0, \\ &\pi_1 P_{12} - \pi_2 (1 - P_{22}) = 0, \\ &\pi_1 + \pi_2 + \pi_3 = 1. \end{aligned} \quad (5)$$

Let us solve the obtained system of equations by Cramer's rule [4]. Accordingly, using (5), we introduce the required set of determinants of matrices:

$$A_0 = det\begin{pmatrix} 1 - P_{11} & - P_{21} & 0 \\ P_{12} & 1 - P_{22} & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A_1 = det\begin{pmatrix} 0 & - P_{21} & 0 \\ 0 & -(1 - P_{21}) & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A_2 = det\begin{pmatrix} 1 - P_{11} & 0 & 0 \\ P_{12} & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A_3 = det\begin{pmatrix} 1 - P_{11} & - P_{21} & 0 \\ P_{12} & 1 - P_{22} & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

From here

$$\begin{aligned} A_0 &= (1 - P_{11})(1 - P_{22}) + P_{12}P_{21} = \\ &= 1 - P_{22} - 1_{11} + P_{11}P_{22} + P_{12}P_{21} = \\ &= P_{11} + P_{12} + P_{13} - P_{11} + P_{11}P_{22} + P_{12}P_{21} = \\ &= P_{12} + 1_{13} + P_{11}P_{22} + P_{12}P_{21}; \end{aligned}$$

$$A_1 = A_2 = 0;$$

$A_3 = A_0$.

There fore

$$\pi_1 = \frac{A_1}{A_0} = 0;\ \pi_2 = \frac{A_2}{A_0} = 0;\ \pi_3 = \frac{A_3}{A_0} = 1. \tag{6}$$

A natural and expected result is obtained: the analysed system has a single absorbing state. Therefore, all components of the final probability distribution of the system states are equal to zero, except for the only probability corresponding to the absorbing state, which is equal to one. Note that in this particular problem this result could be obtained from relation (1), if in the final relation we put the probability of $P_{31} = 0,$ thus making the state with number 3 absorbing. Thus from (1) for $P_{31} = 1$ value $\pi_2 = 0,$ from here $\pi_1 = 0$ and $\pi_3 = 1.$

Consider again a system which at each moment of time can be in one of its possible states. In the process of functioning the system passes from one state to another. Suppose now that some of the possible states are absorbing. It is clear that the final probability distribution of such a system has the following structure: the final probabilities for the returning states are zero, and the final probabilities of the absorbing states sum to one. The probability distribution for absorbing states is of practical interest because the possible consequences of stopping the system in different states can be significantly unequal. A system in which the property of accessibility of any state of the system from any other state is not fulfilled is by definition non-ergodic. To analyse such a system, the standard approach used above, based on the solution of the system of linear algebraic equations of Kolmogorov with respect to the unknown final probabilities of the system states, is unacceptable, which determines the purpose of the work.

Purpose of the study. Development of a method for calculating the set of final state probabilities for a non-ergodic Markov chain.

Development of a method for analysing a non-ergodic Markov system. The simplest approach to solving this problem is to use the step-by-step evolution of the system state vector. Let us introduce an initial vector of state probabilities $\pi^{(0)} = (\pi_1^{(0)}, \pi_2^{(0)}, .., \pi_n^{(0)})$ and matrix $P = (pij)$, of transition probabilities.

Then

$\pi^{(1)} = (\pi_1^{(1)}, \pi_2^{(1)}, .., \pi_n^{(1)}) = P.$

In this case

$\pi_1^{(1)} = \pi_1^{(0)} P_{11} + \pi_2^{(0)} P_{21} + \ldots + \pi_n^{(0)} P_{n1},$

$\pi_2^{(1)} = \pi_1^{(1)} P_{12} + \pi_2^{(0)} P_{22} + \ldots + \pi_n^{(0)} P_{n2},$

....

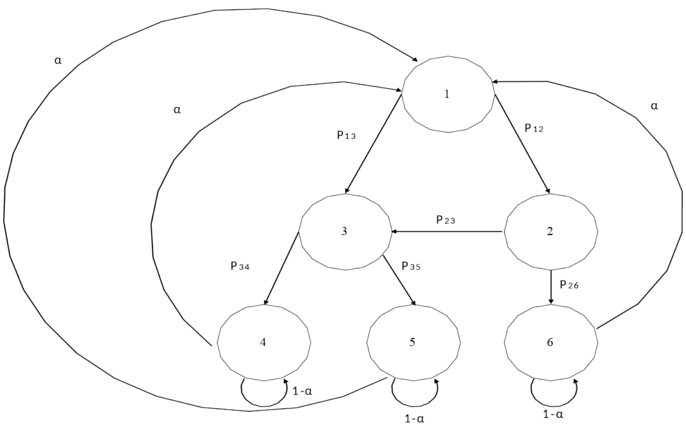
$\pi_n^{(1)} = \pi_1^{(0)} P_{1n} + \pi_2^{(0)} P_{2n} + \ldots + \pi_n^{(0)} P_{nn}.$

Further

$$\pi^{(2)} = \pi^{(1)}P = \pi^{(0)}P^2,$$
$$\pi^{(3)} = \pi^{(2)}P = \pi^{(0)}P^3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\pi^{(k+1)} = \pi^{(k)}P = \pi^{(0)}P^{k+1}.$$

It is clear that each step of this procedure approximates the vector $\pi^{(k+1)}$, obtained at the next step to the final distribution of state probabilities. The procedure continues until the natural stopping criterion is fulfilled $\max\left\{\Delta\pi_j^{(k+1)} = \pi_j^{(k+1)} - \pi_j^{(k)}\right\} < \varepsilon$, ε – a given small number. The disadvantages of this approach are obvious. Firstly, there is no proof of monotonic convergence of the procedure. Second, the convergence rate of the procedure is unpredictable. It is not known how it depends on the dimensionality of the system.

In this connection, to solve this problem, we consider a computational procedure transforming the initial nonergodic system into a pseudoergodic one. For this purpose, for each absorbing state the probability of return to this state, equal to unity, will be reduced by some small value α. It is necessary to add a fictitious transition from this state with the same probability α to any return state, for example, to the first state. The corresponding transition graph for the resulting system and the transition probability matrix are shown in Fig. 3.



$$P = \begin{pmatrix} 0 & P_{12} & P_{13} & 0 & 0 & 0 \\ 0 & 0 & P_{23} & 0 & 0 & P_{26} \\ 0 & 0 & 0 & P_{34} & P_{35} & 0 \\ \alpha & 0 & 0 & 1-\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 & 1-\alpha & 0 \\ \alpha & 0 & 0 & 0 & 0 & 1-\alpha \end{pmatrix}$$

Fig.3. State graph and transition probability matrix
Рис.3. Графік станів та матриця ймовірностей переходу

Let us introduce a vector of final probabilities of the system states and a system of linear algebraic equations to calculate the components of this system:

$$\pi = (\pi_1\pi_2\pi_3\pi_4\pi_5\pi_6); \ \lambda = \pi P. \quad (7)$$

Let us write the system of equations (7) in scalar form, dividing it into two subsystems.

$$\pi_1 = \pi_4\alpha + \pi_5\alpha + \pi_6\alpha = (\pi_4 + \pi_5 + \pi_6)\alpha = S(\alpha)\cdot\alpha_1$$
$$\pi_2 = \pi_1 P_{12} = P_{12}S(\alpha)\alpha. \quad (8)$$
$$\pi_3 = \pi_1 P_{13} + \pi_2 P_{23} = \pi_1 P_{13} + \pi_1 P_{12} P_{23} = (P_{13} + P_{12}P_{23})S(\alpha)\alpha,$$

$$\pi_4 = \pi_3 P_{34} + \pi_4(1-\alpha),$$
$$\pi_5 = \pi_3 P_{35} + \pi_5(1-\alpha), \quad (9)$$
$$\pi_6 = \pi_2 P_{26} + \pi_6(1-\alpha).$$

Solving the subsystem (8), we obtain

$$\pi_1(\alpha) = S(\alpha)\cdot\alpha,$$
$$\pi_2(\alpha) = P_{12}\cdot S(\alpha)\cdot\alpha, \quad (10)$$
$$\pi_3(\alpha) = (P_{13} + P_{12}\cdot P_{23})S(\alpha)\alpha.$$

Further, for the equations of the subsystem (9) we open the brackets and give similar terms. Thus, acting sequentially for each of the equations, we obtain:

$$\pi_4 = \pi_3 P_{34} + \pi_4 - 2\pi_4,$$

from where

$$\alpha\pi_4 = P_{34}\pi_3 = P_{34}(P_{13} + P_{12}\cdot P_{23})S(\alpha)\alpha$$
$$\pi_5 = \pi_3\cdot P_{35} + \pi_5 - \alpha\pi_5, \quad (11)$$

from where

$a p_5 = P_{35} p_3 = P_{35}(P_{13} + P_{12} \times P_{23}) S(a) a$

$p_6 = p_2 \times P_{26} + p_6 (1 - a),$

from where

$$a p_6 = P_{26} \times p_2 = P_{26} P_{12} S(a) a$$

Performing in relations (10) the limit transition by a, we get

$p_1 = p_2 = p_3 = 0.$

Then from the normalisation condition we have

$$p_4 + p_5 + p_6 = 1 - (p_1 + p_2 + p_3) = 1, \quad (12)$$

from where $S = 1$.

Substituting (12) into (11), we obtain the desired result

$p_4 = P_{34}(P_{13} + P_{12} \times P_{23}),$

$p_5 = P_{35}(P_{13} + P_{12} \times P_{23}),$

$p_6 = P_{12} P_{26}.$

It is easy to check that the normalisation condition $p_4 + p_5 + p_6 = 1$ is being fulfilled.

Indeed,

$p_4 + p_5 + p_6 = P_{34}(P_{13} + P_{12} \times P_{23}) + P_{35}(P_{13} + P_{12} \times P_{23}) +$

$+ P_{12} \times P_{26} = (P_{34} + P_{35})(P_{13} + P_{12} \times P_{23}) + P_{26} = P_{13} + P_{12} \times P_{23} +$

$+ P_{12}(1 - P_{23}) = P_{13} + P_{12} P_{23} + P_{12} - P_{12} P_{23} = P_{12} + P_{13} = 1.$

The last equality represents the sum of probabilities of all exits from states 1.

Thus, we obtained a methodology for calculating the final probability distribution for the failure states of a Markov system. The computational procedure is based on the use of a simple mechanism of transformation of the initial non-ergodic system into an ergodic one. We note important advantages of this procedure. First, the computational complexity of the implementation of the procedure for a non-ergodic problem does not exceed the complexity of its implementation for an ergodic problem. Second, this complexity is practically independent of the number of failed states and their share in the total number of states. Note, moreover, that the solution of the problem can be obtained not necessarily by analytical solution of the system of linear algebraic equations (8), (9). The required result can be achieved by solving this system of equations numerically, setting the value of the parameter a with the required accuracy. The direction of further research is to extend the proposed method for solving problems of analysing continuous Markovian [6,7] and semi-Markovian [8.9] systems, including situations when the initial data are fuzzy [10].

Conclusions.

1. A technique for calculating the final state probability distribution for a non-ergodic Markov chain is proposed.
2. The technique is based on the transformation of the original non-ergodic discrete system into an ergodic one.

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